$K(\pi, 1)$ Spaces in Algebraic Geometry

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A dissertation submitted in partial satisfaction of the
requirements for the degree of

Doctor of Philosophy
in
Mathematics

in the
Graduate Division
of the
University of California, Berkeley

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Spring 2015

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Abstract

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The theme of this dissertation is the study of fundamental groups and classifying spaces in the context of the étale topology of schemes. The main result is the existence of $K(\pi, 1)$ neighborhoods in the case of semistable (or more generally log smooth) reduction, generalizing a result of Gerd Faltings. As an application to p-adic Hodge theory, we use the existence of these neighborhoods to compare the cohomology of the geometric generic fiber of a semistable scheme over a discrete valuation ring with the cohomology of the associated Faltings topos. The other results include comparison theorems for the cohomology and homotopy types of several types of Milnor fibers. We also prove an ℓ-adic version of a formula of Ogus, describing the monodromy action on the complex of nearby cycles of a log smooth family in terms of the log structure.

“Proof is hard to come by.”

-Proposition Joe

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Acknowledgements

First of all, I would like to thank my advisor, Professor Arthur Ogus, for his guidance during my years in graduate school. Despite his many duties as the chair of the department, he always managed to find the time to discuss mathematics with me.

I am indebted to Professor Ahmed Abbes for suggesting to me the problem of constructing $K(\pi, 1)$ neighborhoods, which became the crux of this Dissertation.

I would like to thank Professor Martin Olsson for numerous fruitful discussions.

I am lucky to have learned from Professors Abbes, Ogus, and Olsson not only the techniques of algebraic geometry, but also the importance of rigor, cautiousness, and good writing style.

Many thanks to Dorka Budacz, Wolfgang Schmalz, and Masha Vlasenko for providing illustrations for the Introduction, and to Ravi Fernando, Anna Seigal, Minseon Shin, and Alex Youcis for their helpful suggestions regarding the exposition.

Special thanks to The Cheeseboard Collective.

Chapter 1

Introduction

Let X be a sufficiently nice topological space — for example, a CW complex or a manifold (see §2.1.1 for the minimal assumptions we need). Assume that X is connected, and pick a base point $x \in X$. We call X a $K(\pi,1)$ *space* if its higher homotopy groups

$$\pi_i(X,x), \qquad i = 2, 3, \ldots$$

are zero. The homotopy type of such a space is completely determined by its fundamental group $\pi_1(X,x)$, and in particular the cohomology of X with coefficients in every local system (a locally constant sheaf of abelian groups) can be identified with the group cohomology of the corresponding representation of $\pi_1(X,x)$.

Similarly, in the context of algebraic geometry, a connected scheme X with a geometric point $\overline{x}$ is a called a $K(\pi,1)$ *scheme* if the cohomology of every *étale* local system agrees with the cohomology of the corresponding representation of the fundamental group $\pi_1^{\text{ét}}(X,\overline{x})$. The importance of this notion was first revealed in the context of Artin's proof of the comparison theorem [SGA73b, Exp. XI, 4.4] between the étale cohomology of a smooth scheme X over $\mathbf{C}$ and the singular cohomology of the associated analytic space X^{an}. The main step in the proof is the construction of a covering of X by $K(\pi,1)$ open subsets[1] by constructing certain "elementary fibrations".

[1]More precisely, Zariski open subsets U such that U^{an} are $K(\pi,1)$ spaces and each $\pi_1(U^{\text{an}})$ is a "good group" (cf. Definition 2.1.13).

Coverings by $K(\pi,1)$ schemes also play an important role in p-adic Hodge theory. In the course of the proof of the comparison between the p-adic étale cohomology and Hodge cohomology of a smooth proper scheme over a p-adic field K (called the Hodge–Tate decomposition, or the C_{HT} conjecture of Fontaine), Faltings showed [Fal88, Lemma II 2.1] that a smooth scheme X over $\mathcal{O}_K$ can be covered by Zariski open subsets U whose geometric generic fibers $U_{\overline{K}}$ are $K(\pi,1)$ schemes. However, to tackle the more difficult C_{dR} and C_{st} conjectures this way, one needs an analogous statement for X *semistable* over $\mathcal{O}_K$. In his subsequent work on C_{st}, Faltings used a different approach, and remarked [Fal02, Remark on p. 242] that one could use the $K(\pi,1)$ property instead if it was known to hold in the semistable case. Our main result proves that this is indeed true.

Theorem (Special case of Theorem 3.2.1)**.** *Let X be a semistable scheme over $\mathcal{O}_K$. Then X can be covered (in the étale topology) by schemes U such that $U_{\overline{K}}$ is a $K(\pi,1)$ scheme.*

In addition, we provide comparison theorems for the homotopy types of several types of Milnor fibers. Finally, in Theorem 5.4.4, we provide an ℓ-adic version of the computation of the monodromy action on nearby cycles in the log smooth case due to Ogus [Ogu13, Theorem 3.3].

⋆ ⋆ ⋆

We start with a gentle introduction (§1.1) to the relevant concepts, concluding with an informal statement of the results in §1.1.5. Section 1.2 discusses our main theorem: its context, corollaries, and the idea of proof (which itself occupies Chapter 3). In §1.3, we state the remaining results: the comparison theorems for Milnor fibers (Chapter 4) and the “monodromy formula” (Chapter 5).

1.1 A non-technical outline

The results in this thesis all deal with *one-parameter degenerations* of algebraic varieties and the associated notions of *Milnor fibers*, *monodromy*, and *nearby cycles*. We will start by reviewing these concepts in the classical (complex analytic) setting (§1.1.1), then in the algebraic setting (§1.1.2). After discussing how these objects behave in the situation when the degeneration is *semistable* (§1.1.3), and briefly touching on the topic of p-adic Hodge theory (§1.1.4), we state the results of this thesis in §1.1.5.

1.1.1 The classical theory over the complex numbers

In the complex analytic picture, one typically takes a small disc

$$S = \{z : |z| < \delta\} \subseteq \mathbf{C}$$

as a base (parameter space) and considers a holomorphic map $f : X \longrightarrow S$ from a complex manifold (or more generally an analytic space) X. We think of f as of a family of spaces $X_t = f^{-1}(t)$ parametrized by $t \in S$. Let $S^* = S \setminus \{0\}$ be the punctured disc, $X^* = X \setminus f^{-1}(0)$ its preimage. We assume that $f|_{X^*} : X^* \longrightarrow S^*$ is a locally trivial fibration, i.e. that locally on S^*, f topologically looks like the projection $X_t \times S^* \rightarrow S^*$ (in particular all X_t are homeomorphic for $t \neq 0$), and that the "special fiber" $X_0 = f^{-1}(0)$ is a deformation retract of X. These assumptions are satisfied for $\delta \ll 1$ for example if f is proper [GM88, 1.5, 1.7].Usually f^* is assumed to be smooth (i.e., a submersion), so that the fibers X_ε ($\varepsilon \neq 0$) are complex manifolds, while X_0 acquires some singularities (which is why we call f a degeneration), as in Figure 1.1.1.

One can study the topological properties of f in the neighborhood of X_0, both globally and locally, using the notions discussed below.

Milnor fibers

In the local situation, the original approach of Milnor [Mil68] is as follows. Suppose that $X \subseteq \mathbf{C}^N$, and let $x \in X_0$. Let $S_x(\varepsilon)$ be the intersection of X with a sphere (in the Euclidean metric on $\mathbf{C}^N$) of radius $\varepsilon \ll 1$ and center x. Consider the map

$$\varphi_x = \arg f : S_x(\varepsilon) \setminus X_0 \longrightarrow S^* \xrightarrow{\arg} \mathbf{S}^1,$$

called the *Milnor fibration*. Milnor showed that, if X is smooth at x, φ_x is a locally trivial fibration whose fiber $\varphi_x^{-1}(1)$ (the *Milnor fiber*) is independent of ε up to homeomorphism (for ε small enough). These results have been extended to the case of a general X by Lê [Lê77].

From our point of view, it will be more natural to work with the open ball $B_x(\varepsilon)$ rather than the sphere, and get rid of the argument map. Let $\widetilde{S}^* = \{\mathrm{Re}(z) < \log \delta\}$. The map $\exp : \widetilde{S}^* \rightarrow S^*$ is a universal cover of S^*. One can show (cf. Theorem 4.1.5) that for $\varepsilon \ll 1$ the space

$$F_{x,\varepsilon} = \widetilde{S}^* \times_S B_x(\varepsilon) \tag{1.1}$$

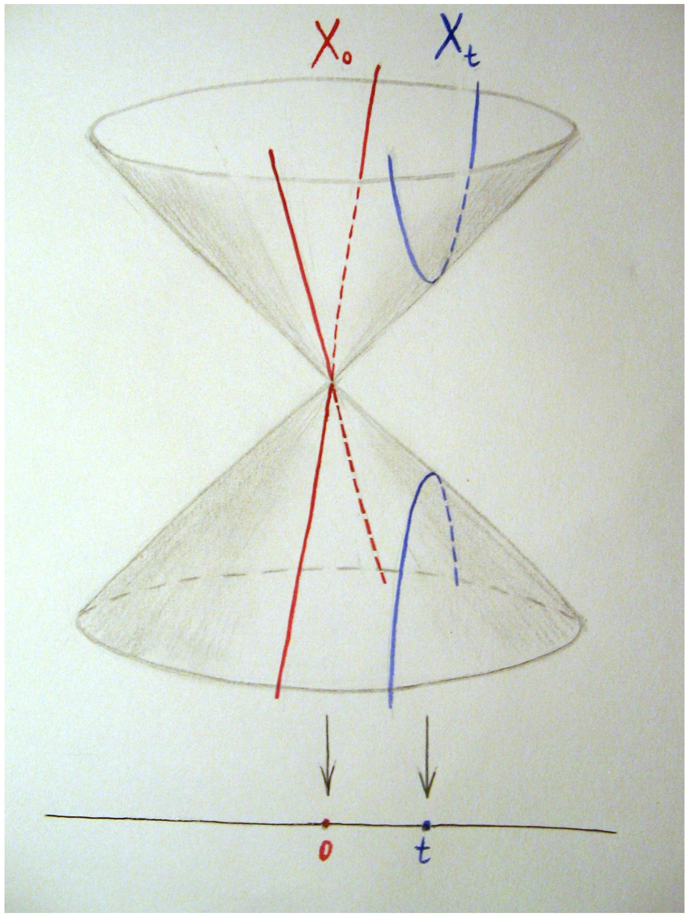

Figure 1.1.1: *A family of hyperbolae $x_1x_2 = t$ degenerating to the union of coordinate axes (illustration by Masha Vlasenko)*

is independent of ε up to homeomorphism, is homotopy equivalent to the Milnor fiber $\varphi_x^{-1}(1)$, and the inclusions $F_{x,\varepsilon'} \subseteq F_{x,\varepsilon}$ are homotopy equivalences for $\varepsilon' < \varepsilon$. We will henceforth abbreviate $F_{x,\varepsilon}$ to F_x, and call F_x the Milnor fiber of f at x.

Our choice of the universal cover of S^* allows us to canonically identify the fundamental group $\pi_1(S^*)$ with the group $\mathbf{Z}(1) = 2\pi i\mathbf{Z}$, acting on $\widetilde{S}^*$ via deck transformations $\zeta \mapsto \zeta + \alpha$, $\alpha \in \mathbf{Z}(1)$. It is clear from our definition of F_x that $\mathbf{Z}(1)$ acts on F_x in a natural way.

Example 1.1.1. In the situation of Figure 1.1.1, the Milnor fiber at $x = (0,0)$ is

$$F_x = \{(x_1, x_2, \zeta) : |x_1|^2 + |x_2|^2 < \varepsilon^2, \exp\zeta = x_1x_2\},$$

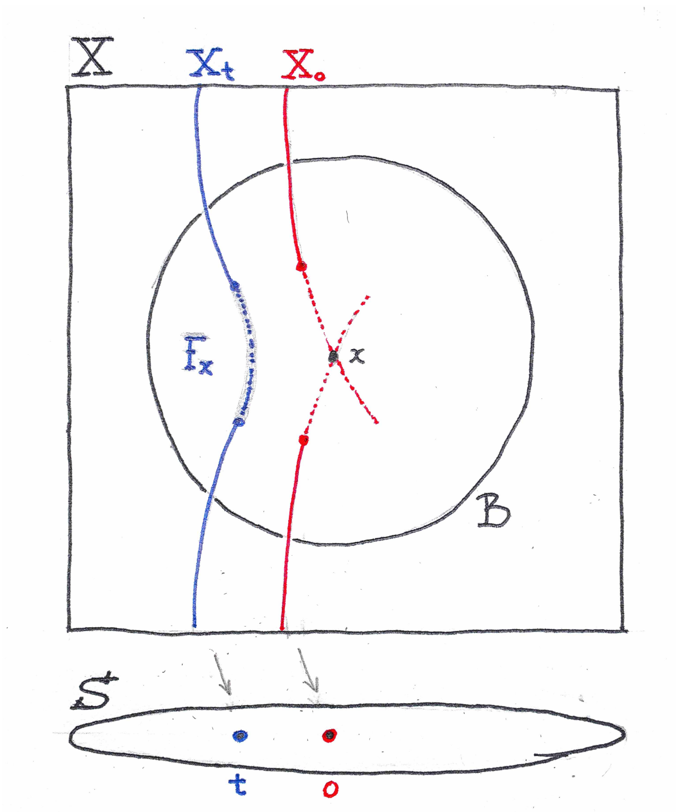

Figure 1.1.2: *The Milnor fiber*

which is homotopy equivalent to $\mathbf{S}^1$ for $\varepsilon \ll 1$ (for example, via the map $(x_1, x_2, \zeta) \mapsto \arg(x_1)$).

Example 1.1.2. Suppose that X is a complex manifold of dimension n and that $f : X \longrightarrow S$ is a holomorphic map having an *isolated* singular point $x \in X$. In this case, *Milnor's bouquet theorem* [Mil68] states that F_x has the homotopy type of a wedge of $\mu(x)$ spheres $\mathbf{S}^{n-1}$, where

$$\mu(x) = \dim_{\mathbf{C}} \mathscr{O}_{X,x} / \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

for a choice of local coordinates $x_1, \ldots, x_n \in \mathscr{O}_{X,x}$. Here $\mathscr{O}_{X,x}$ denotes the ring of germs of holomorphic functions at x. For instance, if X is a neighborhood of $x = 0 \in \mathbf{C}^n$ and $f(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2$, then F_x is homotopy equivalent to $\mathbf{S}^{n-1}$.

The monodromy

Consider the higher direct image sheaf $\mathscr{F} = R^q(f|_{X^*})_*\mathbf{Z}$ on S^* for some $q \geq 0$, which is the sheaf associated to the presheaf

$$(U \subseteq S^*) \mapsto H^q(f^{-1}(U), \mathbf{Z}).$$

As f is a locally trivial fibration over S^*, $\mathscr{F}$ is a locally constant sheaf on S^*, and $\mathscr{F}_t = H^q(X_t, \mathbf{Z})$ for $t \in S^*$. If $\gamma : [0,1] \longrightarrow S^*$ is a path in S^*, the pullback $\gamma^*\mathscr{F}$ is also locally constant, and hence constant. Thus γ induces an isomorphism $\mathscr{F}_{\gamma(0)} \xrightarrow{\sim} \mathscr{F}_{\gamma(1)}$, which depends only on the homotopy class of the path γ. In particular, the fundamental group $\pi_1(S^*, t)$ acts on $\mathscr{F}_t = H^q(X_t, \mathbf{Z})$. This action is called the *monodromy action*, and the action of a chosen generator of $\pi_1(S^*, t)$ is called the *monodromy operator*. Note that since $\pi_1(S^*) = \mathbf{Z}(1) = 2\pi i\mathbf{Z}$ is abelian, we can avoid choosing a base point t of S^*. Similarly, the action of $\pi_1(S^*)$ on F_x induces an action on $H^q(F_x, \mathbf{Z})$ for $x \in X_0$.

Nearby cycles

The topological complexity of the Milnor fiber F_x is an indicator of the singularities of f around $x \in X_0$. To compare X_0 and X_t ($t \neq 0$), one would like to have a global object on X_0 carrying information about all of the Milnor fibers of f. The *sheaves of nearby cycles* $R^q\Psi(\mathbf{Z})$ are sheaves on X_0 whose stalks are the cohomology of the Milnor fibers:

$$(R^q\Psi(\mathbf{Z}))_x \simeq H^q(F_x, \mathbf{Z}). \tag{1.2}$$

Moreover, the monodromy operators on $H^q(F_x, \mathbf{Z})$ can be assembled together to give a monodromy operator on $R^q\Psi(\mathbf{Z})$.

To define $R^q\Psi(\mathbf{Z})$, consider the natural maps[2] $\overline{\jmath} : \widetilde{X}^* = X \times_S \widetilde{S}^* \longrightarrow X$ and $i : X_0 \longrightarrow X$, fitting in a cartesian diagram

$$\begin{tikzcd}
\widetilde{X}^* \arrow[d] \arrow[dr] \arrow[drr, "\overline{\jmath}"] & & & \\
\widetilde{S}^* \arrow[dr] & X^* \arrow[r, "j"'] \arrow[d] & X \arrow[d] & X_0 \arrow[l, "i"'] \arrow[d] \\
 & S^* \arrow[r] & S & 0. \arrow[l]
\end{tikzcd}$$

[2]For typesetting reasons, we chose the notation $\overline{\jmath}$ instead of the commonly used $\bar{\jmath}$. The same for $\bar{\imath}$.

Then define $R^q\Psi(\mathbf{Z}) = i^* R^q \bar{\jmath}_* \mathbf{Z}$. It comes with a natural $\pi_1(S^*)$-action, deduced from the $\pi_1(S^*)$-action on $\widetilde{X}^*$. To check (1.2), observe that

$$R^q\Psi(\mathbf{Z})_x = (R^q\bar{\jmath}_*\mathbf{Z})_x = \varinjlim_{x\in U} H^q(\bar{\jmath}^{-1}(U),\mathbf{Z}) = \varinjlim_{\varepsilon} H^q(\bar{\jmath}^{-1}(B_x(\varepsilon)),\mathbf{Z}) = H^q(F_x,\mathbf{Z})$$

(for the last equality, note that $\bar{\jmath}^{-1}(B_x(\varepsilon))$ equals $F_{x,\varepsilon} = F_x$).

One can also consider the total complex $R\Psi(\mathbf{Z}) := i^* R\bar{\jmath}_*\mathbf{Z}$. If f is proper, one has a natural identification (cf. [SGA73a, Exp. XIV, §1.3.3])

$$R\Gamma(X_0, R\Psi(\mathbf{Z})) = R\Gamma(X, R\bar{\jmath}_*\mathbf{Z}) = R\Gamma(X\times_S \widetilde{S}^*, \mathbf{Z}) = R\Gamma(X_t,\mathbf{Z})$$

(for $t\in S^*$), compatible with the monodromy operators. This allows one to gain information about the global monodromy from the local monodromy operators on $H^q(F_x,\mathbf{Z})$.

Example 1.1.3. Consider the *Dwork family of elliptic curves* (e.g. [Kat09, Ogu13])

$$f : X = \left\{(\varepsilon,(x_0 : x_1 : x_2)) \in S\times \mathbf{P}^2(\mathbf{C}) : \varepsilon(x_0^3 + x_1^3 + x_2^3) = 3x_0x_1x_2\right\} \longrightarrow S.$$

The fibers X_t ($t \neq 0$) are one-dimensional complex tori (in particular, $H^1(X_t,\mathbf{Z}) \simeq \mathbf{Z}^2$), while the special fiber X_0 is a "triangle", the union of the three coordinate lines $D_i = \{x_i = 0\}$ in $\mathbf{P}^2(\mathbf{C})$. The map f is non-submersive at the three "vertices" of that triangle

$$P_0 = (0,(1:0:0)),\quad P_1 = (0,(0:1:0)),\quad P_2 = (0,(0:0:1)).$$

The Milnor fiber F_x ($x \in X_0$) is homotopy equivalent to $\mathbf{S}^1$ if $x \in \{P_0, P_1, P_2\}$, and is contractible otherwise. Thus

$$R^0\Psi(\mathbf{Z}) = \mathbf{Z} \text{ (constant sheaf on } S\text{)},\quad R^1\Psi(\mathbf{Z}) \simeq \mathbf{Z}_{P_0}\oplus\mathbf{Z}_{P_1}\oplus\mathbf{Z}_{P_2},$$

where $\mathbf{Z}_P$ denotes the skyscraper sheaf supported at P. We will see how to calculate the monodromy operator on $H^1(X_t,\mathbf{Z})$ in Example 1.1.9.

1.1.2 The algebraic theory

In the above section, we defined Milnor fibers, the monodromy action, and nearby cycles in the complex analytic context. Little of this machinery is lost when passing to the setting of algebraic geometry. Instead of a small disc, one takes the base S to be a *trait*, i.e. the spectrum of a discrete valuation ring V. Thus S has only two points: the closed point $s = \operatorname{Spec} k$, where k is the residue field of V, playing the role of 0, and the generic point $\eta = \operatorname{Spec} K$, where K is the fraction field of V, playing the role of S^*. If we pick an algebraic closure $\overline{K}$ of K, the corresponding geometric point[3] $\overline{\eta} = \operatorname{Spec} \overline{K}$ will play the role of *both* the universal cover $\widetilde{S}^*$ and a chosen point $t \in S^*$. Intuitively, S is a germ of a smooth curve around a point s, and $\overline{\eta}$ is a point "infinitely close" to s.

We will always assume that

- V is henselian[4],
- $\operatorname{char} K = 0$,
- k is perfect.

Example 1.1.4. Important examples of V as above are

1. $V = \mathbf{C}\{t\}$, the ring of convergent power series — then one can think of $S = \operatorname{Spec} V$ as the limit of discs of radii going to 0. In this case $\overline{K}$ is obtained by adjoining roots of the variable t of all degrees, hence $\operatorname{Gal}(\overline{K}/K) = \hat{\mathbf{Z}}(1) := \varprojlim \mu_n(\mathbf{C})$,

2. $V = \mathbf{C}[[t]]$, the ring of formal power series, corresponding to an "even smaller" base S (the description of the algebraic closure and the Galois group is the same as in the preceding example),

3. $V = \mathscr{O}^h_{X,x}$, the henselization of the local ring of a smooth curve X over a characteristic zero field k at a point $x \in X(k)$ (abstractly, this is always isomorphic to the algebraic closure of $k[x]$ in $k[[x]]$, and the description of the algebraic closure and the Galois group is the same as in the preceding examples),

4. $V = \mathbf{Z}_p$, the ring of p-adic integers, or more generally the integral closure of $\mathbf{Z}_p$ in a finite extension of $\mathbf{Q}_p$. This case has a different flavor than the previous three (in particular, the Galois group $\operatorname{Gal}(\overline{K}/K)$ is much more complicated than in the previous examples), and is of considerable interest in arithmetic geometry,

[3] A geometric point is a morphism from the spectrum of a separably closed field.

[4] A discrete valuation ring V is *henselian* if for every finite extension L of its fraction field, the integral closure of V in L is a discrete valuation ring. Complete discrete valuation rings are henselian.

5. $V = W(k)$, the ring of Witt vectors of a perfect field k of characteristic $p > 0$.

Consider a scheme X of finite type over S. We call $X_s := X \times_S s$, resp. $X_\eta := X \times_S \eta$, resp. $X_{\overline{\eta}} := X \times_S \overline{\eta}$ the *special* (or *closed*), resp. *generic*, resp. *geometric generic* fiber of X.

In place of the singular cohomology $H^q(Y, \mathbf{Z})$, one considers the ℓ-adic cohomology $H^q(Y, \mathbf{Z}_\ell)$, where ℓ is an auxiliary prime (usually assumed to be invertible on Y). These groups are finitely generated $\mathbf{Z}_\ell$-modules if Y is "sufficiently nice." In case Y is a scheme of finite type over $\mathbf{C}$, one can consider the associated analytic space Y^{an}, and one has the following comparison theorem.

Theorem 1.1.5 ([SGA73b, Exp. XI, Theorem 4.4])**.** *The comparison maps*

$$H^q(Y, \mathbf{Z}_\ell) \xrightarrow{\sim} H^q(Y^{\mathrm{an}}, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell$$

are isomorphisms for all $q \geq 0$.

In analogy with the classical complex-analytic picture, one has the notions described below.

Milnor fibers

If $\overline{x}$ is a geometric point of X_s, the natural analogue of a ball around $\overline{x}$ of an unspecified small radius is the scheme $X_{(\overline{x})}$, the localization[5] of X at $\overline{x}$ in the étale topology[6]. Then the base change

$$M_{\overline{x}} = X_{(\overline{x})} \times_{S_{(f(\overline{x}))}} \overline{\eta} \tag{1.3}$$

is called the *Milnor fiber* of X at $\overline{x}$. Note the similarity between the two definitions (1.1) and (1.3).

Remark 1.1.6. To avoid confusion, we might call $M_{\overline{x}}$ the algebraic Milnor fiber, and the space F_x defined in §1.1.1 the classical Milnor fiber.

[5] i.e., the inverse limit of all étale neighborhoods of $\overline{x}$

[6] One could also consider the completion of X at $\overline{x}$. We compare the two options in Theorem 4.2.8.

The monodromy action

In this setting, the inertia subgroup I of the Galois group $\mathrm{Gal}(\overline{\eta}/\eta)$ plays the role of the fundamental group $\pi_1(S^*)$, and the monodromy action is just the natural Galois action on the ℓ-adic cohomology groups $H^q(X_{\overline{\eta}}, \mathbf{Z}_\ell)$ and $H^q(M_{\overline{x}}, \mathbf{Z}_\ell)$. However, some complications appear, as I is much bigger than $\mathbf{Z}$! If $\operatorname{char} k = 0$, then $I \simeq \hat{\mathbf{Z}}(1)$ is (non-canonically) isomorphic the profinite completion of $\mathbf{Z}$ (there is no canonical choice of a topological generator). If $\operatorname{char} k = p > 0$, the I has a very large pro-p part P, called the *wild inertia*. If moreover $\ell = p$, which is the case of interest in p-adic Hodge theory, the restriction of the monodromy representation to P usually does not factor through a finite quotient.

Nearby cycles

Let $\overline{V}$ be the integral closure of V in $\overline{K}$, $\overline{S} = \operatorname{Spec} \overline{V}$, $\overline{X} = X \times_S \overline{S}$, and let $\overline{s}$ be the closed point of $\overline{S}$. We have a cartesian diagram

$$\begin{tikzcd}
\overline{\eta} \arrow[r] \arrow[d] & \overline{S} \arrow[d] & \overline{s} \arrow[l] \arrow[d] \\
\eta \arrow[r] & S & s. \arrow[l]
\end{tikzcd}$$

Let $\overline{X} = X \times_S \overline{S}$, and let $\overline{\jmath} : X_{\overline{\eta}} \longrightarrow \overline{X}$ and $\overline{\imath} : X_{\overline{s}} \longrightarrow \overline{X}$ be the natural maps. As in 1.1.1, one has a cartesian diagram

$$\begin{tikzcd}
X_{\overline{\eta}} \arrow[rr, "\overline{\jmath}"] \arrow[d] \arrow[dr] & & \overline{X} \arrow[d] & & X_{\overline{s}} \arrow[ll, "\overline{\imath}"'] \arrow[d] \arrow[dl] \\
\overline{\eta} \arrow[dr] & X_\eta \arrow[r, "j"'] \arrow[d] & X \arrow[d] & X_s \arrow[l, "i"] \arrow[d] & \overline{s} \arrow[dl] \\
& \eta \arrow[r] & S & s. \arrow[l] &
\end{tikzcd}$$

The sheaves $R^q\Psi(\mathbf{Z}_\ell) := \overline{\imath}^* R^q \overline{\jmath}_* \mathbf{Z}_\ell$, endowed with the natural $\mathrm{Gal}(\overline{\eta}/\eta)$-action, are called the *sheaves of nearby cycles*. One also considers $R\Psi(\mathbf{Z}_\ell) := \overline{\imath}^* R\overline{\jmath}_* \mathbf{Z}_\ell$. As before, we have $(R^q\Psi(\mathbf{Z}_\ell))_{\overline{x}} \simeq H^q(M_{\overline{x}}, \mathbf{Z}_\ell)$ almost by definition.

If X is proper over S, the proper base change theorem implies that $R\Gamma(X_{\bar{s}}, R\Psi(\mathbf{Z}_\ell)) \simeq R\Gamma(X_{\bar{\eta}}, \mathbf{Z}_\ell)$. In particular, we have the *nearby cycles spectral sequence*

$$E_2^{pq} = H^p(X_{\bar{s}}, R^q\Psi\mathbf{Z}_\ell) \quad \Rightarrow \quad H^{p+q}(X_{\bar{\eta}}, \mathbf{Z}_\ell). \tag{1.4}$$

Remark 1.1.7. If $\operatorname{char} k = p > 0$ and $\ell = p$, the sheaves of nearby cycles carry much more information than usual. They are highly nontrivial even if X is smooth over S! The problem of describing these *p-adic nearby cycles* is very closely tied to p-adic Hodge theory (cf. §1.1.4).

1.1.3 Semistable degenerations

The simplest kind of nontrivial degenerations are the semistable ones. A scheme X over S is called *semistable* if X is regular and X_s is a reduced normal crossings divisor on X. Such an X étale locally looks like

$$X = \operatorname{Spec} V[x_1, \ldots, x_n]/(x_1 \cdot \ldots \cdot x_r - \pi)$$

where $r \leq n$ and π is a uniformizer of V. One has an analogous notion in the complex analytic picture (replacing “regular” with “complex manifold” and “étale locally” with “locally”).

Example 1.1.8. In the complex analytic picture, the analogous map

$$f : X = \mathbf{C}^n \longrightarrow \mathbf{C} = S, \quad f(x_1, \ldots, x_n) = x_1 \cdot \ldots \cdot x_r$$

has Milnor fiber at $x \in f^{-1}(0)$ equal to

$$F_x = \left\{ (x'_1, \ldots, x'_n, \zeta) : \sum_{i=1}^n |x'_i - x_i|^2 < \varepsilon^2, \exp \zeta = x_1 \ldots x_r \right\}$$

which is homotopy equivalent to $(\mathbf{S}^1)^{m_x - 1}$, where $m_x = \#\{i \leq r : x_i = 0\}$. In particular,

$$(R^q\Psi(\mathbf{Z}))_x \simeq \bigwedge^q \mathbf{Z}^{m_x - 1}. \tag{1.5}$$

Motivated by this simple example, we would like to give formula (1.5) an intrinsic global meaning. Suppose that X is semistable and that $X_s = D_1 \cup \ldots \cup D_m$ where the divisors D_i are irreducible and have no self-intersections. We introduce the sheaves $\overline{\mathscr{M}}^{\mathrm{gp}}_S = \mathbf{Z}_s$ (skyscraper sheaf $\mathbf{Z}$ supported at the closed point s of S) and $\overline{\mathscr{M}}^{\mathrm{gp}}_X = \bigoplus \mathbf{Z}_{D_i}$ (the sheaf of divisors supported on X_s). (The complicated notation is motivated by logarithmic geometry, as to be explained later.) We have a natural map

$$f^*(\overline{\mathscr{M}}^{\mathrm{gp}}_S) = \mathbf{Z}_{X_s} \longrightarrow \overline{\mathscr{M}}^{\mathrm{gp}}_X$$

sending the generator 1 to the sum of the generators $1 \in \mathbf{Z}_{D_i}$. Finally, let $\overline{\mathscr{M}}^{\mathrm{gp}}_{X/S}$ be its cokernel (i.e., the sheaf of divisors supported on X_s modulo X_s itself), so that we have an extension

$$0 \longrightarrow f^*(\overline{\mathscr{M}}^{\mathrm{gp}}_S) \longrightarrow \overline{\mathscr{M}}^{\mathrm{gp}}_X \longrightarrow \overline{\mathscr{M}}^{\mathrm{gp}}_{X/S} \longrightarrow 0 \tag{1.6}$$

In these terms, the global version of (1.5) is

$$R^q\Psi(\mathbf{Z}_\ell) \simeq \bigwedge^q (\overline{\mathscr{M}}^{\mathrm{gp}}_{X/S} \otimes \mathbf{Z}_\ell(-1)) \tag{1.7}$$

(here (-1) denotes the Tate twist, the twist by the inverse of the ℓ-adic cyclotomic character). Moreover, the construction of (1.7) shows that the monodromy action on $R^q\Psi(\mathbf{Z}_\ell)$ (but not on $R\Psi(\mathbf{Z}_\ell)$!) is trivial. Again, one has an analogous description in the classical setting.

The main reason one is interested in semistable families is as follows. A given smooth and proper X_η over η might not extend to a smooth and proper X over S. If it does, we say that X_η has *good reduction*. One obstruction to good reduction is the non-triviality of the action of the inertia subgroup I of $\mathrm{Gal}(\overline{\eta}/\eta)$ on $H^q(X_{\overline{\eta}}, \mathbf{Z}_\ell)$ (for $\ell \neq \operatorname{char} k$). However, one can often find a *semistable* X after replacing η by a finite extension η' (and X_η by $X_{\eta'}$). In such case we say that $X_{\eta'}$ has *semistable reduction*. It is known that every X_η admits a semistable reduction after a finite extension of η in the following situations:

- if $\operatorname{char} k = 0$ (using resolution of singularities) [KKMSD73, Thm. p.53], or
- if X_η is a curve [Abb00, Théorème 1.1] or an abelian variety [SGA72, XI], [Abb00, Théorème 5.4],

and conjecturally always (granted resolution of singularities in the arithmetic setting).

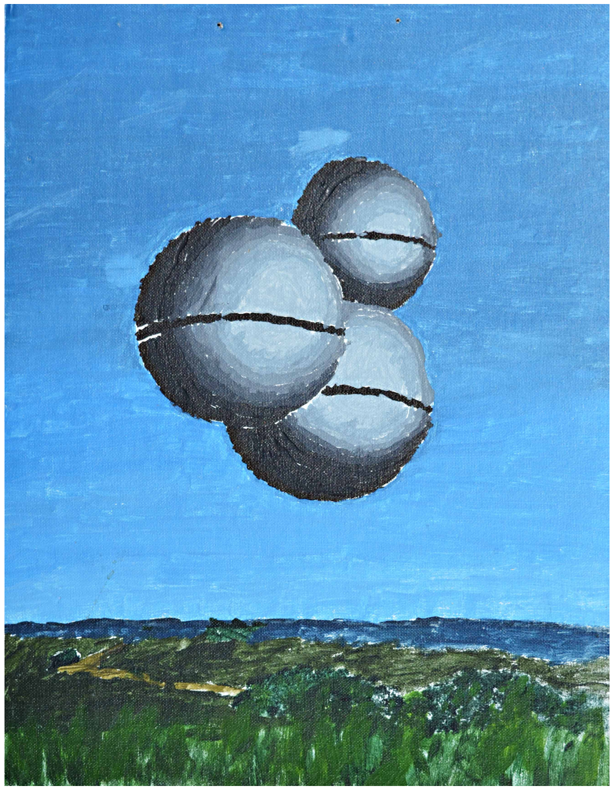

Figure 1.1.3: *René Magritte "The Voice of the Winds", reproduction by Wolfgang Schmalz. The configuration of spheres resembles the special fiber X_0 of the Dwork family of elliptic curves, if one imagines that the three spheres touch.*

Example 1.1.9. The families in Examples 1.1.1 and 1.1.3 are semistable. We will compute the monodromy action on $H^1(X_t, \mathbf{Z})$ for the Dwork family 1.1.3. Let us abbreviate $H^1(X, \mathbf{Z})$ to $H^1(X)$. The exact sequence (1.6) takes the form

$$0 \longrightarrow \mathbf{Z} \longrightarrow \bigoplus \mathbf{Z}_{D_i} \longrightarrow \bigoplus \mathbf{Z}_{P_i} \longrightarrow 0 \tag{1.8}$$

where $[D_i]$ is mapped to $[P_{i+1}] - [P_{i-1}]$ (indices mod 3). Here we implicitly fix an orientation of the nontrivial loop L on X_0. Using (1.7), we now have $R^0\Psi\mathbf{Z} = \mathbf{Z}$ and

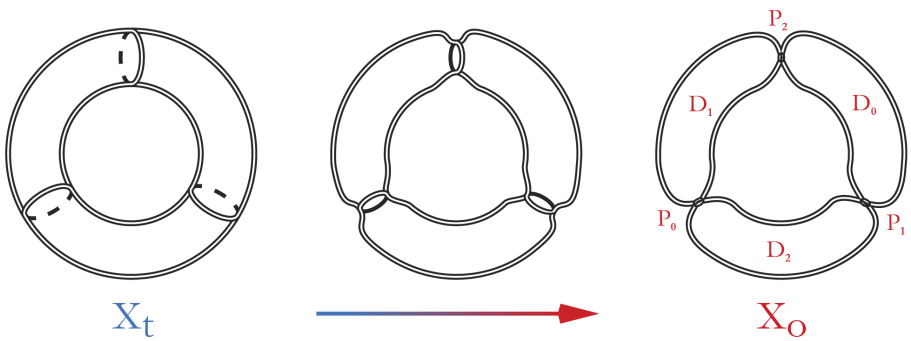

Figure 1.1.4: *The Dwork family of elliptic curves at* $t \longrightarrow 0$ *(illustration by Dorka Budacz)*

$R^1\Psi\mathbf{Z} = \bigoplus \mathbf{Z}_{P_i}$. The E_2-page of the spectral sequence (1.4) takes the form

$$\left[\begin{array}{ccc} H^0(X_0,R^1\Psi\mathbf{Z}) & H^1(X_0,R^1\Psi\mathbf{Z}) & H^2(X_0,R^1\Psi\mathbf{Z}) \\ & & \\ H^0(X_0) & H^1(X_0) & H^2(X_0) \end{array}\right] = \left[\begin{array}{ccc} \mathbf{Z}^3 & & \\ & & \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z}^3 \end{array}\right].$$

The spectral sequence therefore induces an exact sequence

$$0 \longrightarrow H^1(X_0) \xrightarrow{i} H^1(X_t) \xrightarrow{p} H^0(X_0,R^1\Psi\mathbf{Z}) \xrightarrow{\partial} H^2(X_0) \longrightarrow H^2(X_t) \longrightarrow 0$$

of $\pi_1(S^*)$-modules. Let $K \subseteq H^0(X_0,R^1\Psi\mathbf{Z})$ denote the image of the map p. Because $H^1(X_t) \simeq \mathbf{Z}^2$, $H^1(X_0) \simeq \mathbf{Z}$, and $H^0(X_0,R^1\Psi\mathbf{Z})$ is torsion-free, we have $K \simeq \mathbf{Z}$. As explained earlier, the monodromy action on $R^1\Psi\mathbf{Z}$ is trivial. If T is a generator of $\pi_1(S^*)$, the map

$$1 - T : H^1(X_t) \longrightarrow H^1(X_t)$$

factors as

$$H^1(X_t) \xrightarrow{p} K \xrightarrow{\alpha} H^1(X_0) \xrightarrow{i} H^1(X_t)$$

for a certain map α. Let $m \geq 0$ be such that $\operatorname{coker}\alpha \simeq \mathbf{Z}/m\mathbf{Z}$. If we choose for a basis of $H^1(X_t)$ a generator of $H^1(X_0)$ and an element mapping to a generator of K, the monodromy operator T will take the form

$$T = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}.$$

We will show that $m=3$. Let

$$\overline{\alpha}: H^0(X_0, R^1\Psi\mathbf{Z}) = \bigoplus \mathbf{Z}[P_i] \longrightarrow \mathbf{Z}[L] = H^1(X_0)$$

be the connecting homomorphism of the cohomology exact sequence of (1.8). We easily check that $\overline{\alpha}([P_i]) = [L]$. On the other hand, as will see later in Example 5.5.3 using the "monodromy formula", α is the restriction of $\overline{\alpha}$ to K up to a sign depending on the choice of T (that is, the diagram

$$\begin{tikzcd}
H^1(X_t) \arrow[r, "p"] \arrow[d, "1-T"'] & H^0(X_0, R^1\Psi\mathbf{Z}) \arrow[d, "\overline{\alpha}"] \\
H^1(X_t) & H^1(X_0) \arrow[l, "i"]
\end{tikzcd}$$

commutes up to sign). Let $\mathbf{Z}/3\mathbf{Z}$ act on X by permuting the coordinates x_i cyclically. Since the maps above are obviously $\mathbf{Z}/3\mathbf{Z}$-invariant, the subgroup K of $\bigoplus \mathbf{Z}[P_i]$ has to be invariant under cyclic permutation. Since $K \simeq \mathbf{Z}$, K must be generated by a multiple of $[P_1]+[P_2]+[P_3]$. On the other hand, since $H^0(X_0, R^1\Psi\mathbf{Z})/K$ injects into $H^2(X_0)$ which is torsion-free, K is in fact generated by $[P_1]+[P_2]+[P_3]$. Since

$$\overline{\alpha}([P_1]+[P_2]+[P_3]) = 3[L],$$

we deduce that $m=3$, as desired. □

Remark 1.1.10. For many reasons, semistable families are not the most natural notion: for instance, unlike smoothness, semistability is not preserved upon base change. The "right" generalization of semistability is one of the major achievements of *logarithmic geometry*. There one studies *logarithmic schemes*, which are pairs $(X, \mathscr{M}_X)$ with X a scheme and $\mathscr{M}_X$ a certain (étale) sheaf of commutative monoids on X endowed with a map to $\mathscr{O}_X$. The fundamental insight of log geometry is that certain maps of schemes behave like smooth maps if the source and target are given the correct log structure.

1.1.4 $K(\pi,1)$ neighborhoods and p-adic Hodge theory

While classical Hodge theory deals with the cohomology $H^n(X, \mathbf{C})$ of a compact Kähler manifold X, together with its extra structure (the *Hodge structure*), its p-adic counterpart studies the p-adic étale cohomology $H^n(X_{\overline{K}}, \mathbf{Q}_p)$ of a smooth projective variety X over a field K which is a finite extension of $\mathbf{Q}_p$. The extra structure in this case is the action of the Galois group $\mathrm{Gal}(\overline{K}/K)$. The p-adic analogue of the Hodge decomposition

$$H^n(X, \mathbf{C}) \simeq \bigoplus_{i+j=n} H^i(X, \Omega^j_X)$$

(where Ω^j_X is the sheaf of holomorphic j-forms) is the *Hodge–Tate decomposition*.

Theorem 1.1.11 (Hodge–Tate decomposition [Fal88])**.** *Let $\mathbf{C}_K$ be the completion of $\overline{K}$ with respect to the p-adic norm. There exists a canonical isomorphism of* $\mathrm{Gal}(\overline{K}/K)$*-modules*

$$H^n(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_K \simeq \bigoplus_{i+j=n} H^i(X, \Omega^j_Y) \otimes_K \mathbf{C}_K(-j),$$

where $\mathbf{C}_K(-j)$ is $\mathbf{C}_K$ tensored with the j-th power of the inverse of the p-adic cyclotomic character.

When studying differentiable manifolds, one benefits from the fact that the underlying topological space is locally contractible. This is not the case in algebraic geometry: a smooth complex algebraic variety, e.g. a curve of positive genus, often does not admit a Zariski open cover by subvarieties which are contractible in the classical topology. On the other hand, as noticed by Artin in the course of the proof of the aforementioned comparison theorem (Theorem 1.1.5), one can always cover such a variety by Zariski opens which are $K(\pi,1)$ spaces.

The notion of a $K(\pi,1)$ space has a natural counterpart in algebraic geometry, defined in terms of étale local systems. In a similar way as in Artin's comparison theorem, coverings by $K(\pi,1)$ play a role in Faltings' approach to p-adic comparison theorems [Fal88, Fal02, Ols09]. Faltings shows the following analogue of Artin's result.

Theorem 1.1.12 ([Fal88, Lemma II 2.1])**.** *Let $S = \mathrm{Spec}\, V$ as in §1.1.2, and let X be a smooth scheme over S. Then there exists a covering of X by Zariski open subsets U for which $U_{\overline{\eta}}$ is a $K(\pi,1)$ scheme.*

1.1.5 Summary of the results

In brief, this thesis contains the following original results:

R1. If X *semistable* over S, then X can be covered (in the étale topology) by schemes U such that $U_{\overline{\eta}}$ is a $K(\pi,1)$ scheme. In particular, the Milnor fibers $M_{\overline{x}} = X_{(\overline{x})} \times_{S_{(f(\overline{x}))}} \overline{\eta}$ are $K(\pi,1)$ schemes (cf. Theorem 3.2.1).

R2. If $V = \mathbf{C}\{t\}$, so that one can speak about the classical Milnor fibers $F_{\overline{x}}$, the étale homotopy type of the algebraic Milnor fiber $M_{\overline{x}}$ is the profinite completion of the homotopy type of $F_{\overline{x}}$ (cf. Theorem 4.2.9). This means that the categories of local systems of finite groups are canonically equivalent, and that the cohomology groups of corresponding local systems are canonically isomorphic.

R3. In the definition of the algebraic Milnor fiber $M_{\overline{x}}$, one can replace henselization by formal completion, i.e., consider the scheme $(\operatorname{Spec} \hat{\mathscr{O}}_{X,(\overline{x})})_{\overline{\eta}}$ instead, without changing the fundamental group or the cohomology of local systems (in characteristic zero, cf. Theorem 4.2.8).

R4. If $\operatorname{char} k = 0$ and X is semistable over S, it is known that the monodromy action on $R\Psi(\mathbf{Z}_\ell)$ induces the trivial action on the associated cohomology sheaves, and hence defines for every $\gamma \in I$ certain maps in the derived category

$$1-\gamma : R^q\Psi(\mathbf{Z}_\ell) \longrightarrow R^{q-1}\Psi(\mathbf{Z}_\ell)[1].$$

We provide an explicit description of these maps via the isomorphism (1.7) in terms of the extension (1.6) (cf. Theorem 5.4.4).

If $V = \mathbf{C}\{t\}$, the classical Milnor fibers are products of circles as in Example 1.1.8, hence $K(\pi,1)$ spaces, and we can deduce from R2 that $M_{\overline{x}}$ is a $K(\pi,1)$. But in mixed characteristic, the Milnor fibers are complicated even for X smooth (Remark 1.1.7). Thus R1 extends Theorem 1.1.12 to the semistable case, which helps simplify Faltings' proof of the C_{st} conjecture of Fontaine [Fal02], a deep result which in particular implies the Hodge-Tate decomposition (Theorem 1.1.11).

Results R1 and R4 hold more generally for X log smooth and saturated over S.

The last result R4 is an ℓ-adic analogue of a result of Ogus which treats the complex analytic case, using Kato-Nakayama spaces.

1.2 Discussion of the main result

1.2.1 The Fontaine conjectures in p-adic Hodge theory

We continue using the setup and notation of §1.1.2, but assume that the residue field k is of positive characteristic p. We denote by K_0 the fraction field of the ring of Witt vectors $W(k)$. It is a subfield of K in a natural way, and is endowed with a natural Frobenius $\sigma : K_0 \longrightarrow K_0$. For a representation M of Gal($\overline{\eta}/\eta$), we denote by $M(n)$ the twist of M by the n-th power of the p-adic cyclotomic character. Finally, $\mathbf{C}_K$ stands for the completion of $\overline{K}$ with respect to the p-adic metric.

p-adic cohomology theories

Let X be a proper scheme over S whose generic fiber X_η is smooth. One of the principal aims of p-adic Hodge theory is to relate the *p-adic étale cohomology* $H^n(X_{\overline{\eta}}, \mathbf{Q}_p)$ — a finite dimensional $\mathbf{Q}_p$-vector space endowed with a Gal($\overline{K}/K$)-action — to the following invariants of X:

1. the *de Rham cohomology*

$$H^n_{\mathrm{dR}}(X_\eta) = (H^n(X_\eta, \Omega^\bullet_{X_\eta/\eta}), \text{Hodge filtration}),$$

   the hypercohomology of the de Rham complex with its Hodge filtration — a finite dimensional filtered K-vector space,

2. the *Hodge cohomology*

$$H^n_{\mathrm{Hdg}}(X_\eta) = \mathrm{gr}\, H^n_{\mathrm{dR}}(X_\eta/\eta) = \bigoplus_{i+j=n} H^j(X_\eta, \Omega^i_{X_\eta/\eta}),$$

   a finite dimensional graded K-vector space,

3. (if X is smooth over S) the *crystalline cohomology*

$$H^n_{\mathrm{cris}}(X) = (H^n_{\mathrm{cris}}(X_s/W(k)) \otimes_{W(k)} K_0,\ \varphi,\ \rho^*(\text{Hodge filtration}))$$

which is a finite dimensional K_0-vector space endowed with a σ-linear endomorphism

$$\varphi : H^n_{\mathrm{cris}}(X) \longrightarrow H^n_{\mathrm{cris}}(X)$$

induced by the absolute Frobenius map of X, and a filtration on $H^n_{\mathrm{cris}}(X) \otimes_{K_0} K$, inherited from the Hodge filtration on $H^n_{\mathrm{dR}}(X_\eta)$ via the Berthelot–Ogus isomorphism

$$\rho : H^n_{\mathrm{cris}}(X) \otimes_{K_0} K \xrightarrow{\sim} H^n_{\mathrm{dR}}(X_\eta).$$

4. (if X is semistable over S) the *log-crystalline (Hyodo-Kato) cohomology*

$$H^n_{\mathrm{log-cris}}(X) = (H^n_{\mathrm{log-cris}}(X_s/W(k)) \otimes_{W(k)} K_0, \varphi, \rho^*_\pi(\text{Hodge filtration}), N)$$

an object of the same type as $H^n_{\mathrm{cris}}(X)$, together with a linear endomorphism N (the monodromy operator) satisfying $N\varphi = p\varphi N$. Here X_s is endowed with the natural log structure, and the filtration is inherited from the Hodge filtration on $H^n_{\mathrm{dR}}(X_\eta)$ via the Hyodo–Kato isomorphism

$$\rho_\pi : H^n_{\mathrm{log-cris}}(X) \otimes_{K_0} K \xrightarrow{\sim} H^n_{\mathrm{dR}}(X_\eta)$$

(depending on the choice of a uniformizer π of V). See [HK94, 3.4] for the definition of the monodromy operator N.

Note that H_{dR} and H_{Hdg} depend only on the generic fiber, while H_{cris} (without its filtration) depends only on the special fiber. The log-crystalline cohomology $H_{\mathrm{log-cris}}$ (again without the filtration) depends only on the special fiber *with its log structure*, which in turn depends only on $X \otimes_V V/\pi^2$. The filtrations on H^n_{cris} and $H^n_{\mathrm{log-cris}}$ are thus needed to include some information about the lifting of X_s.

The p-adic period rings

The p-adic period rings, defined by Fontaine [Fon82], are certain rings endowed with a $\mathrm{Gal}(\overline{\eta}/\eta)$-action:

1. B_{dR}, a discrete valuation field with residue field $\mathbf{C}_K$,

2. $B_{\mathrm{HT}} = \mathrm{gr}\, B_{\mathrm{dR}} \simeq \bigoplus_{n \in \mathbf{Z}} \mathbf{C}_K(n)$, a graded K-algebra,

3. $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$, a K_0-algebra with a σ-linear homomorphism $\varphi : B_{\mathrm{cris}} \longrightarrow B_{\mathrm{cris}}$,

4. $B_{\mathrm{st}} \supseteq B_{\mathrm{cris}}$, a B_{cris}-algebra together with an extension of φ and an operator N satisfying $N\varphi = p\varphi N$.

Moreover, a choice of a uniformizer π of V induces an injective B_{cris}-algebra homomorphism $B_{\mathrm{st}} \longrightarrow B_{\mathrm{dR}}$. This in turns gives a filtration on B_{st}, the restriction of the valuation filtration on B_{dR}.

The p-adic comparison theorems

The following result, due to Faltings [Fal02] and Tsuji [Tsu99], lies at the heart of p-adic Hodge theory.

Theorem 1.2.1 (The C_{st} conjecture of Fontaine). *Let X be a proper semistable scheme over S. There exists a* $\mathrm{Gal}(\overline{\eta}/\eta)$*-equivariant isomorphism*

$$H^n(X_{\overline{\eta}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{st}} \simeq H^n_{\log-\mathrm{cris}}(X) \otimes_{K_0} B_{\mathrm{st}},$$

compatible with φ, N, and the filtrations after tensoring with B_{dR}.

Here we regard $H^n(X_{\overline{\eta}}, \mathbf{Q}_p)$ as having $N = 0$ and $\varphi = id$. Moreover, we use the same uniformizer π in the Hyodo-Kato isomorphism and in the choice of the injection $B_{\mathrm{st}} \longrightarrow B_{\mathrm{dR}}$.

As $N = 0$ on B_{cris}, in the smooth case this specializes to the following result.

Theorem 1.2.2 (C_{cris}). *Let X be a smooth proper scheme over S. There exists a* $\mathrm{Gal}(\overline{\eta}/\eta)$*-equivariant isomorphism*

$$H^n(X_{\overline{\eta}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}} \simeq H^n_{\mathrm{cris}}(X) \otimes_{K_0} B_{\mathrm{cris}},$$

compatible with φ and the filtrations after tensoring with B_{dR}.

In case we are given only the generic fiber X_η, we can use alterations to reduce to the semistable case and deduce the following theorem.

Theorem 1.2.3 (C_{dR})**.** *Let X_η be a smooth proper scheme over η. There exists a* $\mathrm{Gal}(\overline{\eta}/\eta)$*-equivariant isomorphism*

$$H^n(X_{\overline{\eta}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}} \simeq H^n_{\mathrm{dR}}(X_\eta) \otimes_K B_{\mathrm{dR}},$$

compatible with the filtrations.

Passing to the associated graded vector spaces and using the degeneration of the Hodge to the Rham spectral sequence, we can then deduce the Hodge-Tate decomposition (Theorem 1.1.11, also called the C_{HT} conjecture).

1.2.2 Faltings' topos and coverings by $K(\pi,1)$'s

Let Y be a connected scheme with a geometric point $\overline{y}$. If $\mathscr{F}$ is a locally constant constructible abelian sheaf on $Y_{\text{ét}}$, the stalk $\mathscr{F}_{\overline{y}}$ is a representation of the fundamental group $\pi_1(Y,\overline{y})$, and we have natural maps

$$\rho^q : H^q(\pi_1(Y,\overline{y}), \mathscr{F}_{\overline{y}}) \longrightarrow H^q(Y_{\text{ét}}, \mathscr{F}).$$

Definition 1.2.4. We call Y a $K(\pi,1)$ if for every n invertible on Y, and every $\mathscr{F}$ as above with $n \cdot \mathscr{F} = 0$, the maps ρ^q are isomorphisms for all $q \geq 0$.

(See Section 2.1.2 for a slightly more general definition and a discussion of this notion.)

Let X_η be a smooth and proper scheme over η. As a step towards C_{HT} and C_{st}, under the assumption that there is a smooth proper model X over S, Faltings defines an intermediate cohomology theory $\mathscr{H}^\bullet(X)$ as the cohomology of a certain topos $\widetilde{E}$. Following Abbes and Gros [AG11], we call it the *Faltings' topos*. This is the topos associated to a site E whose objects are morphisms $V \longrightarrow U$ over $X_{\overline{K}} \longrightarrow X$ with $U \longrightarrow X$ étale and $V \longrightarrow U_{\overline{K}}$ finite étale (see 3.6.1 for the definition). The association $(V \longrightarrow U) \mapsto V$ induces a morphism of topoi

$$\Psi : X_{\overline{K},\text{ét}} \longrightarrow \widetilde{E}.$$

To compare $H^n(X_{\overline{K}}, \mathbf{Q}_p)$ and $\mathscr{H}^n(X)$, the first step is to investigate the higher direct images $R^q\Psi_*$. Then the *existence of coverings by* $K(\pi,1)$*'s* (Theorem 1.1.12) implies that $R^q\Psi_*\mathscr{F} = 0$ *for* $q > 0$ *and every locally constant constructible abelian sheaf* $\mathscr{F}$ *on* $X_{\overline{K}}$. It is these two results that we are going to generalize.

For a scheme X over S and an open subscheme $X^\circ \subseteq X$, we denote by $\widetilde{E}$ the Faltings' topos of $X^\circ_{\overline{\eta}} \longrightarrow X$ (see Definition 3.6.1), and by $\Psi : X^\circ_{\overline{\eta},\text{ét}} \longrightarrow \widetilde{E}$ the morphism of topoi 3.6.1(c). Consider the following four statements:

(A) *X has a basis of the étale topology consisting of U for which $U \times_X X^\circ_{\overline{\eta}}$ is a $K(\pi,1)$,*

(B) *for every geometric point $\overline{x} \in X$, $M_{\overline{x}} \otimes_X X^\circ$ is a $K(\pi,1)$,*

(C) *$R^q\Psi_*\mathscr{F} = 0$ ($q > 0$) for every locally constant constructible abelian sheaf $\mathscr{F}$ on $X^\circ_{\overline{\eta}}$,*

(D) *for every locally constant constructible abelian sheaf $\mathscr{F}$ on $X^\circ_{\overline{\eta}}$, the natural maps*

$$H^q(\widetilde{E}, \Psi_*(\mathscr{F})) \longrightarrow H^q(X^\circ_{\overline{\eta},\text{ét}}, \mathscr{F}).$$

are isomorphisms for all $q \geq 0$.

Then $(A) \Rightarrow (B) \Rightarrow (C) \Rightarrow (D)$, and the aforementioned theorem of Faltings (1.1.12) states that (A) holds if X is smooth over S (and $X^\circ = X$). Faltings has also shown [Fal88, Lemma II 2.3] that (B) is true if X is smooth over S and X° is the complement of a normal crossings divisor relative to S.

1.2.3 $K(\pi,1)$ neighborhoods in the log smooth case

It is natural to ask whether these results remain true if we do not require that X be smooth over S (we still want X_η, or at least X°_η, to be smooth over η). In general, the answer is no, even for X regular (see Section 3.3 for a counterexample).

The most natural and useful generalization, hinted at in [Fal02, Remark on p. 242], and brought to our attention by Ahmed Abbes, seems to be the case of X log smooth over S, where we endow S with the "standard" log structure $\mathscr{M}_S \longrightarrow \mathscr{O}_S$, i.e. the compactifying log structure induced by the open immersion $\eta \hookrightarrow S$. Our first main result confirms this expectation:

Theorem (3.2.1)**.** *Assume that* $\operatorname{char} k = p > 0$. *Let $(X, \mathscr{M}_X)$ be a log smooth log scheme over $(S, \mathscr{M}_S)$ such that X_η is smooth over η. Then (A) holds for $X^\circ = X$.*

Note that in the applications, in the above situation one usually cares about the case $X^\circ = (X, \mathcal{M}_X)_{\mathrm{tr}}$ (the biggest open on which the log structure is trivial). While the theorem deals with $X^\circ = X$, we are able to deduce corollaries about the other case as well (see below).

The strategy is to reduce to the smooth case (idea due to R. Lodh) by finding an étale neighborhood U' of $\overline{x}$ in X and a map

$$f : U' \longrightarrow W'$$

to a smooth S-scheme W' such that $f_\eta : U'_\eta \longrightarrow W'_\eta$ is finite étale. In such a situation, by Faltings' result, there is an open neighborhood W of $f(x)$ (x being the underlying point of $\overline{x}$) in W' such that W_η is a $K(\pi, 1)$. Then $U = f^{-1}(W)$ is an étale neighborhood of $\overline{x}$, and since $U_\eta \longrightarrow W_\eta$ is finite étale, U_η is a $K(\pi, 1)$ as well.

The proof of the existence of f makes use of the technique of Nagata's proof of the Noether normalization lemma, combined with the observation that the exponents used in that proof can be taken to be divisible by high powers of p (see §3.1.1). Therefore our proof applies only in mixed characteristic. While we expect the result to be true regardless of the characteristic, we point out an additional difficulty in equal characteristic zero in §3.2.4.

We also treat the equicharacteristic zero case and the case with boundary. More precisely, we use Theorem 3.2.1 and log absolute cohomological purity to prove the following:

Theorem (3.6.5+3.6.6). *Let $(X, \mathcal{M}_X)$ be a log smooth log scheme over $(S, \mathcal{M}_S)$ such that X_η is smooth over η, and let $X^\circ = (X, \mathcal{M}_X)_{\mathrm{tr}}$ be the biggest open subset on which $\mathcal{M}_X$ is trivial. If* $\mathrm{char}\, k = 0$, *assume moreover that $(X, \mathcal{M}_X)$ is saturated. Then (B)–(D) above hold for X and X°.*

1.3 Other results

1.3.1 Comparison theorems for Milnor fibers

In the situation of §1.1.2, suppose that $V = \mathbf{C}\{t\}$. Let X be an S-scheme of finite type and $x \in X_0(\mathbf{C})$ a point in the special fiber. We would like to compare the homotopy type of the classical Milnor fiber F_x with its algebraic version $M_x = (X_{(x)})_{\overline{\eta}}$. For this, we need a notion of "homotopy equivalence after profinite completion".

Definition 1.3.1 (cf. [AM69]). A morphism $f : X \longrightarrow Y$ of topoi is a $\natural$-*isomorphism* if for every locally constant sheaf $\mathscr{F}$ of finite sets (resp. finite groups, resp. finite abelian groups) on Y, the pullback map

$$f^* : H^q(Y, \mathscr{F}) \longrightarrow H^q(X, f^*\mathscr{F})$$

is an isomorphism for $q = 0$ (resp. for $q = 0, 1$, resp. for $q \geq 0$).

For a map of schemes $f : X \longrightarrow Y$, the induced map of étale topoi is a $\natural$-isomorphism if and only if its induces an isomorphism of the profinite completions of their étale homotopy types. Moreover, for a scheme X of finite type over $\mathbf{C}$, the natural map $X^{\mathrm{an}} \longrightarrow X_{\text{ét}}$ is a $\natural$-isomorphism [AM69, 12.9]. We extend this to Milnor fibers as follows:

Theorem (4.2.9). *The topoi F_x and M_x are canonically $\natural$-isomorphic (meaning that there exists a canonical chain of $\natural$-isomorphisms of topoi connecting the two).*

At first, this seems quite unlike the comparison between X^{an} and $X_{\text{ét}}$ for X of finite type over $\mathbf{C}$, as M_x does not admit an analytification, and F_x is not an analytic space (and is only well-defined up to homotopy). But in fact, the difficulty of defining the chain of $\natural$-isomorphisms aside, this result can almost be deduced from [SGA73b, Exp. XVI, 4.1] and [SGA73a, Exp. XIV, 2.8]. We give two proofs, whose intermediate results may be of independent interest. The first one (Theorem 4.3.1) uses [SGA73b, Exp. XVI, 4.1] in the key step. The second one (Theorem 4.2.9) derives the theorem from the following result:

Theorem (4.2.2). *Let R be a normal Noetherian henselian local $\mathbf{Q}$-algebra and let $I \subseteq R$ be an ideal. Let $X = \mathrm{Spec}(R) \setminus V(I)$, $Y = \mathrm{Spec}(\hat{R}) \setminus V(I \cdot \hat{R})$ where $\hat{R}$ is the completion of R at the maximal ideal. Then $Y_{\text{ét}} \longrightarrow X_{\text{ét}}$ is a $\natural$-isomorphism.*

The proof uses resolution of singularities and logarithmic geometry (the Kummer étale topology).

1.3.2 The monodromy formula

The following is an ℓ-adic version of the "monodromy formula" of Ogus [Ogu]. See the introduction to Chapter 5 for more context.

Let $f : (X, \mathcal{M}_X) \longrightarrow (S, \mathcal{M}_S)$ be a log smooth and saturated morphism, where $\mathcal{M}_S \to \mathcal{O}_S$ is the standard log structure, let ℓ be a prime invertible on S, and let Λ be either $\mathbf{Z}/\ell^h\mathbf{Z}$ for some $h \geq 1$, $\mathbf{Z}_\ell$, or $\mathbf{Q}_\ell$.We assume moreover that f is *vertical*, i.e., the log structure $\mathcal{M}_X$ is the compactifying log structure associated to the inclusion $X_\eta \subseteq X$. The purity theorem implies that we have natural isomorphisms

$$\bigwedge^q(\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S} \otimes \Lambda(-1)) \xrightarrow{\sim} R^q\Psi(\Lambda). \tag{1.9}$$

The naturality of (1.9) implies that monodromy action of the inertia subgroup $I \subseteq \mathrm{Gal}(\overline{\eta}/\eta)$ on $R\Psi(\Lambda)$ is trivial on the cohomology sheaves $R^q\Psi(\Lambda)$, and hence induces for each $\gamma \in I$ a morphism in the derived category

$$1 - \gamma : R^q\Psi(\Lambda) \longrightarrow R^{q-1}\Psi(\Lambda)[1]. \tag{1.10}$$

Using the fundamental extension

$$0 \longrightarrow f^*(\overline{\mathcal{M}}^{\mathrm{gp}}_S) \longrightarrow \overline{\mathcal{M}}^{\mathrm{gp}}_X \longrightarrow \overline{\mathcal{M}}^{\mathrm{gp}}_{X/S} \longrightarrow 0,$$

we construct for every $\gamma \in I$ a natural map

$$\bigwedge^q(\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S} \otimes \Lambda(-1)) \longrightarrow \bigwedge^{q-1}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S} \otimes \Lambda(-1))[1] \tag{1.11}$$

and prove that, under the above identification, it corresponds to (1.10).

Theorem (5.5.1)**.** *Under the above assumptions, the diagram*

$$\begin{tikzcd}
\bigwedge^q(\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S} \otimes \Lambda(-1)) \arrow[r, "(1.11)"] \arrow[d, "(1.9)"'] & \bigwedge^{q-1}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S} \otimes \Lambda(-1))[1] \arrow[d, "(1.9)"] \\
R^q\Psi(\Lambda) \arrow[r, "(1.10)"'] & R^{q-1}\Psi(\Lambda)[1]
\end{tikzcd}$$

commutes.

We also provide a version over a standard log point (Theorem 5.4.4), as well as a new proof of the original result of Ogus.

1.4 Conventions and notation

For future reference, we record our basic setup. Unless stated otherwise, V is a discrete valuation ring with perfect residue field k and fraction field K of characteristic zero. We choose an algebraic closure $\overline{K}$ of K, and define

$$S = \operatorname{Spec} V, \quad s = \operatorname{Spec} k, \quad \eta = \operatorname{Spec} K, \quad \overline{\eta} = \operatorname{Spec} \overline{K}.$$

Moreover, we endow S with the compactifying log structure $\mathscr{M}_S \longrightarrow \mathscr{O}_S$ induced by the open immersion $\eta \longrightarrow S$.

For a scheme X over S, we call $X_s := X \times_S s$, resp. $X_\eta := X \times_S \eta$, resp. $X_{\overline{\eta}} := X \times_S \overline{\eta}$ the *special* (or *closed*), resp. *generic*, resp. *geometric generic* fiber of X. If $\overline{x}$ is a geometric point of X, we denote by $X_{(\overline{x})}$ the henselization of X at $\overline{x}$.

Chapter 2

Preliminaries

2.1 $K(\pi,1)$ spaces

This section recalls the definition of a $K(\pi,1)$ space in algebraic topology and algebraic geometry, establishes some basic properties that apparently do not appear in the literature, and states the theorems of Artin and Faltings which assert the existence of coverings of smooth schemes by $K(\pi,1)$'s.

2.1.1 $K(\pi,1)$ spaces in algebraic topology

Let us start by recalling the classical theory of the fundamental group seen through covering spaces (cf. e.g. [Spa81]). Let X be a path connected, semi-locally 1-connected[1] topological space. A choice of a base point $x \in X$ allows us to define the category of pointed covering spaces (X', x') over (X, x). This category has an initial object $p : (\widetilde{X}, \widetilde{x}) \longrightarrow (X, x)$ called the *(pointed) universal cover* of (X, x), and the group $\pi_1(X, x)$ of automorphisms of $\widetilde{X}$ (without the base point) over X is called the *fundamental group* of (X, x). This group can be identified with the automorphism group of the *fiber functor*, which is the functor sending a covering space $f : X' \longrightarrow X$ of X to the fiber $f^{-1}(x)$. Thus $\pi_1(X, x)$ acts on any such $f^{-1}(x)$ (say on the right), and the fiber functor induces

[1]this means that every $x \in X$ is in an open $U \subseteq X$ for which $\pi_1(U, x) \longrightarrow \pi_1(X, x)$ is trivial.

an equivalence between the category of covering spaces of X and the category of right $\pi_1(X,x)$-sets. Under this equivalence, $\widetilde{X}$ corresponds to $\pi_1(X,x)$ with action by multiplication on the right. Thus $\widetilde{X}$ is a principal $\pi_1(X,x)$-bundle over X. For a discrete group G, the functor $X \mapsto$ (isomorphism classes of principal G-bundles on X) is represented by (the homotopy type of) a space BG (also denoted $K(G,1)$) with a principal G-bundle $EG \longrightarrow BG$ (called the universal G-bundle). Therefore there is a map (unique up to homotopy) $\rho : X \longrightarrow B\pi_1(X,x)$ to the classifying space of the fundamental group for which $\widetilde{X}$ is the pullback of the universal $\pi_1(X,x)$-bundle.

We now turn to describing local systems on X. There is a functor

$$\rho^* : \pi_1(X,x)\text{-sets} \longrightarrow \text{Sheaves}(X)$$

associating to a $\pi_1(X,x)$-set F a locally constant sheaf $\mathscr{F}$ with $\mathscr{F}_x = F$. It can be constructed as follows: the constant sheaf $\underline{F}$ on $\widetilde{X}$ associated to F carries a $\pi_1(X,x)$ action compatible with the $\pi_1(X,x)$-action on $\widetilde{X}$. We define $\mathscr{F}$ as the subsheaf of $\pi_1(X,x)$-invariants of $p_*\underline{F}$. This functor is fully faithful and its essential image is the subcategory of locally constant sheaves on X. In fact, ρ^* can be identified with pullback of locally constant sheaves along $\rho : X \longrightarrow B\pi_1(X,x)$.

We have $\Gamma(X,\rho^*(-)) \simeq (-)^G$, hence from the "universal δ-functor" formalism we get for every $\pi_1(X,x)$-module M a system of maps

$$\rho^q : H^q(\pi_1(X,x),M) \longrightarrow H^q(X,\rho^*(M)).$$

Even though ρ^* is exact, these maps do not have to be isomorphisms because ρ^* does not always send injective $\pi_1(X,x)$-modules to acyclic sheaves (e.g. $X = 2$).

Proposition 2.1.1. *The following are equivalent:*

(a) we have $\pi_i(X) = 0$ for $i > 1$,

(b) the universal cover $\widetilde{X}$ is weakly contractible,

(c) for every locally constant sheaf $\mathscr{F}$ on X, the pullback maps $H^q(X,\mathscr{F}) \longrightarrow H^q(\widetilde{X}, p^*\mathscr{F})$ are zero for $q > 0$.

(d) for every locally constant sheaf $\mathscr{F}$ on X and any class $\omega \in H^q(X,\mathscr{F})$, $q > 0$ there exists a covering $f : X' \longrightarrow X$ such that $f^*\omega = 0$ in $H^q(X', f^*\mathscr{F})$.

(e) *the natural transformation* $id \longrightarrow R\rho_*\rho^*$ *is an isomorphism,*

(f) *the maps* ρ^q *are isomorphisms for any* G*-module* M*,*

(g) *the map* $\rho : X \longrightarrow B\pi_1(X,x)$ *is a weak homotopy equivalence.*

Proof. The map $\rho : X \longrightarrow B\pi_1(X,x)$ is an isomorphism on π_1 and $\pi_i(B\pi_1(X,x)) = 0$ for $i > 0$, hence (g) and (a) are equivalent. They are equivalent to (b) by looking at the fibration exact sequence of p. Condition (b) implies (c) because $H^q(\widetilde{X}, f^*\mathscr{F}) = 0$ for $i > 0$ as $f^*\mathscr{F}$ is locally constant. Conversely, using $\mathscr{F} = \mathbf{Z}$ we see that (c) implies that $H^q(\widetilde{X},\mathbf{Z}) = 0$ for $q > 0$. The universal coefficient theorem for cohomology then implies that $\operatorname{Ext}^1(H_i(\widetilde{X},\mathbf{Z}),\mathbf{Z}) = 0$ and $\operatorname{Hom}(H_i(\widetilde{X},\mathbf{Z}),\mathbf{Z}) = 0$ for $i > 0$, thus $H_i(\widetilde{X},\mathbf{Z}) = 0$ for $i > 0$. By the Hurewicz theorem, $\pi_i(\widetilde{X}) = 0$ for $i > 0$, i.e., $\widetilde{X}$ is weakly contractible. Naturally (c) is equivalent to (d), as p factors through every $f : X' \longrightarrow X$ with X' connected. Since the sheaves $R^q\rho_*\mathscr{F}$ are formed by sheafifying the presheaf $(X' \to X) \mapsto H^q(X', f^*\mathscr{F})$, we see that (e) is equivalent to both (d) and (f). $\square$

Definition 2.1.2. We call a space X satisfying the equivalent conditions of Proposition 2.1.1 a $K(\pi,1)$ *space*.

Because of condition (g), a $K(\pi,1)$ space is determined by its fundamental group up to weak homotopy equivalence.

Example 2.1.3. While for every group π there exists a CW complex X which is a $K(\pi,1)$, it is difficult to find finite dimensional examples of $K(\pi,1)$ spaces. Some examples of interest are:

(1) the circle $\mathbf{S}^1$, tori $(\mathbf{S}^1)^n$,

(2) 2-dimensional manifolds other than $\mathbf{S}^2$,

(3) hyperbolic manifolds (complete Riemannian manifolds of constant negative sectional curvature), by the Cartan–Hadamard theorem,

(4) complements of hyperplane arrangements in $\mathbf{C}^d$ which are complexifications of simplicial arrangements in $\mathbf{R}^d$ [Del72].

2.1.2 $K(\pi, 1)$ spaces in algebraic geometry

We will often consider schemes X satisfying the following condition:

$$X \text{ is coherent and has finitely many connected components} \tag{2.1}$$

(see [AG11, 9.6] for some criteria). A scheme is called *coherent* if it is quasi-compact and quasi-separated.

Let Y be a scheme satisfying (2.1). We denote by $\mathrm{Fét}(Y)$ the full subcategory of the étale site $\mathrm{Ét}(Y)$ consisting of *finite* étale maps $Y' \longrightarrow Y$, endowed with the induced topology, and by $Y_{\text{fét}}$ the corresponding topos (cf. [AG11, 9.2]). Note that the maps in $\mathrm{Fét}(Y)$ are also finite étale. The inclusion functor induces a morphism of topoi (cf. [AG11, 9.2.1])

$$\rho : Y_{\text{ét}} \longrightarrow Y_{\text{fét}}.$$

The pullback ρ^* identifies $Y_{\text{fét}}$ with the category of sheaves on $Y_{\text{ét}}$ equal to the union of their locally constant subsheaves (cf. [Ols09, 5.1], [AG11, 9.17]). If Y is connected and $\overline{y} \longrightarrow Y$ is a geometric point, we have an equivalence of topoi $Y_{\text{fét}} \simeq B\pi_1(Y, \overline{y})$. Here, for a profinite group G, BG denotes the classifying topos of G, i.e., the category of continuous left G-sets.

Definition 2.1.4 ([SGA73b, Exp. IX, Définition 1.1])**.** Let $\wp$ be a set of prime numbers, let T be a topos, $\mathscr{F}$ a sheaf of abelian groups on T. We say that $\mathscr{F}$ is a $\wp$-torsion sheaf if the canonical morphism

$$\varinjlim_n {}_n\mathscr{F} \longrightarrow \mathscr{F}$$

is an isomorphism, where n ranges over all integers n with $\operatorname{ass} n \subseteq \wp$, ordered by divisibility, and where ${}_n\mathscr{F} = \ker(n : \mathscr{F} \longrightarrow \mathscr{F})$.

Proposition 2.1.5. *Let $\wp$ be a set of prime numbers, Y be a scheme satisfying (2.1). The following conditions are equivalent.*

(K a) For every $\wp$-torsion abelian sheaf F on $Y_{\text{fét}}$, the adjunction map

$$F \longrightarrow R\rho_*\rho^* F$$

is an isomorphism.

(K b) *For every $\wp$-torsion lcc abelian sheaf $\mathscr{F}$ on $Y_{\text{ét}}$, we have $R^q\rho_*\mathscr{F}=0$ for $q>0$.*

(K c) *For every $\wp$-torsion lcc abelian sheaf $\mathscr{F}$ on $Y_{\text{ét}}$, every finite étale $(f:X\longrightarrow Y)\in$ $\mathrm{Fét}(Y)$, and every $\zeta\in H^q(X,f^*\mathscr{F})$ $(q>0)$, there exists a cover*

$$\{g_i:(f_i:X_i\longrightarrow Y)\longrightarrow(f:X\longrightarrow Y)\}_{i\in I}\quad \textit{in}\quad \mathrm{Fét}(Y)$$

such that $f_i^*\zeta=0\in H^q(X_i,g_i^*f^*\mathscr{F})=H^q(X_i,f_i^*\mathscr{F})$.

(K d) *For every $\wp$-torsion lcc abelian sheaf $\mathscr{F}$ on $Y_{\text{ét}}$, and every class $\zeta\in H^q(Y,\mathscr{F})$ with $q>0$, there exists a finite étale surjective map $f:Y'\longrightarrow Y$ such that $f^*(\zeta)=0\in$ $H^q(Y',f^*\mathscr{F})$.*

(K e) *For every $\wp$-torsion lcc abelian sheaf $\mathscr{F}$ on $Y_{\text{ét}}$, the maps*

$$\rho^*:H^q(Y_{\text{fét}},\rho_*\mathscr{F})\longrightarrow H^q(Y_{\text{ét}},\mathscr{F})$$

are isomorphisms for all $q\geq 0$.

(K f) *For every geometric point $\overline{y}$ of Y and every $\wp$-torsion lcc abelian sheaf $\mathscr{F}$ on $Y_{\text{ét}}$, the maps*

$$H^q(\pi_1(Y,\overline{y}),\mathscr{F}_{\overline{y}})\longrightarrow H^q(Y_{\text{ét}},\mathscr{F})$$

are isomorphisms for all $q\geq 0$.

Moreover, these conditions are equivalent to the analogous conditions (K a')–(K f') where "$\wp$-torsion sheaf" is replaced by "sheaf of $\mathbf{Z}/p\mathbf{Z}$-modules for some $p\in\wp$".

Proof. The equivalence of (K a) and (K b) is [AG11, 9.17]. Conditions (K b) and (K c) are equivalent because $R^q\rho_*\mathscr{F}$ is the sheaf associated to the presheaf

$$(f:X\longrightarrow Y)\quad\mapsto\quad H^q(X,f^*\mathscr{F}).$$

We show that (K c) implies (K d). In case Y is connected, each g_i with X_i nonempty is finite étale surjective, and we get (K d)) by considering $Y'=Y$. The general case of this implication follows by considering the connected components of Y separately (as (2.1) implies that Y is the disjoint union of its connected components).

We prove that (K d) implies (K c). In the situation of (K c), let $\mathscr{F}_0 = f^*\mathscr{F}$ for brevity, and consider the sheaf $f_*\mathscr{F}_0$. As f is finite étale, $f_*\mathscr{F}_0$ is locally constant constructible and $R^j f_*\mathscr{F}_0 = 0$ for $j > 0$, therefore the natural map (2.8)

$$\mu : H^q(Y, f_*\mathscr{F}_0) \longrightarrow H^q(X, \mathscr{F}_0), \tag{2.2}$$

is an isomorphism. Let $\zeta' \in H^q(Y, f_*\mathscr{F})$ map to ζ under (2.2). By (K d), there exists a finite étale surjective map $g : Y' \longrightarrow Y$ with $g^*\zeta' = 0 \in H^q(Y', g^*f_*\mathscr{F}_0)$. Form a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y. \end{tikzcd}$$

Then g' is finite étale and surjective. Moreover, by Proposition 2.3.1, the diagram

$$\begin{tikzcd} & & \zeta' \arrow[r, mapsto] & \zeta \\ & & \rotatebox{-90}{$\in$} & \rotatebox{-90}{$\in$} \\ \zeta' \arrow[d, mapsto] & \in & H^q(Y, f_*\mathscr{F}_0) \arrow[r, "\mu"] \arrow[d, "g^*"'] & H^q(X, \mathscr{F}_0) \arrow[d, "g'^*"] \\ 0 & \in & H^q(Y', g^*f_*\mathscr{F}_0) \arrow[r, "\mu'"'] & H^q(X', g'^*\mathscr{F}_0) \end{tikzcd}$$

commutes, hence $g'^*\zeta = 0$.

Clearly (K b) implies (K e) and (K e) implies (K d). Finally, (K e) and (K f) are equivalent because Y is the disjoint union of its connected components, and if Y is connected then $Y_{\text{fét}}$ can be identified with $B\pi_1(Y, \bar{y})$.

By the same reasoning, conditions (a')–(g') are equivalent, and obviously (a) implies (a') etc. We show that (b') implies (b). By the Chinese remainder theorem, a sheaf as in (b) is a direct sum of sheaves of $\mathbf{Z}/p^h\mathbf{Z}$-modules for various $p \in \wp$ and $h \geq 1$. Thus it is sufficient to prove (b) for sheaves of $\mathbf{Z}/p^h\mathbf{Z}$-modules. We prove this by induction on h, the case $h = 1$ being (b'). For $h \geq 2$, we consider the extension

$$0 \longrightarrow {}_p\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/{}_p\mathscr{F} \longrightarrow 0.$$

Here both ${}_p\mathscr{F}$ and $\mathscr{F}/{}_p\mathscr{F}$ are both sheaves of $\mathbf{Z}/p^{h-1}\mathbf{Z}$-modules, and hence the assertion of (b) holds for them. Considering the long exact sequence for $R^q\rho_*$ shows that the same is true for $\mathscr{F}$. $\square$

Definition 2.1.6 (cf. [Ols09, Definition 5.3])**.** Let $\wp$ be a set of prime numbers. A scheme Y satisfying (2.1) is called a $K(\pi,1)$ *for $\wp$-torsion coefficients* if the equivalent conditions of Proposition 2.1.5 are satisfied. If $\wp$ is the set of primes invertible on Y, we simply call Y a $K(\pi,1)$.

Lemma 2.1.7. *The assertions of (K b)–(K f) are always satisfied for $q=1$.*

Proof. Let us check (K d). A class $\zeta \in H^1(Y,\mathscr{F})$ corresponds to an isomorphism class of an $\mathscr{F}$-torsor $f : Y' \longrightarrow Y$. The pullback $Y' \times_Y Y' \longrightarrow Y'$ has a section, and hence is a trivial $f^*\mathscr{F}$-torsor, thus the corresponding class $f^*\zeta \in H^1(Y', f^*\mathscr{F})$ is zero. □

Proposition 2.1.8. *Let $\wp$ be a set of prime numbers, and let Y be a scheme satisfying 2.1.*

(a) Let $f : X \longrightarrow Y$ be a finite étale surjective map. Then X satisfies 2.1, and Y is a $K(\pi,1)$ for $\wp$-adic coefficients if and only if X is.

(b) Suppose that Y is of finite type over a field F and that F' is a field extension of F. Denote $X = Y_{F'}$. Then X satisfies 2.1, and Y is a $K(\pi,1)$ if and only if X is.

Proof. (a) If X is a $K(\pi,1)$, Y is a $K(\pi,1)$ as well by condition (K d). Suppose that Y is a $K(\pi,1)$ and let $\mathscr{F}_0$ be a locally constant constructible $\wp$-torsion sheaf on $X_{\text{ét}}$, $\zeta \in H^q(X,\mathscr{F}_0)$ $(q>0)$. Apply the same reasoning as in the proof of Proposition 2.1.5, equivalence of (K c) and (K d).

(b) If F'/F is a finite separable extension, this follows from (a) as then $X \longrightarrow Y$ is finite étale and surjective. If F' is a separable closure of F, the assertion follows from the characterization in (a) and usual limit arguments. If F'/F is finite and purely inseparable, $X \longrightarrow Y$ induces equivalences $X_{\text{ét}} \simeq Y_{\text{ét}}$ and $X_{\text{fét}} \simeq Y_{\text{fét}}$, so there is nothing to prove. If F' and F are both algebraically closed, the assertion follows from [SGA73b, Exp. XVI, 1.6]. If F'/F is arbitrary, pick an algebraic closure $\overline{F}'$ and let $\overline{F}$ be the algebraic closure of F in $\overline{F}'$. We now have a "path" from F to F' of the form

$$F \subseteq F^{\text{sep}} \subseteq \overline{F} \subseteq \overline{F}' \supseteq (F')^{\text{sep}} \supseteq F'$$

and the assertion follows from the preceding discussion. □

Theorem 2.1.9 (M. Artin, follows from [SGA73b, Exp. XI, 3.3], cf. [Ols09, Lemma 5.5])**.** *Let Y be a smooth scheme over a field of characteristic zero, y a point of Y. There exists an open neighborhood U of y which is a $K(\pi,1)$.*

Theorem 2.1.10 (Faltings, [Fal88, Lemma 2.1], cf. [Ols09, Theorem 5.4])**.** *Let S be as in 1.4, let Y be a smooth S-scheme, and let y be a point of Y. There exists an open neighborhood U of y for which U_η is a $K(\pi,1)$.*

Example 2.1.11. The following are examples of $K(\pi,1)$ schemes:

(1) Spec k for a field k,

(2) a scheme of cohomological dimension ≤ 1 (Lemma 2.1.7),

(3) a smooth connected curve X over a field k such that $X_{\bar{k}}$ is not isomorphic to $\mathbf{P}^1_{\bar{k}}$,

(4) an abelian variety over a field.

Example 2.1.12. Let p be a prime. Then every connected affine $\mathbf{F}_p$-scheme X is a $K(\pi,1)$ for p-torsion coefficients. We check condition (K d'). Let $\mathscr{F}$ be an lcc $\mathbf{Z}/p\mathbf{Z}$-sheaf on X, and let $\zeta \in H^q(X,\mathscr{F})$ ($q>0$). We need to find a finite étale cover $X' \longrightarrow X$ killing ζ. First, we can assume that $\mathscr{F}$ is constant, as there exists a finite étale cover $X' \longrightarrow X$ such that the pullback of $\mathscr{F}$ to X' is constant, with X' affine and connected. Second, we can reduce to the case $\mathscr{F} = \mathbf{F}_p$. In this case, the Artin–Schreier sequence on $X_{\text{ét}}$

$$0 \longrightarrow \mathbf{F}_p \longrightarrow \mathscr{O}_X \xrightarrow{1-F} \mathscr{O}_X \longrightarrow 0$$

together with Serre vanishing ($H^q(X_{\text{ét}},\mathscr{O}_X) = H^q(X,\mathscr{O}_X) = 0$ for $q>0$) shows that $H^q(X,\mathbf{F}_p) = 0$ for $i > 1$. Thus is $q>1$, we are done. If $q=1$, then ζ corresponds to an $\mathbf{F}_p$-torsor on $X_{\text{ét}}$, which again can be made trivial by a finite étale $X' \longrightarrow X$.

This example has been recently used by Scholze [Sch13, 4.9] to show that any Noetherian affinoid adic space over Spa$(\mathbf{Q}_p,\mathbf{Z}_p)$ is a $K(\pi,1)$ for p-adic coefficients.

2.1.3 Complements and examples

Definition 2.1.13 ([Ser63, §2.6])**.** Let G be a discrete group, and let $\iota : G \longrightarrow \hat{G}$ denote its pro-finite completion. We call G a *good group* if for every finite $\hat{G}$-module M, the maps

$$\iota^* : H^q(\hat{G},M) \longrightarrow H^q(G,M)$$

are isomorphisms for all $q \geq 0$.

Example 2.1.14. Some examples of good groups:

(1) finite groups,

(2) finitely generated free groups and finitely generated free abelian groups,

(3) iterated extensions of groups of type (1) and (2),

(4) braid groups (a special case of (3)),

(5) the Bianchi groups $PSL(2,\mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers in an imaginary quadratic number field $\mathbf{Q}(\sqrt{-d})$ [GJZZ08, Theorem 1.1].

Arithmetic groups are not good in general (e.g. $\mathrm{Sp}(2n,\mathbf{Z})$ is not a good group for $n>1$ [GJZZ08]). It is not known whether the mapping class groups $\Gamma_{g,n}$ (the orbifold fundamental group of $\mathcal{M}_{g,n}$) are good groups [LS06, 3.4].

Proposition 2.1.15. *Let X be a connected scheme of finite type over* $\mathbf{C}$, *X^{an} the associated analytic space, and $x\in X(\mathbf{C})$. Consider the following three conditions.*

(1) X is a $K(\pi,1)$ scheme.

(2) X^{an} is a $K(\pi,1)$ space.

(3) The fundamental group $\pi_1(X^{\mathrm{an}},x)$ is a good group.

Then (1)&(2) $\Rightarrow$ (3) *and* (2)&(3) $\Rightarrow$ (1).

Proof. Denote by ε the natural map of topoi $X^{\mathrm{an}}\longrightarrow X_{\text{ét}}$. Pullback by ε induces an equivalence of categories of locally constant sheaves of finite abelian groups on $X_{\text{ét}}$ and X^{an} and identifies $\pi_1(X,x)$ with the profinite completion of $\pi_1(X^{\mathrm{an}},x)$. Let $\mathcal{F}$ be an lcc sheaf on $X_{\text{ét}}$, and let $q\geq 0$. We have a commutative diagram

$$\begin{tikzcd} H^q(\pi_1(X^{\mathrm{an}},x),\mathcal{F}_x) \arrow[r] & H^q(X^{\mathrm{an}},\varepsilon^*\mathcal{F}) \\ H^q(\pi_1(X,x),\mathcal{F}_x) \arrow[r] \arrow[u] & H^q(X_{\text{ét}},\mathcal{F}). \arrow[u] \end{tikzcd}$$

The map $\varepsilon^*: H^q(X_{\text{ét}},\mathcal{F})\longrightarrow H^q(X^{\mathrm{an}},\varepsilon^*\mathcal{F})$ is an isomorphism by [SGA73b, Exp. XVI Théoreme 4.1]. Thus out of the three remaining maps, if two are isomorphisms, so is the third. $\square$

Note that we do not get (1)&(3) $\Rightarrow$ (2), as neither (1) nor (3) gives us any information about the cohomology of X^{an} with non-torsion coefficients. Still, we see that (1)&(3) implies that X^{an} is a "$K(\pi,1)$ for local systems of finite groups".

Remark 2.1.16. We expect the same statement to hold for Deligne–Mumford stacks of finite type over $\mathbf{C}$, with their orbifold fundamental groups. In particular, the open question whether $\Gamma_{g,n}$ is a good group [LS06, 3.4] would be equivalent to the question whether the stack $\mathscr{M}_{g,n}$ is a $K(\pi,1)$ in the algebraic sense. Similarly, as the orbifold fundamental group of $\mathscr{A}_g$ is $\mathrm{Sp}(2g,\mathbf{Z})$, while the orbifold universal cover of $\mathscr{A}_g$ is the Siegel upper-half space (which is contractible), the stack $\mathscr{A}_g$ ($g>1$) gives a probable example of a smooth Deligne–Mumford stack which is a $K(\pi,1)$ in the analytic sense but not in the algebraic sense.

2.2 Logarithmic geometry

In this section, we review the relevant facts from log geometry and investigate the local structure of a log smooth S-scheme (with the standard log structures on X and S). We also state the logarithmic version of absolute cohomological purity, used in the remaining chapters.

2.2.1 Conventions about log geometry

If P is a monoid, $\overline{P}$ denotes the quotient of P by its group P^* of invertible elements, and $P \longrightarrow P^{\mathrm{gp}}$ is the universal (initial) morphism form P into a group. P is called *fine* if it is integral (i.e., $P \longrightarrow P^{\mathrm{gp}}$ is injective) and finitely generated. A *face* of a monoid P is a submonoid $F \subseteq P$ satisfying $x+y \in F \Rightarrow x,y \in F$. For an integral monoid P and face F, the *localization* of P at F is the submonoid P_F of P^{gp} generated by P and $-F$. It satisfies the obvious universal property. If Q is a submonoid of an integral monoid P, the quotient P/Q is defined to be the image of P in $P^{\mathrm{gp}}/Q^{\mathrm{gp}}$. A monoid P is called *saturated* if it is integral and if whenever $nx \in P$ for some $n>0$, $x \in P^{\mathrm{gp}}$, we have $x \in P$.

For a monoid P, $\mathbf{A}_P = \mathrm{Spec}(P \longrightarrow \mathbf{Z}[P])$ is the log scheme associated to P; for a homomorphism $\theta : P \longrightarrow Q$, $\mathbf{A}_\theta : \mathbf{A}_Q \longrightarrow \mathbf{A}_P$ is the induced morphism of log schemes. A morphism $(X,\mathscr{M}_X) \longrightarrow (Y,\mathscr{M}_Y)$ of log schemes is *strict* if the induced map

$f^\flat : f^*\mathcal{M}_Y \longrightarrow \mathcal{M}_X$ is an isomorphism. A strict map to some $\mathbf{A}_P$ is called a *chart*. A log scheme is *fine* if étale locally it admits a chart with target $\mathbf{A}_P$ for a fine monoid P. If $j : U \longrightarrow X$ is an open immersion, the *compactifying log structure* on X associated to U is the preimage of $j_*\mathcal{O}_U^*$ under the restriction map $\mathcal{O}_X \longrightarrow j_*\mathcal{O}_U$. For a log scheme $(X,\mathcal{M}_X)$, we denote by $(X,\mathcal{M}_X)_{\mathrm{tr}}$ the complement of the support of $\overline{\mathcal{M}}_X$. It is an open subset of X if $(X,\mathcal{M}_X)$ is fine, the biggest open subset on which $\mathcal{M}_X$ is trivial. *References:* [Kat89, Ogu, ACG+13].

We recall Kato's structure theorem for log smooth morphisms (which for our purposes might as well serve as a definition):

Theorem 2.2.1 (cf. [Kat89, Theorem 3.5]). *Let $f : (X,\mathcal{M}_X) \longrightarrow (S,\mathcal{M}_S)$ be a morphism of fine log schemes. Assume that we are given a chart $\pi : (S,\mathcal{M}_S) \longrightarrow \mathbf{A}_Q$ with Q a fine monoid. Then f is log smooth if and only if, étale locally on X, there exists a fine monoid P, a map $\rho : Q \longrightarrow P$ such that the kernel and the torsion part of the cokernel of $\rho^{\mathrm{gp}} : Q^{\mathrm{gp}} \longrightarrow P^{\mathrm{gp}}$ are finite groups of order invertible on S, and a commutative diagram*

$$\begin{tikzcd}
(X,\mathcal{M}_X) \arrow[r] \arrow[dr, "f"'] & \mathbf{A}_{P,\rho,\pi} \arrow[r] \arrow[d] & \mathbf{A}_P \arrow[d, "\mathbf{A}_\rho"] \\
& (S,\mathcal{M}_S) \arrow[r, "\pi"'] & \mathbf{A}_Q,
\end{tikzcd}$$

where the square is cartesian (in the category of log schemes) and $(X,\mathcal{M}_X) \longrightarrow \mathbf{A}_{P,\rho,\pi}$ is strict, and étale (as a morphism of schemes).

2.2.2 Charts

Suppose that $f : (X,\mathcal{M}_X) \longrightarrow \mathbf{A}_P$ is a chart with P a fine monoid, and $\overline{x} \rightarrow X$ is a geometric point. Let $F \subseteq P$ be the preimage of 0 under the induced homomorphism $P \longrightarrow \overline{\mathcal{M}}_{X,\overline{x}}$. Then F is a face of P, and P injects into the localization P_F. Moreover, the induced map $P/F \longrightarrow \overline{\mathcal{M}}_{X,\overline{x}}$ is an isomorphism.

As P is finitely generated, F is finitely generated as a face, hence the natural map $\mathbf{A}_{P_F} \longrightarrow \mathbf{A}_P$ is an open immersion: if F is generated as a face by an element $a \in P$, then

$\mathbf{A}_{P_F} = D(a)$. Let us form a cartesian diagram

$$\begin{tikzcd} (U, \mathscr{M}_X|_U) \arrow[r] \arrow[d] & \mathbf{A}_{P_F} \arrow[d] \\ (X, \mathscr{M}_X) \arrow[r, "f"'] & \mathbf{A}_P. \end{tikzcd}$$

Then $U = D(f^*a)$ and $\overline{x}$ lies in U because f^*a is an invertible element of $\mathscr{O}_{X,x}$ by the construction of F.

It follows that any chart $f : (X, \mathscr{M}_X) \longrightarrow \mathbf{A}_P$ as above can be locally replaced by one for which the homomorphism $\overline{P} \longrightarrow \overline{\mathscr{M}}_{X,\overline{x}}$ is an isomorphism, without changing the local properties of f (e.g. without sacrificing étaleness if f is étale).

2.2.3 Absolute cohomological purity

We will need the following result (cf. [Nak98, Proposition 2.0.2]): let $(X, \mathscr{M}_X)$ be a regular [Kat94, Definition 2.1] (in particular, fine and saturated) log scheme such that X is locally Noetherian. Let $X^\circ = (X, \mathscr{M}_X)_{\mathrm{tr}}$ be the biggest open subset on which $\mathscr{M}_X$ is trivial, and let $j : X^\circ \longrightarrow X$ be the inclusion. Let n be an integer invertible on X. Then for any $q \geq 0$, we have a natural isomorphism

$$R^q j_*(\mathbf{Z}/n\mathbf{Z}) \simeq \bigwedge^q \overline{\mathscr{M}}^{\mathrm{gp}}_X \otimes \mathbf{Z}/n\mathbf{Z}(-1). \tag{2.3}$$

We will have a closer look at how these isomorphisms are constructed in Chapter 5.

2.2.4 Saturated morphisms

Before defining saturated morphisms (a notion introduced by K. Kato and developed by T. Tsuji [Tsu97]), let us discuss a certain simple phenomenon related to fiber products in log geometry. We will need the assumption that a morphism of log schemes is saturated on several occasions, for a reason closely related to this phenomenon.

Let $(X, \mathscr{M}_X) = \mathbf{A}_{\mathbf{N}} \otimes \mathbf{C}$, which is the affine line $\mathbf{A}^1$ over $\mathbf{C}$ equipped with the log structure induced by inclusion of $U = \mathbf{A}^1 \setminus \{0\}$. The double cover $V = \mathbf{A}^1 \setminus \{0\} \longrightarrow U$, $z \mapsto z^2$, extends to a Kummer étale map $(Y, \mathscr{M}_Y) = \mathbf{A}_{\mathbf{N}} \otimes \mathbf{C} \longrightarrow (X, \mathscr{M}_X)$. Note that V is an étale $\mathbf{Z}/2\mathbf{Z}$-torsor over U, and hence $V \times_U V$ has a section over V (in particular, it is disconnected). Since $(Y,\mathscr{M}_Y)/(X,\mathscr{M}_X)$ should behave like V/U, we shall investigate the fiber product $(Z,\mathscr{M}_Z):=(Y,\mathscr{M}_Y)\times_{(X,\mathscr{M}_X)}(Y,\mathscr{M}_Y)$ in the category of log schemes.

The scheme Z is the union of two copies of $\mathbf{A}^1$ glued together transversally at the origins, and the log structure $\mathscr{M}_Z$ is trivial away from the origin. The stalk of $\overline{\mathscr{M}}_Z$ at the origin is the pushout

$$\begin{tikzcd} \mathbf{N} \arrow[r, "2"] \arrow[d, "2"'] & \mathbf{N} \arrow[d] \\ \mathbf{N} \arrow[r] & \overline{\mathscr{M}}_{Z,0}. \end{tikzcd}$$

This is the submonoid of $\mathbf{N}\times\mathbf{Z}/2\mathbf{Z}$ generated by $(1,0)$ and $(1,1)$. It is therefore non-saturated, as $(0,1)+(0,1)=(0,0)\in\overline{\mathscr{M}}_{Z,0}$ while $(0,1)\notin\overline{\mathscr{M}}_{Z,0}$.

If the fiber product is taken in the category of fs log schemes instead, $(Z,\mathscr{M}_Z)$ is replaced by its saturation $(Z',\mathscr{M}_{Z'})$, where Z' is the *disjoint* union of two affine lines $\mathbf{A}_{\mathbf{N}}$. The moral of this is that it is more natural to take fiber products in the category of fs log schemes. Saturated morphisms are morphisms for which there is no distinction between the two types of fiber product.

Definition 2.2.2. A morphism $f:(X,\mathscr{M}_X)\longrightarrow(S,\mathscr{M}_S)$ of fs log schemes is called *saturated* if for every map $(S',\mathscr{M}_{S'})\longrightarrow(S,\mathscr{M}_S)$ of fs log schemes, the fiber product $(X',\mathscr{M}_{X'})=(X,\mathscr{M}_X)\times_{(S,\mathscr{M}_S)}(S',\mathscr{M}_{S'})$ (in the category of log schemes) is an fs log scheme.

This definition is equivalent to the original definition [Tsu97, Definition II 2.10] for morphisms of fs log schemes. This follows from [Tsu97, Proposition II 2.13(2)] and [Kat89, 4.3.1].

The situation of interest for us will be that of a smooth log scheme over a standard log point or log trait. In these situations, one has a simple characterization of saturated morphisms.

Theorem 2.2.3 (cf. [IKN05, Remark 6.3.3]). *Let $(S,\mathscr{M}_S)$ be a trait with the standard log structure (resp. a standard log point), and let $f:(X,\mathscr{M}_X)\longrightarrow(S,\mathscr{M}_S)$ be a smooth morphism of fs log schemes. Then f is an integral morphism [Kat89, 4.3]. Furthermore, the following conditions are equivalent:*

(1) f is saturated,

(2) X_s is reduced (resp. X is reduced),

(3) $\overline{\mathscr{M}}^{\mathrm{gp}}_{X/S}$ is torsion-free.

Proof. That f is integral follows from [Kat89, Corollary 4.4(ii)]. The equivalence of (1) and (2) is [Tsu97, Theorem II 4.2] (see also [Ogu, Theorem 4.8.14]). The equivalence of (1) and (3) follows from Kato's structure theorem Theorem 2.2.1 and [Ogu, Theorem 4.8.14]. □

2.3 Functoriality of cohomology pullback maps

This section checks a certain functoriality property of cohomology of topoi, needed in Section 2.1.2. The result we need (Proposition 2.3.1) states that given a commutative diagram of topoi

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd} \tag{2.4}$$

and a sheaf $\mathscr{F}$ on X, there exist certain natural commutative diagrams (2.9)

$$\begin{tikzcd} H^q(Y, f_*\mathscr{F}) \arrow[r] \arrow[d] & H^q(X, \mathscr{F}) \arrow[d] \\ H^q(Y', g^* f_*\mathscr{F}) \arrow[r] & H^q(X', g'^*\mathscr{F}) \end{tikzcd}$$

for all $q \geq 0$.

2.3.1 Base change morphisms

Suppose we are given a commutative diagram of morphisms of topoi as in (2.4), that is, a diagram of morphisms together with a chosen isomorphism

$$\iota : f_* g'_* \simeq g_* f'_*. \tag{2.5}$$

By adjunction, this also induces an isomorphism (also denoted ι)

$$\iota : f'^* g^* \simeq g'^* f^*. \tag{2.6}$$

Applying f_* to the unit $\eta : id \longrightarrow g'_* g'^*$ and composing with (2.5) yields a map

$$f_* \longrightarrow f_* g'_* g'^* \simeq g_* f'_* g'_*,$$

which (using the adjunction between g^* and g_*) gives us a map

$$\varphi : g^* f_* \longrightarrow f'_* g'^* \tag{2.7}$$

called the *base change morphism*.

Similarly, applying g'^* to the counit $\varepsilon : f^* f_* \longrightarrow id$, and composing with (2.6) yields a map

$$f'^* g^* f_* \simeq g'^* f^* f_* \longrightarrow g'^*,$$

which (using the adjunction between f'^* and f'_*) gives us another map $g^* f_* \longrightarrow f'_* g'^*$ that is equal to (2.7) by [SGA73b, Exp. XVII, Proposition 2.1.3].

2.3.2 Cohomology pullback morphisms

Recall that if $f : X \longrightarrow Y$ is a morphism of topoi, there is a natural map of δ-functors from the category of abelian sheaves on Y to the category of abelian groups:

$$f^* : H^q(Y,-) \longrightarrow H^q(X, f^*(-)).$$

Indeed, the right hand side is a δ-functor because f^* is exact, the transformation is defined for $q = 0$, and $H^q(Y,-)$ is a universal δ-functor.

The formation of this map is compatible with composition, that is, if $g : Z \longrightarrow X$ is another map, the diagram (of δ-functors of the above type)

$$\begin{tikzcd} H^q(Y,-) \arrow[r, "f^*"] \arrow[d, "(fg)^*"'] & H^q(X, f^*(-)) \arrow[d, "g^*"] \\ H^q(Z, (fg)^*(-)) \arrow[r, equal] & H^q(Z, g^* f^*(-)) \end{tikzcd}$$

commutes.

Applying this to the situation of 2.3.1 and composing with the map induced by the counit $\varepsilon : f^* f_* \longrightarrow id$, we get a system of natural transformations

$$\mu : H^q(Y, f_*(-)) \xrightarrow{f^*} H^q(X, f^* f_*(-)) \xrightarrow{\varepsilon} H^q(X,-). \tag{2.8}$$

These coincide with the edge homomorphisms in the Leray spectral sequence for f by [Gro61, 0_{III} 12.1.7].

2.3.3 Compatibility of base change and pullback

In the situation of the previous subsection, let μ' be the following composition

$$\mu' : H^q(Y', g^*f_*(-)) \xrightarrow{\varphi} H^q(Y', f'_*g'^*(-)) \xrightarrow{f'^*} H^q(X', f'^*f'_*g'^*(-)) \xrightarrow{\varepsilon'} H^q(X', g'^*(-)),$$

where $\varepsilon' : f'^*f'_* \longrightarrow id$ is the counit. The goal of this section is to prove the following compatibility.

Proposition 2.3.1. *The diagram*

$$\begin{tikzcd}
H^q(Y, f_*(-)) \arrow[r, "\mu"] \arrow[d, "g^*"'] & H^q(X, (-)) \arrow[d, "g'^*"] \\
H^q(Y', g^*f_*(-)) \arrow[r, "\mu'"'] & H^q(X', g'^*(-))
\end{tikzcd} \tag{2.9}$$

commutes.

Proof. The assertion will follow from the commutativity of the following diagram

$$\begin{tikzcd}
H^q(Y, f_*(-)) \arrow[rr, "f^*"] \arrow[dd, "g^*"'] & & H^q(X, f^*f_*(-)) \arrow[r, "\varepsilon"] \arrow[d, "g'^*"'] & H^q(X, -) \arrow[d, "g'^*"] \\
& \arrow[r, phantom, "\mathrm{(I)}"] & H^q(X', g'^*f^*f_*(-)) \arrow[r, "g'^*(\varepsilon)"'] \arrow[ur, phantom, "\mathrm{(III)}"] & H^q(X', g'^*(-)) \\
H^q(Y', g^*f_*(-)) \arrow[r, "f'^*"] \arrow[d, "\varphi"'] & H^q(X', f'^*g^*f_*(-)) \arrow[ur, equal, "\iota"] \arrow[d, "f'^*(\varphi)"] \arrow[r, phantom, "\mathrm{(IV)}"] & & \\
H^q(Y', f'^*g'^*(-)) \arrow[r, "f'^*"'] \arrow[ur, phantom, "\mathrm{(II)}"] & H^q(X', f'^*f'_*g'^*(-)). \arrow[uurr, bend right, "\varepsilon'"'] & &
\end{tikzcd}$$

Square (I) commutes by the functoriality of f^* (2.3.2). Squares (II) and (III) commute simply because f^* is a natural transformation.

It remains to prove that (IV) commutes. This in turn will follow from the commutativity of

$$\begin{tikzcd}
f'^*g^*f_* \arrow[r, equal, "\iota"] \arrow[d, "f'^*(\varphi)"'] & g'^*f^*f_* \arrow[d, "g'^*(\varepsilon)"] \\
f'^*f'_*g'^* \arrow[r, "\varepsilon'"'] & g'^*.
\end{tikzcd}$$

By the discussion of 2.3.1, the composition $g'^*(\varepsilon)\circ\iota$ above is adjoint (under the adjunction between f'^* and f'_*) to the base change map $\varphi : g^* f_* \longrightarrow f'_* g'^*$. It suffices to show that $\varepsilon \circ f'^*(\varphi)$ is also adjoint to the base change map. This follows precisely from the triangle identities for the adjunction between f'^* and f'_*. $\square$

Chapter 3

$K(\pi, 1)$ neighborhoods and p-adic Hodge theory

Recall the setting of §1.4. The goal of this chapter is to prove the main result of this thesis.

Theorem (3.2.1)**.** *Assume that* $\operatorname{char} k = p > 0$*. Let* $(X, \mathscr{M}_X)$ *be a log smooth log scheme over* $(S, \mathscr{M}_S)$ *such that* X_η *is smooth over* η*. Then there exists an étale cover of* X *by schemes* U *such that* $U_{\overline{\eta}}$ *is a* $K(\pi, 1)$*.*

Corollary (3.6.6)**.** *Let* $(X, \mathscr{M}_X)$ *be a log smooth log scheme over* $(S, \mathscr{M}_S)$ *such that* X_η *is smooth over* η*, and let* X° *be the biggest open subset on which* $\mathscr{M}_X = \mathscr{O}_X^*$*. If* $\operatorname{char} k = 0$*, assume moreover that* $(X, \mathscr{M}_X)$ *is saturated. Let* $\widetilde{E}$ *denote the Faltings' topos of* $X^\circ_{\overline{\eta}} \longrightarrow X$ *(cf. Definition 3.6.1),* $\Psi : X^\circ_{\overline{\eta}} \longrightarrow \widetilde{E}$ *the natural morphism of topoi. Then for every locally constant constructible abelian sheaf* $\mathscr{F}$ *on* $X^\circ_{\overline{\eta}}$*, the natural maps*

$$H^q(\widetilde{E}, \Psi_*(\mathscr{F})) \longrightarrow H^q(X^\circ_{\overline{\eta},\text{ét}}, \mathscr{F})$$

are isomorphisms for all $q \geq 0$*.*

3.1 η-étale maps and Noether normalization

This section contains the key technical point used in the proof of Theorem 3.2.1. First, we prove a (slightly spiced-up) relative version of the Noether Normalization Lemma (Proposition 3.1.3). Then we study η-étale maps $f : X \longrightarrow Y$ over S, that is, maps which are étale in an open neighborhood of the closed fiber X_s of X. The main result is Proposition 3.1.8, which asserts that *in mixed characteristic* we can often replace an η-étale map $f' : X \longrightarrow \mathbf{A}^d_S$ by a quasi-finite η-étale map $f : X \longrightarrow \mathbf{A}^d_S$.

3.1.1 Relative Noether normalization

Lemma 3.1.1. *Let F be a field, and let $a \in F[x_1,\dots,x_n]$ be a nonzero polynomial. For large enough m, the polynomial*

$$a(x_1 - x_n^m, x_2 - x_n^{m^2}, \dots, x_{n-1} - x_n^{m^{n-1}}, x_n),$$

treated as a polynomial in x_n over $F[x_1,\dots,x_{n-1}]$, has a constant leading coefficient.

Proof. Standard, cf. e.g. [Mum99, §1] or [Sta14, Tag 051N]. □

Definition 3.1.2. Let $f : X \longrightarrow Y$ be a map of schemes over some base scheme S. We call f *fiberwise finite relative to S* if for every point $s \in S$, the induced map $X_s \longrightarrow Y_s$ is finite.

Let $V, S, \dots$ be as in 1.4 (the assumptions on K and k are unnecessary here). The following is a relative variant of Noether normalization. In the applications we will take N to be a high power of p.

Proposition 3.1.3. *Let $X = \operatorname{Spec} R$ be a flat affine S-scheme of finite type, let $d \geq 0$ be an integer such that $\dim(X/S) \leq d$, and let $x_1,\dots,x_d \in R$. For any integer $N \geq 1$, there exist $y_1,\dots,y_d \in R$ such that the following conditions hold.*

(i) The map $f = (f_1,\dots,f_d) : X \longrightarrow \mathbf{A}^d_S$, $f_i = x_i + y_i$, is fiberwise finite relative to S.

(ii) The y_i belong to the subring generated by N-th powers of elements of R.

Proof. Write $R = V[x_1,\ldots,x_d,x_{d+1},\ldots,x_n]/I$. The proof is by induction on $n-d$. If $n=d$, then the map $(x_1,\ldots,x_d) : X \longrightarrow \mathbf{A}^d_S$ is a closed immersion, and we can take $y_i = 0$. Suppose that $n>d$.

Let $a \in V[x_1,\ldots,x_n]$ be an element of I with nonzero image in $k[x_1,\ldots,x_n]$. Such an element exists, for otherwise X_s is equal to $\mathbf{A}^n_s$, hence cannot be of dimension $\leq d$ as $n>d$.

For an integer $m\geq 1$, consider the elements

$$z_i = x_i + x_n^{(Nm)^i}, \quad i=1,\ldots,n-1,$$

and let $R' \subset R$ be the V-subalgebra generated by $z_1,\ldots,z_{n-1}$. By Lemma 3.1.1 applied to the image of a in $K[x_1,\ldots,x_n]$ (with $F=K$) resp. $k[x_1,\ldots,x_n]$ (with $F=k$), there exists an m such that the images of x_n in $R\otimes K$, resp. $R\otimes k$ will be integral over $R'\otimes K$, resp. $R'\otimes k$. As $x_i = z_i - x_n^{(Nm)^i}$, the other x_i will have the same property, which is to say, $\operatorname{Spec} R \longrightarrow \operatorname{Spec} R'$ is fiberwise finite over S.

We check that it is possible to apply the induction assumption to $X' = \operatorname{Spec} R'$ and $z_1,\ldots,z_d \in R'$. Since R' is a subring of R and R is torsion-free, R' is torsion-free as well, hence flat over V. As $R'\otimes_V K \longrightarrow R\otimes_V K$ is finite and injective, we have $\dim X'_\eta = \dim X_\eta$. Since R' is flat over V, we have $\dim X'_s \leq \dim X'_\eta$, so $\dim X'_s \leq d$ as well. Finally, R' is generated as a V-algebra by $n-1$ elements with $z_1,\ldots,z_d$ among them.

By the induction assumption applied to $X' = \operatorname{Spec} R'$ and $z_1,\ldots,z_d \in R$, there exists a fiberwise finite map $f' = (f_1,\ldots,f_d) : \operatorname{Spec} R' \longrightarrow \mathbf{A}^d_S$ with $f_i = z_i + y'_i$, where the y'_i belong to the subring of R' generated by N-th powers of elements of R'. As the composition of fiberwise finite maps is clearly fiberwise finite, the composition $f = (f_1,\ldots,f_n) : X = \operatorname{Spec} R \longrightarrow \mathbf{A}^d_S$ is fiberwise finite (i). We have $f_i = x_i + y_i$, $y_i = y'_i + (x_n^{N^{i-1}m^i})^N$, so (ii) is satisfied as well. $\square$

It would be interesting to have a generalization of this result to a general Noetherian local base ring V.

3.1.2 η-étale maps

We now assume that $\operatorname{char} k = p > 0$. Let $f : X \longrightarrow Y$ be a map of S-schemes of finite type.

Definition 3.1.4. We call f *η-étale at a point* $x \in X_s$ if there is an open neighborhood U of x in X such that $f_\eta : U_\eta \longrightarrow Y_\eta$ is étale. We call f *η-étale* if it is η-étale at all points $x \in X_s$, or equivalently, if there is an open neighborhood U of X_s in X such that $f_\eta : U_\eta \longrightarrow Y_\eta$ is étale.

We warn the reader not to confuse "f is η-étale" with "f_η is étale" (the latter is a stronger condition).

Lemma 3.1.5. *Consider the following properties.*

(i) The map f is η-étale.

(ii) There exists an $n \geq 0$ such that $(p^n \Omega^1_{X/Y})|_{X_s} = 0$ (pullback as an abelian sheaf).

(iii) There exists an $n \geq 0$ such that $(p^n \Omega^1_{X/Y}) \otimes_V k = 0$.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii), and the three properties are equivalent if X_η and Y_η are smooth of the same relative dimension d over S.

Proof. The equivalence of (ii) and (iii) follows from Nakayama's lemma.

Suppose that f is η-étale, and let $U \subseteq X$ be an open subset containing X_s such that $f|_{U_\eta}$ is étale. In particular, $f|_{U_\eta}$ is unramified, hence $\Omega^1_{X/Y}|_{U_\eta} = 0$.

Recall that if $\mathscr{F}$ is a coherent sheaf on a Noetherian scheme U and $a \in \Gamma(U, \mathscr{O}_U)$, then $\mathscr{F}|_{D(a)} = 0$ if and only if $a^n \mathscr{F} = 0$ for some $n \geq 0$. Applying this to $\mathscr{F} = \Omega^1_{X/Y}|_U$ and $a = p$ (noting that $U_\eta = D(p)$), we get that $(p^n \Omega^1_{X/Y})|_U = 0$, hence in particular $(p^n \Omega^1_{X/Y})|_{X_s} = 0$.

Suppose now that $(p^n \Omega^1_{X/Y})|_{X_s} = 0$ for some $n \geq 0$. By Nakayama's lemma, $(p^n \Omega^1_{X/Y})_x = 0$ for every $x \in X_s$, hence there is an open subset U containing X_s such that $(p^n \Omega^1_{X/Y})|_U = 0$. As p is invertible on U_η, we get that $\Omega^1_{X/Y}|_{U_\eta} = 0$, that is, $f|_{U_\eta}$ is unramified. If X_η and Y_η are smooth of the same dimension over S, this is enough to guarantee that $f|_{U_\eta}$ is étale. □

Lemma 3.1.6. *Suppose that f is closed and η-étale. There exists an open neighborhood W of Y_s in Y such that $f : f^{-1}(W)_\eta \longrightarrow W_\eta$ is étale.*

Proof. Let $Z \subseteq X_\eta$ be the locus where f_η is not étale. Then Z is closed in X. Since f is a closed, $f(Z)$ is closed in Y, and of course $f(Z) \cap Y_s = \varnothing$. Take $W = Y \setminus f(Z)$. □

We now consider the case $Y = \mathbf{A}^d_S$.

Lemma 3.1.7. *Suppose that $f' : X \longrightarrow \mathbf{A}^d_S$ is such that $(p^n\Omega^1_{f'})|_{X_s} = 0$ and that $y_1, \ldots, y_d \in \Gamma(X, \mathscr{O}_X)$ are polynomials in p^{n+1}-powers of elements of $\Gamma(X, \mathscr{O}_X)$. If $f = f' + (y_1, \ldots, y_d)$, then $(p^n\Omega^1_f)|_{X_s} = 0$ as well.*

Proof. Let $S_n = \operatorname{Spec} V/p^{n+1}V$, $X_n = X \times_S S_n$. The presentations

$$\mathscr{O}^d_X \xrightarrow{df'_i} \Omega^1_{X/S} \longrightarrow \Omega^1_{f'} \longrightarrow 0, \quad \mathscr{O}^d_X \xrightarrow{df_i} \Omega^1_{X/S} \longrightarrow \Omega^1_f \longrightarrow 0$$

give after base change to S_n the short exact sequences

$$\mathscr{O}^d_{X_n} \xrightarrow{df'_i} \Omega^1_{X_n/S_n} \longrightarrow \Omega^1_{f'}/p^{n+1} \longrightarrow 0, \quad \mathscr{O}^d_{X_n} \xrightarrow{df_i} \Omega^1_{X_n/S_n} \longrightarrow \Omega^1_f/p^{n+1} \longrightarrow 0.$$

By the assumption on the y_i, we have $dy_i \in p^{n+1}\Omega^1_{X/S}$, therefore the two maps $\mathscr{O}^d_{X_n} \longrightarrow \Omega^1_{X_n/S_n}$ above are the same. It follows that $\Omega^1_f/p^{n+1} \simeq \Omega^1_{f'}/p^{n+1}$. The assumption that $(p^n\Omega^1_{f'})|_{X_s} = 0$ means that $p^n\Omega^1_{f'}/\pi p^n\Omega^1_{f'} = 0$ for a uniformizer π of V. As $p = u\pi^e$ for a unit $u \in V$ and $e \geq 1$ an integer, we have $p^n\Omega^1_{f'}/p^{n+1}\Omega^1_{f'} = 0$. Since $\Omega^1_f/p^{n+1} \simeq \Omega^1_{f'}/p^{n+1}$, the same holds for Ω^1_f. We thus have $(p^n\Omega^1_f)|_{X_n} = 0$, hence $(p^n\Omega^1_f)|_{X_s} = 0$. □

Proposition 3.1.8. *Assume that X affine and flat over S, that X_η is smooth of relative dimension d over S, and that $f' : X \longrightarrow \mathbf{A}^d_S$ is η-étale. There exists an $f : X \longrightarrow \mathbf{A}^d_S$ which is η-étale and fiberwise finite over S.*

Proof. By Lemma 3.1.5, there exists an n such that $(p^n\Omega^1_{f'})|_{X_s} = 0$. Apply Proposition 3.1.3 to $x_i = f'_i$ and $N = p^{n+1}$, obtaining a fiberwise finite $f : X \longrightarrow \mathbf{A}^d_S$ which differs from f' by some polynomials in p^{n+1}-powers. Then f is η-étale by Lemma 3.1.7 and Lemma 3.1.5. □

3.2 Existence of $K(\pi, 1)$ neighborhoods

3.2.1 Charts over a trait

In the situation of §1.4, let $f : (X, \mathscr{M}_X) \longrightarrow (S, \mathscr{M}_S)$ be a log smooth morphism. Applying Theorem 2.2.1 to a chart $\pi : (S, \mathscr{M}_S) \longrightarrow \mathbf{A}_{\mathbf{N}}$ given by a uniformizer π of V, we conclude that, étale locally on X, there exists a strict étale $g:(X,\mathcal{M}_X)\longrightarrow \mathbf{A}_{P,\rho,\pi}$ where

$$\mathbf{A}_{P,\rho,\pi}=\operatorname{Spec}\left(P\longrightarrow \frac{V[P]}{(\pi-\rho)}\right) \tag{3.1}$$

for a fine monoid P and a non-invertible element $\rho\in P$ with the property that $(P/\rho)^{\mathrm{gp}}$ is p-torsion free.

Assume that X_η is smooth over η. Localizing P, we can assume that the scheme underlying $(\mathbf{A}_{P,\rho,\pi})_\eta$ is smooth over η as well. But $(\mathbf{A}_{P,\rho,\pi})_\eta$ is isomorphic to $\operatorname{Spec}(P/\rho\longrightarrow K[P/\rho])$. Note that for a fine monoid M, $\operatorname{Spec}K[M]$ is smooth over K if and only if $\overline{M}$ is a free monoid. It follows that $\overline{P}/\rho$ is free, and therefore the stalks of $\mathcal{M}_{X/S}:=\mathcal{M}_X/f^\flat\mathcal{M}_S$ are free monoids. Moreover, every geometric point $\overline{x}$ of X has an étale neighborhood U such that $(U_\eta,\mathcal{M}_X|_{U_\eta})_{\mathrm{tr}}$ is the complement of a divisor with strict normal crossings on U_η.

3.2.2 Proof of the main theorem

Theorem 3.2.1. *Assume that* $\operatorname{char}k=p>0$. *Let* $(X,\mathcal{M}_X)$ *be a log smooth log scheme over* $(S,\mathcal{M}_S)$ *such that* X_η *is smooth over* η, *and let* $\overline{x}\longrightarrow X$ *be a geometric point. There exists an étale neighborhood* U *of* $\overline{x}$ *such that* U_η *is a* $K(\pi,1)$.

Proof. If $\overline{x}$ is contained in X_η, the existence of such a neighborhood follows from Theorem 2.1.9. We are therefore going to restrict ourselves to the case where $\overline{x}$ is a geometric point of the closed fiber X_s. The question being étale local around X, we are allowed to shrink X around $\overline{x}$ if needed.

Let π be a uniformizer of V, inducing a chart $(S,\mathcal{M}_S)\longrightarrow \mathbf{A}_{\mathbf{N}}$. By the discussion of 3.2.1, in an étale neighborhood of $\overline{x}$ there exists a fine monoid P, an element $\rho\in P$ such that $\overline{P/\rho}$ is a free monoid, and an étale map

$$g:X\longrightarrow \mathbf{A}_{P,\rho,\pi}=\operatorname{Spec}\left(P\longrightarrow \frac{V[P]}{(\pi-\rho)}\right)$$

over S.

We can replace X by an étale neighborhood of $\overline{x}$ for which the above data exist. Shrinking X further, we can also assume that X is affine.

Let us denote by $P[\rho^{-1}]$ the submonoid of P^{gp} generated by P and ρ^{-1}, and by P/ρ the quotient of $P[\rho^{-1}]$ by the subgroup generated by ρ. Since $\overline{P[\rho^{-1}]} = \overline{P/\rho}$ is free, there is an isomorphism $P[\rho^{-1}] \simeq P[\rho^{-1}]^* \oplus \overline{P[\rho^{-1}]}$. Picking an isomorphism $\overline{P[\rho^{-1}]} \simeq \mathbf{N}^b$ and a decomposition of $P[\rho^{-1}]^*$, we can write $P[\rho^{-1}] \simeq T \oplus \mathbf{Z} \oplus \mathbf{Z}^a \oplus \mathbf{N}^b$ where T is a finite abelian group and ρ corresponds to an element of the $\mathbf{Z}$ summand. Dividing by ρ, we obtain an isomorphism $P/\rho \simeq T \oplus \mathbf{Z}^a \oplus \mathbf{N}^b$. Let $d = a+b$, and let $\chi_0 : \mathbf{N}^d = \mathbf{N}^a \oplus \mathbf{N}^b \longrightarrow T \oplus \mathbf{Z}^a \oplus \mathbf{N}^b \simeq P/\rho$ be the map implied by the notation. As the source of χ_0 is free and $P \longrightarrow P/\rho$ is surjective, we can choose a lift $\chi : \mathbf{N}^d \longrightarrow P$ of χ_0:

$$\begin{tikzcd} & \mathbf{N}^d \arrow[dl, dotted, "\chi"'] \arrow[d, "\chi_0"] \\ P \arrow[r, two heads] & P/\rho. \end{tikzcd}$$

Then χ induces a map $h : \mathbf{A}_{P,\rho,\pi} \longrightarrow \mathbf{A}^d_S$ over S.

I claim that h_η is étale. Note first that h_η is the pullback under $\pi : \eta \longrightarrow \mathbf{A}_{\mathbf{Z}}$ of the horizontal map in the diagram

$$\begin{tikzcd} \mathbf{A}_{P[\rho^{-1}]} \arrow[rr, "\mathbf{A}_{\rho\oplus\chi}"] \arrow[dr] & & \mathbf{A}_{\mathbf{Z}} \times \mathbf{A}_{\mathbf{N}^d} \arrow[dl] \\ & \mathbf{A}_{\mathbf{Z}} & \end{tikzcd}$$

We want to check that the horizontal map becomes étale after base change to $\mathbf{Q}$. Since the base is $\mathbf{A}_{\mathbf{Z}} = \mathbf{G}_m$ and the map is $\mathbf{G}_m$-equivariant, it suffices to check this on one fiber. If we set $\rho = 1$, the resulting map is none other than the map induced by

$$\mathbf{N}^d = \mathbf{N}^a \oplus \mathbf{N}^b \hookrightarrow \mathbf{Z}^a \oplus \mathbf{N}^b \hookrightarrow T \oplus \mathbf{Z}^a \oplus \mathbf{N}^b \simeq P/\rho = P[\rho^{-1}]/\rho,$$

which is étale after adjoining $1/\#T$.

Let $f' = h \circ g : X \longrightarrow \mathbf{A}^d_S$. This map is η-étale, therefore by Proposition 3.1.8, there exists a map $f : X \longrightarrow \mathbf{A}^d_S$ which is η-étale and fiberwise finite over S (hence quasi-finite). As f is quasi-finite, we can perform an étale localization at $\overline{x}$ and $\overline{y}$ which will make it finite. More precisely, we apply [Gro67, Théorème 18.12.1] (or [Sta14, Tag 02LK]) and conclude that there exists a commutative diagram

$$\begin{tikzcd} \overline{x} \arrow[r] \arrow[d] & U' \arrow[r] \arrow[d, "f"'] & X \arrow[d, "f"'] \\ \overline{y} \arrow[r] & W' \arrow[r] & \mathbf{A}^d_S \end{tikzcd}$$

with $U' \longrightarrow X'$ and $W' \longrightarrow \mathbf{A}^d_S$ étale and $f : U' \longrightarrow W'$ finite. It follows that $f : U' \longrightarrow W'$ is also η-étale.

By Lemma 3.1.6 applied to $f : U' \longrightarrow W'$, we can shrink W' around $\overline{y}$ (and shrink U' accordingly to be the preimage of the new W') so that $U'_\eta \longrightarrow W'_\eta$ is finite étale.

Since W' is smooth over S, by Faltings' theorem (2.1.10) there is an open neighborhood W of $\overline{y}$ in W' such that W_η is a $K(\pi,1)$. Let U be the preimage of W in U' under $f : U' \longrightarrow W'$. The induced map $f_\eta : U_\eta \longrightarrow W_\eta$ is finite étale, hence U_η is a $K(\pi,1)$ as well by Proposition 2.1.8(a). □

3.2.3 Relatively smooth log structures

A reader familiar with the notion of a relatively smooth log structure (cf. [NO10, Definition 3.6], [Ogu09]) might appreciate the fact that the above proof applies to relatively smooth X/S as well. Recall that we call $(X,\mathscr{F})/(S,\mathscr{M}_S)$ *relatively log smooth* if, étale locally on X, there exists a log smooth log structure $(X,\mathscr{M})/(S,\mathscr{M}_S)$ and an inclusion $\mathscr{F} \subseteq \mathscr{M}$ as a finitely generated sheaf of faces, for which the stalks of $\mathscr{M}/\mathscr{F}$ are free monoids. We can then apply Theorem 3.2.1 to $(X,\mathscr{M})$ instead of $(X,\mathscr{F})$.

Important examples of relatively log smooth X/S appear in the Gross–Siebert program in mirror symmetry [GS06, GS10, GS11] as so-called *toric degenerations*. Degenerations of Calabi–Yau hypersurfaces in toric varieties are instances of such. For example, the *Dwork families*

$$X = \operatorname{Proj} V[x_0,\ldots,x_n] \Big/ \left((n+1)x_0 \cdot \ldots \cdot x_n - \pi \cdot \sum_{i=0}^{n} x_i^{n+1} \right)$$

(with the standard compactifying log structure) are relatively log smooth over S if $n+1$ is invertible on S, but not log smooth for $n > 2$ ([Ogu09, Proposition 2.2]).

3.2.4 Obstacles in characteristic zero

The need for the positive residue characteristic assumption in our proof of Theorem 3.2.1 can be traced down to the application of Proposition 3.1.8: one can perform relative Noether normalization on an η-étale map $f' : X \longrightarrow \mathbf{A}^d_S$ without sacrificing η-étaleness. One might think that this is too crude and that one could replace that part with a Bertini-type argument. After all, we only need η-étaleness at one point! Unfortunately, this is bound to fail in characteristic zero even in the simplest example, that of a semistable curve:

Proposition 3.2.2 (cf. [Ste06]). *Let X be an open subset of $\operatorname{Spec} V[x,y]/(xy-\pi) \subseteq \mathbf{A}^2_S$ containing the point $P=(0,0)\in \mathbf{A}^2_k$ and let $f : X \longrightarrow \mathbf{A}^1_S$ be an S-morphism. If $f_\eta : X_\eta \longrightarrow \mathbf{A}^1_\eta$ is étale, then df is identically zero on one of the components of X_s. In particular, if $\operatorname{char} k = 0$, then f has to contract one of the components of X_s, hence is not quasi-finite.*

Proof. Let Z be the support of $\Omega^1_{X/\mathbf{A}^1_S}$. As f_η is étale, $Z \subseteq X_s$. On the other hand, the short exact sequence

$$\mathscr{O}_X^2 \xrightarrow{\begin{bmatrix} f_x & y \\ f_y & x \end{bmatrix}} \mathscr{O}_X^2 \xrightarrow{\begin{bmatrix} dx & dy \end{bmatrix}} \Omega^1_{X/\mathbf{A}^1_S} \longrightarrow 0$$

shows that Z is the set-theoretic intersection of two divisors in $\mathbf{A}^2_S$ (given by the equations $xy=\pi$ and $xf_x - yf_y = 0$), each of them passing through P, for if $g\cdot\Omega^1_{X/\mathbf{A}^1_S}=0$, then

$$\begin{bmatrix} f_x & y \\ f_y & x \end{bmatrix} \cdot C = \begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix}$$

for some matrix C, hence $g^2 = (xf_x - yf_y)\cdot\det(C) \in (xf_x - yf_y)$. Since $\mathbf{A}^2_S$ is regular, by the dimension theorem we know that each irreducible component of Z passing through P has to be of positive dimension. Therefore Z has to contain one of the components of X_s. $\square$

3.3 A counterexample

The following is an example of an X/S where X is *regular*, but for which $K(\pi,1)$ neighborhoods do not exist. We use the notation of 1.4.

Proposition 3.3.1. *Suppose the characteristic of k is 0. Let π be a uniformizer of V and let $X \subseteq \mathbf{A}^3_S$ be given by the equation $xy = z^2 - \pi$. Let $P = (0,0,0) \in X_k$ and let U be an open neighborhood of P in X. Then U_η is not a $K(\pi,1)$.*

Out proof of Proposition 3.3.1 relies on a number of claims (Remark 3.3.5) regarding the étale topology of rigid analytic spaces. Unfortunately, proofs for at least some of these claims seem to be missing from the literature. We provide a complete proof of a similar statement over $\mathbf{C}$ in §3.3.2 (cf. Proposition 3.3.7), using complex analytic rather than rigid analytic methods.

Remark 3.3.2. This is example not too surprising. Let us consider an analogous analytic family. It is easy to see that the Milnor fiber [Mil68] at 0 of the function

$$f : \mathbf{C}^3 \longrightarrow \mathbf{C}, \quad f(x,y,z) = z^2 - xy,$$

(which is $x^2+y^2+z^2$ after a change of variables) is homotopy equivalent to the 2-sphere. Moreover, its inclusion into $X \setminus f^{-1}(0)$ induces an isomorphism on π_2. It follows that for any open $U \subseteq \mathbf{C}^3$ containing 0, for some Milnor fiber the above isomorphism will factor through $\pi_2(U \setminus f^{-1}(0))$, therefore $U \setminus f^{-1}(0)$ cannot be $K(\pi,1)$. The proof of Proposition 3.3.1 given below is an algebraic analog of this observation.

On the other hand, for $f : \mathbf{C}^n \longrightarrow \mathbf{C}$ given by a monomial $x_1 \dots x_r$, the Milnor fiber at 0 is homotopy equivalent to a torus $(S^1)^r$, which is a $K(\pi,1)$. This explains why one should expect Theorems 3.2.1 and 3.4.1 to be true.

3.3.1 Proof via rigid geometry

Lemma 3.3.3. *There is an isomorphism $\mathbf{P}^1_{K(\sqrt{\pi})} \times \mathbf{P}^1_{K(\sqrt{\pi})} \setminus$ (diagonal) $\longrightarrow X_{K(\sqrt{\pi})}$.*

Proof. Let the coordinates on $\mathbf{P}^1 \times \mathbf{P}^1$ be $((a:b),(c:d))$. The diagonal in $\mathbf{P}^1 \times \mathbf{P}^1$ is defined by the equation $\delta := ad - bc = 0$. Let $f((a:b),(c:d)) = \sqrt{\pi}(2ac/\delta, 2bd/\delta, (bc +$

$ad)/\delta)$. The inverse map is given by $g(x,y,z) = ((x : z-\sqrt{\pi}),(z-\sqrt{\pi} : y))$ on $\{z \neq \sqrt{\pi}\}$ and $g(x,y,z) = ((z+\sqrt{\pi} : y),(x : z+\sqrt{\pi}))$ on $\{z \neq -\sqrt{\pi}\}$. □

Let C be the completed algebraic closure of K. Denote by B the tube of P in X_C:

$$B = \{(x,y,z) : xy = z^2 - \pi, |x| < 1, |y| < 1, |z| < 1\} \subseteq X_C^{an},$$

which we treat as a rigid analytic space over C. Since $P \in U$, we have $B \subseteq U_C^{an}$. We denote by $p : B \longrightarrow \mathbf{P}_C^{1,an}$ the restriction to B of the composition of $g : X_C^{an} \longrightarrow \mathbf{P}_C^{1,an} \times \mathbf{P}_C^{1,an}$ defined in the proof of Lemma 3.3.3 with the first projection $\mathbf{P}_C^1 \times \mathbf{P}_C^1 \longrightarrow \mathbf{P}_C^1$. Explicitly,

$$p(x,y,z) = \begin{cases} (x : z - \sqrt{\pi}) & \text{if } z \neq \sqrt{\pi}, \\ (z + \sqrt{\pi} : y) & \text{if } z \neq -\sqrt{\pi}. \end{cases}$$

Our goal is to prove that $p : B \longrightarrow \mathbf{P}_C^{1,an}$ is a fibration in open discs. If we were working with manifolds, we would deduce that p is a homotopy equivalence which factors through U, hence $\mathbf{P}^1$ is a homotopy retract of U, and deduce that U is not a $K(\pi,1)$ (in the usual sense of algebraic topology), because $\pi_2(U)$ contains $\pi_2(\mathbf{P}^1) \simeq \mathbf{Z}$ as a direct summand. In the rigid analytic setting, we do not have such tools at our disposal, but it is enough to show that B is simply connected and that p is injective on the second étale cohomology groups (see Corollary 3.3.6 below).

Let $U_+ = \{(a : b) \in \mathbf{P}_C^{1,an} : |a| \leq |b|\}$, $U_- = \{(a : b) \in \mathbf{P}_C^{1,an} : |a| \geq |b|\}$. This is an admissible cover of $\mathbf{P}_C^{1,an}$. Let $B_+ = p^{-1}(U_+)$, $B_- = p^{-1}(U_-)$. Explicitly,

$$B_+ = \left\{(x,y,z) : xy = z^2 - \pi,\ |x| < 1,\ |y| < 1,\ |z| < 1,\ |x| \leq |z - \sqrt{\pi}|,\ |z + \sqrt{\pi}| \leq |y|\right\},$$
$$B_- = \left\{(x,y,z) : xy = z^2 - \pi,\ |x| < 1,\ |y| < 1,\ |z| < 1,\ |x| \geq |z - \sqrt{\pi}|,\ |z + \sqrt{\pi}| \geq |y|\right\}.$$

This is an admissible cover of B. Let $U_0 = U_+ \cap U_-$ and $B_0 = B_+ \cap B_- = p^{-1}(U_0)$.

Lemma 3.3.4. *There are isomorphisms*

$$\begin{aligned} & h_+ : U_+ \times \{x : |x| < 1\} \xrightarrow{\sim} B_+, \\ & h_- : U_- \times \{y : |y| < 1\} \xrightarrow{\sim} B_-, \\ \textit{and}\quad & h_0 : U_0 \times \{z : |z| < 1\} \xrightarrow{\sim} B_0, \end{aligned}$$

commuting with the projection p.

Proof. Let

$$h_+((a:b),x) = \left(x, \frac{a}{b}\left(\frac{a}{b}x + 2\sqrt{\pi}\right), \frac{a}{b}x + \sqrt{\pi}\right),$$
$$h_-((a:b),y) = \left(\frac{b}{a}\left(\frac{b}{a}y - 2\sqrt{\pi}\right), y, \frac{b}{a}y - \sqrt{\pi}\right),$$
$$h_0((a:b),z) = \left(\frac{b}{a}\left(z - \sqrt{\pi}\right), \frac{a}{b}\left(z + \sqrt{\pi}\right), z\right).$$

Their inverses are given by $p \times x$, $p \times y$, $p \times z$, respectively. □

Remark 3.3.5 (Étale topology of rigid analytic spaces)**.** We will need the following facts (under the char $k = 0$ assumption):

- the closed disc is simply connected [Ber93, 6.3.2],
- the open disc is simply connected (by the above and a limit argument),
- a product of (closed and open) discs has no nontrivial finite étale covers,
- the closed disc has trivial cohomology with $\mathbf{Z}/\ell$ coefficients ([Ber93, 6.1.3] combined with [Ber93, 6.3.2]),
- the open disc has trivial cohomology with $\mathbf{Z}/\ell$ coefficients (by [Ber93, 6.3.12] applied to the above),
- étale cohomology of rigid analytic spaces satisfies the Künneth formula.

Corollary 3.3.6. *Suppose that* char $k = 0$*. Then B is simply connected and the map*

$$p^* : H^2(\mathbf{P}^{1,an}_{C,\text{ét}}, \mathbf{Z}/\ell) \longrightarrow H^2(B_{\text{ét}}, \mathbf{Z}/\ell)$$

is an isomorphism for any prime number ℓ.

Proof. Since B_+ and B_- are both the product of an open disc and a closed disc and char $k = 0$, they are simply connected. Their intersection B_0 is connected, hence their union B is simply connected (consider the sheaf of sections of a covering $B' \longrightarrow B$).

Consider the two Mayer–Vietoris sequences for the étale cohomology of $\mathbf{Z}/\ell$ with respect to the two open covers $\{U_+, U_-\}$ and $\{B_+, B_-\}$. Since the latter cover is the pullback by p of the former, p induces a morphism of Mayer–Vietoris sequences

$$\begin{tikzcd}
\dots \arrow[r] & H^1(U_{+,\text{ét}},\mathbf{Z}/\ell)\oplus H^1(U_{-,\text{ét}},\mathbf{Z}/\ell) \arrow[r] \arrow[d, "p^*"] & H^1(U_{0,\text{ét}},\mathbf{Z}/\ell) \arrow[r] \arrow[d, "p^*"] & H^2(\mathbf{P}^{1,an}_{C,\text{ét}},\mathbf{Z}/\ell) \arrow[r] \arrow[d, "p^*"] & \dots \\
\dots \arrow[r] & H^1(B_{+,\text{ét}},\mathbf{Z}/\ell)\oplus H^1(B_{-,\text{ét}},\mathbf{Z}/\ell) \arrow[r] & H^1(B_{0,\text{ét}},\mathbf{Z}/\ell) \arrow[r] & H^2(B_{\text{ét}},\mathbf{Z}/\ell) \arrow[r] & \dots
\end{tikzcd}$$

Because $H^i(U_{\pm,\text{ét}},\mathbf{Z}/\ell) = H^i(B_{\pm,\text{ét}},\mathbf{Z}/\ell) = 0$ for $i = 1,2$, the above diagram gives us a commutative diagram whose horizontal maps are isomorphisms

$$\begin{tikzcd}
H^1(U_{0,\text{ét}},\mathbf{Z}/\ell) \arrow[r, "\sim"] \arrow[d, "p^*"] & H^2(\mathbf{P}^{1,an}_{C,\text{ét}},\mathbf{Z}/\ell) \arrow[d, "p^*"] \\
H^1(B_{0,\text{ét}},\mathbf{Z}/\ell) \arrow[r, "\sim"] & H^2(B_{\text{ét}},\mathbf{Z}/\ell).
\end{tikzcd}$$

But B_0 is the product of U_0 with a disc, hence the left vertical map is an isomorphism. We conclude that the right map is an isomorphism as well. □

We can now prove Proposition 3.3.1:

Proof of Proposition 3.3.1. By Proposition 2.1.8(b), it suffices to show that U_C is not a $K(\pi,1)$. Let $U' \longrightarrow U_C$ be a finite étale morphism and let $B' \longrightarrow B$ be the induced covering space of B. Pick a prime number ℓ and consider the diagram of pullback maps

$$\begin{tikzcd}
H^2(B'_{\text{ét}},\mathbf{Z}/\ell) & H^2(U'^{an}_{\text{ét}},\mathbf{Z}/\ell) \arrow[l] & \\
H^2(B_{\text{ét}},\mathbf{Z}/\ell) \arrow[u] & H^2(U^{an}_{\text{ét}},\mathbf{Z}/\ell) \arrow[l] \arrow[u] & H^2(\mathbf{P}^{1,an}_{C,\text{ét}},\mathbf{Z}/\ell). \arrow[l]
\end{tikzcd}$$

Because B is simply connected, B' is a disjoint union of copies of B, hence $H^2(B_{\text{ét}},\mathbf{Z}/\ell) \longrightarrow H^2(B'_{\text{ét}},\mathbf{Z}/\ell)$ is injective. Because $H^2(\mathbf{P}^{1,an}_{C,\text{ét}},\mathbf{Z}/\ell) \longrightarrow H^2(B_{\text{ét}},\mathbf{Z}/\ell)$ is an isomorphism, we conclude that $H^2(\mathbf{P}^{1,an}_{C,\text{ét}},\mathbf{Z}/\ell) \longrightarrow H^2(U'^{an}_{\text{ét}},\mathbf{Z}/\ell)$ is injective. As the spaces involved are smooth over C, by the rigid-étale comparison theorem [dJvdP96, Theorem 7.3.2], $H^2(\mathbf{P}^1_{C,\text{ét}},\mathbf{Z}/\ell) \longrightarrow H^2(U'_{\text{ét}},\mathbf{Z}/\ell)$ is also injective. This shows that U_C is not a $K(\pi,1)$, because we cannot kill the image of $H^2(\mathbf{P}^1_{C,\text{ét}},\mathbf{Z}/\ell) \simeq \mathbf{Z}/\ell$ in $H^2(U_{C,\text{ét}},\mathbf{Z}/\ell)$ by a finite étale $U' \longrightarrow U_C$. □

3.3.2 Proof via complex geometry

Proposition 3.3.7. *Let $k = \mathbf{C}$, $S_0 = \mathbf{A}^1_k$ with coordinate π, and let*

$$X_0 = \operatorname{Spec} k[\pi, x_0, \dots, x_n]/(\pi - f), \quad f = x_0^2 + \dots + x_n^2 \quad (n > 1).$$

Let S be the henselization of S_0 at 0, η its generic point, $\overline{\eta}$ a geometric point above η. Finally let $X = X_0 \times_{S_0} S$ and $\overline{x} = (0,0,\dots,0) \in X$. Then $(X_{(\overline{x})})_{\overline{\eta}}$ is not a $K(\pi,1)$. In particular, there does not exist a basis of étale neighborhoods of $\overline{x}$ in X whose generic fibers are $K(\pi,1)$. However, X is regular.

Proof. Note that $X_{(\overline{x})} = (X_0)_{(\overline{x})}$, $(X_{(\overline{x})})_\eta = (X_0)_{(\overline{x})} \setminus \{\pi = 0\}$. It is enough to show that

(1) $H^n((X_{(\overline{x})})_{\overline{\eta}}, \mathbf{Z}/\ell) \simeq \mathbf{Z}/\ell$,

(2) The scheme $(X_{(\overline{x})})_{\overline{\eta}}$ is simply connected.

Fact (1) follows from the computation of vanishing cycles [SGA73a, XV 2.2.5]. For (2), it suffices to prove that $(X_{(\overline{x})})_\eta \to \eta$ induces an isomorphism on fundamental groups, or equivalently on $H^1(-, G)$ for every finite group G. By the comparison theorem [SGA73b, XVI 4.1] applied to the inclusion $j : X_0 \setminus \{\pi = 0\} \hookrightarrow X_0$, $q = 1$ and a finite group G, we have

$$H^1((X_{(\overline{x})})_\eta, G) \simeq (R^1 j_{\text{ét}*} G)_{\overline{x}} \simeq (R^1 j_{cl*} G)_{\overline{x}} \simeq \varinjlim_{\varepsilon} H^1(B(\varepsilon) \setminus f^{-1}(0), G)$$

where $B(\varepsilon) = \{(x_0, \dots, x_n) \in \mathbf{C}^n : \sum |x_i|^2 < \varepsilon\}$. But by the Milnor fibration and bouquet theorems [Mil68] (see Example 1.1.2), the homotopy fiber of

$$f : B(\varepsilon) \longrightarrow D(\varepsilon) := \{z \in \mathbf{C} : |z| < \varepsilon\}$$

has type $\mathbf{S}^n$ and hence is simply connected as $n > 1$. The long exact sequence of homotopy groups of that fibration shows that $\pi_1(B(\varepsilon)) \simeq \pi_1(S^1)$, hence $H^1(B(\varepsilon) \setminus f^{-1}(0), G) \simeq H^1(S^1, G) \simeq H^1(D(\varepsilon) \setminus \{0\}, G)$. Because the diagram

$$\begin{tikzcd}
H^1((X_{(\overline{x})})_\eta, G) \arrow[r] & \varinjlim_\varepsilon H^1(B(\varepsilon) \setminus f^{-1}(0), G) \\
H^1(\eta, G) \arrow[u] \arrow[r] & \varinjlim_\varepsilon H^1(D(\varepsilon) \setminus \{0\}, G) \arrow[u]
\end{tikzcd}$$

commutes, we conclude that $H^1(\eta, G) \longrightarrow H^1((X_{(\overline{x})})_\eta, G)$ is an isomorphism as claimed. $\square$

3.4 The equicharacteristic zero case

Theorem 3.4.1. *Let $(X, \mathcal{M}_X)$ be a regular (cf. [Kat94, Definition 2.1]) log scheme over* $\mathbf{Q}$ *such that X is locally Noetherian, and let $\overline{x} \longrightarrow X$ be a geometric point. Let $\mathscr{F}$ be a locally constant constructible abelian sheaf on $X^\circ := (X, \mathcal{M}_X)_{\mathrm{tr}}$, the biggest open subset on which $\mathcal{M}_X$ is trivial, and let $\zeta \in H^q(X^\circ, \mathscr{F})$ for some $q > 0$. There exists an étale neighborhood U of $\overline{x}$ and a finite étale surjective map $V \longrightarrow U^\circ$ such that ζ maps to zero in $H^q(V, \mathscr{F})$.*

Proof. In proving the assertion, I claim that we can assume that $\mathscr{F}$ is constant. Let $Y \longrightarrow X^\circ$ be a finite étale surjective map such that the pullback of $\mathscr{F}$ to Y is constant. By the logarithmic version of Abhyankar's lemma [GR04, Theorem 10.3.43], $Y = X'^\circ$ for a finite and log étale $f : (X', \mathcal{M}_{X'}) \longrightarrow (X, \mathcal{M}_X)$. Then $(X', \mathcal{M}_{X'})$ is also log regular (by [Kat94, Theorem 8.2]). Choose a geometric point $\overline{x}' \longrightarrow X'$ mapping to $\overline{x}$, and let $\mathscr{F}' = f^{\circ *}\mathscr{F}$, which is a constant sheaf on X'°.

Suppose that we found an étale neighborhood U' of x' and a finite étale surjective map $V' \longrightarrow U'^\circ$ killing $\zeta' := f^{\circ *}(\zeta) \in H^q(X'^\circ, \mathscr{F}')$. Let X'' be the normalization of U' in V', and choose a geometric point $\overline{x}''$ mapping to $\overline{x}'$. By [Gro67, Théorème 18.12.1] (or [Sta14, Tag 02LK]), there exists a diagram

$$\begin{tikzcd} \overline{x}'' \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & X'' \arrow[d] \\ \overline{x} \arrow[r] & U \arrow[r] & X \end{tikzcd}$$

with $U \longrightarrow X$ and $V \longrightarrow X''$ étale and $V \longrightarrow U$ finite. It follows that $V^\circ \longrightarrow U^\circ$ is also étale, and that the pullback of ζ to V° is zero.

In proving the theorem, we can therefore assume that $\mathscr{F} \simeq \mathbf{Z}/n\mathbf{Z}$ for some integer n, by considering the direct summands.

The question being étale local around $\overline{x}$, we can assume that there exists a chart $g : (X, \mathcal{M}_X) \longrightarrow \mathbf{A}_P$ for a fine saturated monoid P, which we use to form a cartesian diagram

$$\begin{tikzcd} (X', \mathcal{M}_{X'}) \arrow[r, "f"] \arrow[d, "g'"'] & (X, \mathcal{M}_X) \arrow[d, "g"] \\ \mathbf{A}_P \arrow[r, "\mathbf{A}_{\cdot n}"'] & \mathbf{A}_P. \end{tikzcd} \tag{3.2}$$

Then $(X',\mathscr{M}_{X'})$ is log regular, f is finite, and $f: X'^\circ \longrightarrow X^\circ$ is étale. Choose a geometric point $\overline{x}' \longrightarrow X'$ mapping to $\overline{x}$. We have a commutative diagram

$$\begin{tikzcd} \overline{\mathscr{M}}^{\mathrm{gp}}_{X,\overline{x}} \arrow[r, "f^*"] & \overline{\mathscr{M}}^{\mathrm{gp}}_{X',\overline{x}'} \\ \overline{P} \arrow[u] \arrow[r, "\cdot n"'] & \overline{P} \arrow[u] \end{tikzcd}$$

where the vertical maps are surjections induced by the strict morphisms g and g'. We conclude that the map

$$f^\flat \otimes \mathbf{Z}/n\mathbf{Z} : \overline{\mathscr{M}}^{\mathrm{gp}}_{X,\overline{x}} \otimes \mathbf{Z}/n\mathbf{Z} \longrightarrow \overline{\mathscr{M}}^{\mathrm{gp}}_{X'\overline{x}'} \otimes \mathbf{Z}/n\mathbf{Z} \tag{3.3}$$

is zero.

Denote the inclusion $X^\circ \hookrightarrow X$ (resp. $X'^\circ \hookrightarrow X'$) by j (resp. j'). By log absolute cohomological purity (2.3), there is a functorial isomorphism

$$R^q j_*(\mathbf{Z}/n\mathbf{Z}) \simeq \bigwedge^q(\overline{\mathscr{M}}^{\mathrm{gp}}_X \otimes \mathbf{Z}/n\mathbf{Z}(-1)).$$

In our situation, this means that there is a commutative diagram

$$\begin{tikzcd} H^q(X'^\circ,\mathbf{Z}/n\mathbf{Z}) \arrow[r, "\mathrm{sp}_{\overline{x}'}"] & R^q j'_*(\mathbf{Z}/n\mathbf{Z})_{\overline{x}'} \arrow[r, "\sim"] & \bigwedge^q(\overline{\mathscr{M}}^{\mathrm{gp}}_{X',\overline{x}'} \otimes \mathbf{Z}/n\mathbf{Z}) \\ H^q(X^\circ,\mathbf{Z}/n\mathbf{Z}) \arrow[u, "f^*"] \arrow[r, "\mathrm{sp}_{\overline{x}}"'] & R^q j_*(\mathbf{Z}/n\mathbf{Z})_{\overline{x}} \arrow[u, "f^*"] \arrow[r, "\sim"] & \bigwedge^q(\overline{\mathscr{M}}^{\mathrm{gp}}_{X,\overline{x}} \otimes \mathbf{Z}/n\mathbf{Z}) \arrow[u, "\bigwedge^q (3.3)"'] \end{tikzcd}$$

where the rightmost map is zero for $q > 0$ because (3.3) is zero.

It follows that ζ maps to zero in $R^q j'_*(\mathbf{Z}/n\mathbf{Z})_{\overline{x}'}$, hence there exists an étale neighborhood U' of $\overline{x}'$ such that ζ maps to zero in $H^q(U'^\circ,\mathbf{Z}/n\mathbf{Z})$. Applying once again the argument of the second paragraph yields an étale neighborhood U of $\overline{x}$ and a finite étale map $V \longrightarrow U$ killing ζ, as desired. □

3.5 Abhyankar's lemma and extension of lcc sheaves

In the proof of Theorem 3.6.5, we will need the following standard argument (Proposition 3.5.2).

Proposition 3.5.1. *Let X be an integral normal scheme, $j : U \longrightarrow X$ an open immersion, $\mathscr{F}$ an lcc étale sheaf on U. The following conditions are equivalent.*

(i) The sheaf $j_*\mathscr{F}$ is locally constant.

(ii) There exists an lcc étale sheaf $\mathscr{F}'$ on X and an isomorphism $j^*\mathscr{F}' \simeq \mathscr{F}$.

(iii) There exists a finite group G and a G-torsor $f : V \longrightarrow U$ over U such that $f^*\mathscr{F}$ is constant and the normalization of X in V is étale over X.

Proof. This is clear if U is empty. Assume U non-empty and pick a geometric point $\overline{x}$ of U. Clearly (i) implies (ii). For the converse, note that the isomorphism $j^*\mathscr{F}' \longrightarrow \mathscr{F}$ defines by adjunction a map $\mathscr{F}' \longrightarrow j_*\mathscr{F}$ of lcc sheaves which induces an isomorphism on stalks at $\overline{x}$, and hence an is isomorphism. We prove that (ii) implies (iii). Because $\mathscr{F}'$ is lcc, there exists a finite group G and a G-torsor $g : Y \longrightarrow X$ such that $g^*\mathscr{F}'$ is constant. Then the restriction $f = g|_V : V \longrightarrow U$ of g to $V = g^{-1}(U)$ is a G-torsor such that $f^*\mathscr{F} = g^*\mathscr{F}'|_V$ is constant. Moreover, since U is dense in X, Y is normal, and g is finite, Y coincides with the normalization of X in V. Finally, we deduce (ii) from (iii). Let $g : Y \longrightarrow X$ be the normalization of X in V. Then the action of G on V extends to an action on Y, making $g : Y \longrightarrow X$ a G-torsor. If $M = \mathscr{F}_{\overline{x}}$, then $\mathscr{F}$ is represented by the quotient of $V \times_U M$ by the diagonal action of G. We can then define $\mathscr{F}'$ to be the sheaf represented by the analogous quotient of $Y \times_X M$. □

Proposition 3.5.2. *Let X be a regular integral scheme, $D = D_1 \cup \ldots \cup D_r \subseteq X$ a divisor with simple normal crossings, $U = X \setminus D$ its complement, $\mathscr{F}$ an lcc abelian sheaf on U, tamely ramified along D. There exists an integer n such that for every finite morphism $g : Y \longrightarrow X$, such that the restriction $f = g|_V : V \longrightarrow U$ to $V = f^{-1}(U)$ is étale, tamely ramified along D and with ramification indices along the D_i divisible by n, the pullback $f^*\mathscr{F}$ extends to an lcc sheaf on Y.*

Proof. Let η_i be the generic point of D_i, $\overline{\eta}_i$ a geometric point over η_i. For each i, the sheaf $\mathscr{F}$ defines a representation $M_i = \mathscr{F}_{\overline{\eta}_i}$ of the Galois group $G_i = \mathrm{Gal}(\overline{\eta}_i/\eta_i)$. By assumption, there is an open subgroup $I_i' \subseteq I_i$ of the inertia group, containing the wild inertia subgroup, acting trivially on M_i. Let n_i denote its index, and let $n = \mathrm{lcm}(n_1, \ldots, n_r)$.

Let $g : Y \longrightarrow X$ be as described, let $v : V \longrightarrow Y$ be the inclusion, and let $\mathscr{G} = f^*\mathscr{F}$. By Proposition 3.5.1, we only need to prove that $v_*\mathscr{G}$ is lcc. This question is local on Y. Let $\overline{y}$ be a geometric point of Y, $\overline{x} = f(\overline{y})$. Let X' be the strict henselization of X at $\overline{x}$, $D_i' = D_i \times_X X'$, $U' = U \times_X X'$, $\mathscr{F}'$ the preimage of $\mathscr{F}$ on X', $g' : Y' = Y \times_X X' \longrightarrow X'$ the pullback of g, $f' : V' = V \times_X X' \longrightarrow U'$. Since g is quasi-finite, Y' coincides with the strict henselization of Y at $\overline{y}$. The preimage of $\mathscr{G}$ in Y' coincides with $\mathscr{G}' = (f')^*\mathscr{F}'$. By Abhyankar's lemma [SGA03, Exp. XIII, Appendice I, Proposition 5.2, Corollaire 5.3], there exist equations $f_i \in \Gamma(X', \mathscr{O}_{X'})$ for D_i' and integers m_i invertible on X such that $Y' \simeq \operatorname{Spec} \mathscr{O}_{X'}[T_i]/(T_i^{m_i} - f_i)$. By assumption, each m_i is divisible by n, and hence by n_i. By definition of the n_i, the pullback of $\mathscr{F}'$ to $Z' = \operatorname{Spec} \mathscr{O}_{X'}[T_i]/(T_i^{n_i} - f_i)$ extends along the D_i, and hence to all of Z' by Zariski–Nagata purity. Since $Y' \longrightarrow X'$ factors through $Z' \longrightarrow X'$, $\mathscr{G}'$ extends to an lcc sheaf on Y'. □

3.6 The comparison theorem

In [AG11], Abbes and Gros have developed a theory of generalized co-vanishing topoi, of which the Faltings' topos is a special case. This topos has first been introduced in [Fal02], though the definition of [AG11] is different. For reader's convenience, let us recall the definitions, adapting them to our setup.

Definition 3.6.1. Let $f : Y \longrightarrow X$ be a morphism of schemes.

(a) The *Faltings' site* E associated to f is the site with

- OBJECTS morphisms $V \longrightarrow U$ over $f : Y \longrightarrow X$ with $U \longrightarrow X$ étale and $V \longrightarrow U \times_X Y$ finite étale,
- MORPHISMS commutative squares over $f : Y \longrightarrow X$,
- TOPOLOGY generated by coverings of the following form:
  - (V, for vertical) $\{(V_i \longrightarrow U) \longrightarrow (V \longrightarrow U)\}$ with $\{V_i \longrightarrow V\}$ a covering,
  - (C, for cartesian) $\{(V \times_U U_i \longrightarrow U_i) \longrightarrow (V \longrightarrow U)\}$ with $\{U_i \longrightarrow U\}$ a covering.

(b) The *Faltings' topos* $\widetilde{E}$ is the topos associated to E.

(c) We denote by $\Psi : Y_{\text{ét}} \longrightarrow \widetilde{E}$ the morphism of topoi induced by the continuous map of sites $(V \longrightarrow U) \mapsto V : E \longrightarrow \acute{E}t_{/Y}$.

Lemma 3.6.2. *If* $\dim H^0(X_{\text{ét}}, \mathbf{F}_\ell)$ *is finite then* X *has a finite number of connected components.*

Proof. The projection $\alpha : X_{\text{ét}} \to X$ induces an injection $\mathbf{F}_\ell \to \alpha_* \mathbf{F}_\ell$, so $H^0(X, \mathbf{F}_\ell)$ is finite as well. Since the map sending a clopen subset to its characteristic function is injective, the set of clopen subsets of X must be finite. If x is a point in X, let U_x be the intersection of all the clopen sets containing x. Then U_x is open and closed. Furthermore it is connected, so it must be the connected component of X containing x. Hence every connected component of X is clopen, and X has finitely many connected components. □

Proposition 3.6.3. *In the notation of 1.4, let* X *be a scheme of finite type over* S*. Let* $X^\circ \subseteq X$ *be an open subset such that the inclusion* $u : X^\circ \hookrightarrow X$ *is an affine morphism, and let* $Y = X^\circ_{\overline{\eta}}$*. Finally, let* $\wp$ *be a set of prime numbers. The following conditions are equivalent:*

(a) for every étale U *over* X *and every* $\wp$*-torsion locally constant constructible abelian sheaf* $\mathscr{F}$ *on* $U \times_X Y$*, we have* $R^i \Psi_{U*} \mathscr{F} = 0$ *for* $i > 0$*, where* $\Psi_U : U \times_X Y \longrightarrow \widetilde{E}_U$ *is the morphism 3.6.1(c) for* $U \times_X Y \longrightarrow U$*,*

(b) for every étale U *over* X*, every* $\wp$*-torsion locally constant constructible abelian sheaf* $\mathscr{F}$ *on* $U \times_X Y$*, every class* $\zeta \in H^i(U \times_X Y, \mathscr{F})$ *with* $i > 0$*, and every geometric point* $\overline{x} \longrightarrow U$*, there exists an étale neighborhood* U' *of* $\overline{x}$ *in* U *and a finite étale surjective map* $V \longrightarrow U \times_X Y$ *such that the image of* ζ *in* $H^i(V, \mathscr{F})$ *is zero.*

(c) for every geometric point $\overline{x} \longrightarrow X$*,* $(X_{(\overline{x})} \times_{S_{(f(\overline{x}))}} \overline{\eta}) \times_X X^\circ$ *is a* $K(\pi, 1)$ *for* $\wp$*-adic coefficients.*

Proof. The equivalence of (a) and (b) follows from the fact that $R^i \Psi_* \mathscr{F}$ is the sheaf associated to the presheaf $(V \longrightarrow U) \mapsto H^i(V, \mathscr{F})$ on E and the argument in Proposition 2.1.8(a). Note that $(X_{(\overline{x})} \times_{S_{(f(\overline{x}))}} \overline{\eta}) \times_X X^\circ$ is affine, therefore coherent. In case $\overline{x} \in X_s$, the finiteness of the number connected components follows from Lemma 3.6.2 and the finiteness of $\Gamma((X_{(\overline{x})} \times_{S_{(f(\overline{x}))}} \overline{\eta}) \times_X X^\circ, \mathbf{F}_\ell)$, which is the stalk at $\overline{x}$ of the 0-th nearby cycle functor of $u_* \mathbf{F}_\ell$ [Del77, Th. finitude 3.2]. If $\overline{x} \in X_\eta$, this follows similarly from the constructibility of $u_*\mathbf{F}_\ell$. The equivalence of (b) and (c) is then clear in the view of Proposition 2.1.8. $\square$

Corollary 3.6.4. *Suppose that X has a basis for the étale topology consisting of U for which $U \times_X Y$ is a $K(\pi,1)$ for $\wp$-adic coefficients. Then the conditions of Proposition 3.6.3 are satisfied.*

Theorem 3.6.5. *Let $(X,\mathscr{M}_X)$ be a log smooth log scheme over $(S,\mathscr{M}_S)$ such that X_η is smooth over η, and let $X^\circ = (X,\mathscr{M}_X)_{\mathrm{tr}}$. If $\mathrm{char}\, k = 0$, assume moreover that $(X,\mathscr{M}_X)$ is saturated. Then for every geometric point $\overline{x}$ of X, $(X_{(\overline{x})} \times_{S_{(f(\overline{x}))}} \overline{\eta}) \times_X X^\circ$ is a $K(\pi,1)$.*

Proof. We should first note that $(X_{(\overline{x})} \times_{S_{(f(\overline{x}))}} \overline{\eta}) \times_X X^\circ$ satisfies condition 2.1 by the argument used in the proof of Proposition 3.6.3.

In case $\mathrm{char}\, k = 0$, as $(X,\mathscr{M}_X)$ is regular by [Kat94, Theorem 8.2], Theorem 3.4.1 implies condition (b) of Proposition 3.6.3, hence $X_{(\overline{x})} \times_X X^\circ_\eta$ is a $K(\pi,1)$ (note that $X^\circ \subseteq X_\eta$). As $(X_{(\overline{x})} \times_{S_{(f(\overline{x}))}} \overline{\eta}) \times_X X^\circ$ is a limit of finite étale covers of $X_{(\overline{x})} \times_X X^\circ_\eta$, it is a $K(\pi,1)$ as well.

We will now assume that $\mathrm{char}\, k = p > 0$ and follow [Fal88, Lemma II 2.3] (see also [Ols09, 5.10–5.11]). As before, if $\overline{x} \in X_\eta$, this follows from Theorem 3.4.1 applied to $X_{\overline{\eta}}$, so let us assume that $\overline{x} \in X_s$. For simplicity, we can also replace S by $S_{(f(\overline{x}))}$ and X by the suitable base change. By Theorem 3.2.1 and Corollary 3.6.4, we know that $Z := (X_{(\overline{x})})_{\overline{\eta}}$ is a $K(\pi,1)$. Since X_η is smooth, Z is regular and $Z^\circ = X_{(\overline{x})} \times_X X^\circ_{\overline{\eta}}$ is obtained from Z by removing divisor with strictly normal crossings $D = D_1 \cup \ldots \cup D_r$. Let $\mathscr{F}$ be a locally constant constructible abelian sheaf on Z°, and pick a $\zeta \in H^q(Z^\circ,\eta)$ $(q > 0)$. We want to construct a finite étale cover of Z° killing ζ.

By Proposition 3.5.2, there is an integer n such that if $f : Z' \longrightarrow Z$ is a finite cover with ramification indices along the D_i nonzero and divisible by n, then $f^{\circ *}\mathscr{F}$ extends to a locally constant constructible sheaf on Z'. I claim that we can choose Z' which is a $K(\pi,1)$. By the previous considerations, it suffices to find Z' equal to $(X'_{(\overline{x}')})_{\overline{\eta}}$ for some X'/S' satisfying the same assumptions as X. We can achieve this by choosing a chart $X \longrightarrow \mathbf{A}_P$ around $\overline{x}$ as before and taking a fiber product as in (3.2) (and $S' = \mathrm{Spec}\, V[\pi']/(\pi'^n - \pi)$).

Now that we can assume that $\mathscr{F} = j^*\mathscr{F}'$ where $\mathscr{F}'$ is locally constant constructible on Z and $j : Z^\circ \hookrightarrow Z$ is the inclusion, we choose a finite étale cover $g : Y \longrightarrow Z$, Galois with group G, for which $g^*\mathscr{F}'$ is constant.

Let $f : Z' \longrightarrow Z$ be a finite cover with ramification indices along the D_i nonzero and divisible by some integer n. I claim that for any $b \geq 0$, the base change map

$$f^* R^b j_* \mathscr{F} \longrightarrow R^b j'_*(f^{\circ *}\mathscr{F}) \tag{3.4}$$

is divisible by n^b. In case $\mathscr{F}$ is constant, this follows once again from logarithmic absolute cohomological purity (2.3), and in general can be checked étale locally, e.g. after pulling back to Y, where $\mathscr{F}$ becomes constant. Consider the Leray spectral sequence for j:

$$E_2^{a,b} = H^a(Z, R^b j_* \mathscr{F}) \quad \Rightarrow \quad H^{a+b}(Z^\circ, \mathscr{F}),$$

inducing an increasing filtration F^b on $H^q(Z^\circ, \mathscr{F})$. Let $b(\zeta)$ be the smallest $b \geq 0$ for which $\zeta \in F^b$. We prove the assertion by induction on $b(\zeta)$. If $b(\zeta) = 0$, then ζ is in the image of a $\zeta' \in H^q(Z, j_*\mathscr{F})$, and since $j_*\mathscr{F}$ is locally constant and Z is a $K(\pi, 1)$, we can kill ζ' by a finite étale cover of Z. For the induction step, let n be an integer annihilating $\mathscr{F}$, and pick a ramified cover $f : Z' \longrightarrow Z$ as in the previous paragraph, such that again $Z' = (X'_{(\bar{x}')})_{\overline{\eta}}$ for some X'/S' satisfying the assumptions of the theorem. Note that since (3.4) is divisible by n, it induces the zero map on $E_2^{a,b}$ for $b > 0$, hence $b(f^*\zeta) < b(\zeta)$ and we conclude by induction. $\square$

Corollary 3.6.6. *Let $(X, \mathscr{M}_X)$ be as in Theorem 3.6.5, and let $X^\circ = (X, \mathscr{M}_X)_{\mathrm{tr}}$. Consider the Faltings' topos $\widetilde{E}$ of $X^\circ_{\overline{\eta}} \longrightarrow X$ and the morphism of topoi*

$$\Psi : X^\circ_{\overline{\eta},\text{ét}} \longrightarrow \widetilde{E}.$$

Let $\mathscr{F}$ be a locally constant constructible abelian sheaf on $X^\circ_{\overline{\eta}}$. Then $R^q\Psi_*\mathscr{F} = 0$ for $q > 0$, and the natural maps (2.8)

$$\mu : H^q(\widetilde{E}, \Psi_*(\mathscr{F})) \longrightarrow H^q(X^\circ_{\overline{\eta},\text{ét}}, \mathscr{F})$$

are isomorphisms.

Chapter 4

Milnor fibers

This chapter grew out of the author's desire to sort out the rather confusing multitude of possible "Milnor fiber-like objects". Let S be a henselian trait, $f : X \longrightarrow S$ a scheme of finite type over S, and $\overline{x}$ a geometric point in the special fiber X_s (we preserve the notation of §1.4). Each of the following geometric objects might in a certain context be called the Milnor fiber of f at $\overline{x}$:

1. the scheme $M_{\overline{x}} = X_{(\overline{x})} \times_{S_{f(\overline{x})}} \overline{\eta}$, where $X_{(\overline{x})}$ is the localization of X at $\overline{x}$, the *algebraic Milnor fiber*,

2. the scheme $\hat{M}_{\overline{x}} = (\hat{X}_{(\overline{x})})_{\overline{\eta}}$, where $\hat{X}_{(\overline{x})} = \operatorname{Spec} \hat{\mathcal{O}}_{X,(\overline{x})}$ (completion of the local ring for the étale topology),

3. (if $S = \operatorname{Spec} \mathbf{C}\{t\}$) the *classical Milnor fiber* F_x (which itself has several definitions, cf. Theorem 4.1.5),

4. (if S is complete) the rigid analytic *tube* of $\overline{x}$:

$$B_x = \{y \in X(\hat{\overline{K}}) : \operatorname{sp}(y) = \overline{x}\},$$

considered as a rigid analytic space over $\hat{\overline{K}}$.

Our goal is to compare their fundamental groups and cohomology groups of local systems. The comparison between 1. and 3. follows quite easily from the comparison theorem [SGA73b, Exp. XVI Théoreme 4.1]; we take extra care to compare the homotopy types. The comparison of 1. and 2. (possible in characteristic zero only) uses resolution of singularities and log étale topology. We refrain from discussing the the comparison between 1. and 4., which is follows from the Gabber–Fujiwara theorem [Fuj95].

4.1 Preliminaries

4.1.1 Classical Milnor fibers

Let $X \subseteq \mathbf{C}^N$ be a locally closed analytic set, let $f : X \longrightarrow \mathbf{C}$ be a holomorphic function, and let $x \in X$ be a point with $f(x) = 0$. For $\varepsilon > 0$, we denote by $B_x(\varepsilon)$ (resp. $S_x(\varepsilon)$) the intersection of X with an open ball (resp. sphere) in the Euclidean metric on $\mathbf{C}^N$ with radius ε and center x. Let $X_0 = f^{-1}(0)$.

Definition 4.1.1. For a topological space Y, we define the *open cone*

$$C^\circ(Y) = Y \times [0,1)/Y \times \{0\}.$$

We call the point corresponding to $Y \times \{0\}$ the *vertex* and denote by $d : C^\circ(Y) \longrightarrow [0,1)$ the function "distance from the vertex" $(y,t) \mapsto t$.

Theorem 4.1.2 ("Conic structure lemma" [BV72, 3.2], [GM88, 1.4]). *There exists an $\varepsilon > 0$ and a homeomorphism of triples*

$$u : (B_x(\varepsilon), B_x(\varepsilon) \cap X_0, x) \xrightarrow{\sim} (C^\circ(S_x(\varepsilon)), C^\circ(S_x(\varepsilon) \cap X_0), \text{vertex})$$

with the additional property that for $y \in B_x(\varepsilon)$, $\operatorname{dist}(x,y) = \varepsilon \cdot d(u(y))$.

Remark 4.1.3. In the situation of the theorem, if $\varepsilon' < \varepsilon$, then the compatibility of the homeomorphism u with the distance function implies that the restriction of u to $B_x(\varepsilon')$ induces a homeomorphism of triples

$$u' : (B_x(\varepsilon'), B_x(\varepsilon') \cap X_0, x) \xrightarrow{\sim} (C^\circ(S_x(\varepsilon')), C^\circ(S_x(\varepsilon') \cap X_0), \text{vertex}).$$

In particular, the assertion of the theorem holds for any $\varepsilon' < \varepsilon$.

Consider the map $p : Y \longrightarrow X$ fitting inside a cartesian diagram

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[d] & X \arrow[d, "f"] \\ \mathbf{C} \arrow[r, "\exp"'] & \mathbf{C}. \end{tikzcd}$$

The map exp is a $\mathbf{Z}(1)$-torsor over $\mathbf{C}^* \subseteq \mathbf{C}$ (recall that $\mathbf{Z}(1) = 2\pi i\mathbf{Z}$), and hence the map p factors through a $\mathbf{Z}(1)$-torsor over $X \setminus X_0$. We will show that, for $\varepsilon \ll 1$, the pullback of $p : Y \longrightarrow X$ to $B_x(\varepsilon)$,

$$F_{x,\varepsilon} := Y \times_X B_x(\varepsilon)$$

does not depend on ε up to homeomorphism. Moreover, we will see that for $\varepsilon' < \varepsilon$, the inclusion $F_{x,\varepsilon'} \hookrightarrow F_{x,\varepsilon}$ is a homotopy equivalence. Choose ε and a homeomorphism u as in Theorem 4.1.2. Let $S = S_x(\varepsilon) \setminus X_0$, $F = Y \times_X S \longrightarrow S$ the induced $\mathbf{Z}(1)$-torsor. Then u induces a homeomorphism $u' : B_x(\varepsilon) \setminus X_0 \simeq S \times (0,1)$ satisfying

$$d(x,y) = \varepsilon \cdot (\text{second coordinate of } u'(y)).$$

Since $S \times (0,1) \longrightarrow S$ induces an isomorphism on fundamental groups, there exists an isomorphism (of $\mathbf{Z}(1)$-torsors on $B_x(\varepsilon) \setminus X_0$) between $F_{x,\varepsilon} \longrightarrow B_x(\varepsilon) \setminus X_0$ and the pullback of $F \times (0,1) \longrightarrow S \times (0,1)$ under u'. Under this isomorphism, $F_{x,\varepsilon'} \subseteq F_{x,\varepsilon}$ (for $\varepsilon' < \varepsilon$) is identified with $F \times (0, \varepsilon'/\varepsilon) \subseteq F \times (0,1)$.

Definition 4.1.4. We call the space $F_{x,\varepsilon}$ for $\varepsilon \ll 1$ defined above the *(classical) Milnor fiber* of f at x. It is independent of the choice of ε up to homeomorphism.

The following result relates our definition of the Milnor fiber to the more common definitions of [Mil68, Lê77, CMSS09].

Theorem 4.1.5. *(a) For $\eta \ll \varepsilon \ll 1$, the map*

$$f : \overline{B}_x(\varepsilon) \cap f^{-1}(D_\eta \setminus \{0\}) \longrightarrow D_\eta \setminus \{0\},$$

where $\overline{B}_x(\varepsilon)$ is the closed ball of radius ε and $D_\eta = \{z \in \mathbf{C} : |z| < \eta\}$, is a topological fibration, whose fiber is homotopy equivalent to $F_{x,\varepsilon}$.

(b) For $\varepsilon \ll 1$, the map

$$\arg(f) : S_x(\varepsilon) \setminus X_0 \xrightarrow{f} \mathbf{C}^* \xrightarrow{\arg} \mathbf{S}^1$$

is a topological fibration, whose fiber is homotopy equivalent to $F_{x,\varepsilon}$.

(c) The space $F_{x,\varepsilon}$ has the homotopy type of a finite CW complex.

Proof. The maps in (a) and (b) are fibrations with homotopy equivalent fibers by [Lê77]. To compare the fiber of (b) with $F_{x,\varepsilon}$, we first note that the fiber F of a fibration $X \longrightarrow \mathbf{S}^1$ is homotopy equivalent to the fiber product $X \times_{\mathbf{S}^1} \widetilde{\mathbf{S}}^1$, where $\widetilde{\mathbf{S}}^1 \longrightarrow \mathbf{S}^1$ is a universal cover of $\mathbf{S}^1$. The latter is isomorphic to the space $F = Y \times_X S$ defined in the preceding discussion, and we showed using the conic structure lemma that $F_{x,\varepsilon} \simeq F \times (0,1)$.

To show (c), we use the description (a) of the Milnor fiber. The map in (a) is a proper map whose fibers are compact manifolds with boundary, and hence have the homotopy type of finite CW complexes by [KS69, Theorem III]. □

4.1.2 Milnor fibers in the étale topology

Let $S = \operatorname{Spec} V$ be a henselian trait. We preserve the notation of §1.4. Let $f : X \longrightarrow S$ a morphism of finite type, and let $\overline{x}$ be a geometric point of X lying in the closed fiber.

Definition 4.1.6. We call the scheme $M_{\overline{x}} = X_{(\overline{x})} \times_{S_{f(\overline{x})}} \overline{\eta}$ the *algebraic Milnor fiber* of f at $\overline{x}$.

Proposition 4.1.7. *The scheme $M_{\overline{x}}$ satisfies condition 2.1.*

Proof. Since $M_{\overline{x}}$ is affine, it is coherent. The finiteness of the number connected components follows from Lemma 3.6.2 and the finiteness of $\Gamma((X_{(\overline{x})} \times_{S_{(f(\overline{x}))}} \overline{\eta}) \times_X X^\circ, \mathbf{F}_\ell)$, which is the stalk at $\overline{x}$ of the 0-th nearby cycle functor of $u_* \mathbf{F}_\ell$ [Del77, Th. Finitude 3.2]. □

In fact, we can show more:

Theorem 4.1.8. *Let X be a V-scheme of finite type, $\overline{x} \in X(k)$. Then the Milnor fiber $M_{\overline{x}}$ is a Noetherian scheme.*

Proof (Following an idea due to Will Sawin). Without loss of generality, we can assume that V is strictly henselian and that $X = \operatorname{Spec} A$ for a V-algebra A of finite type. The geometric point $\overline{x}$ corresponds to a surjective V-algebra homomorphism $A \longrightarrow k$ whose kernel we denote by $\mathfrak{m}$. Let B be the henselization of A at $\mathfrak{m}$, so that $X_{(\overline{x})} = \operatorname{Spec} B$ and $M_{\overline{x}} = \operatorname{Spec}(B \otimes_V \overline{K})$.

The proof is by induction on $n = \dim B - 1$. If $n = -1$, $M_{\overline{x}}$ is empty and there is nothing to prove. For the induction step, suppose that the assertion is true whenever $\dim B \leq n$. By Noether normalization, there exists a finite map $B' \longrightarrow B$ where B' is the henselization of $V[x_1, \ldots, x_n]$ at the ideal $\mathfrak{m}_V + (x_1, \ldots, x_n)$. Then $B \otimes_V \overline{K}$ is of finite type over $B' \otimes_V \overline{K}$, and hence is Noetherian if $B' \otimes_V \overline{K}$ is. We can therefore assume that $A = V[x_1, \ldots, x_n]$ and $\mathfrak{m} = \mathfrak{m}_V + (x_1, \ldots, x_n)$.

Let $f \in B \otimes_V \overline{K}$ be a nonzero element. By definition, there exists an étale A-algebra A' of finite type, an extension $A' \longrightarrow k$ of $\overline{x} : A \longrightarrow k$ (equivalently, an ideal $\mathfrak{m}'$ of A' whose intersection with A equals $\mathfrak{m}$), and a finite Galois extension L of K such that $f \in A' \otimes_V L$. Considering the L/K-norm $N_{L/K}(f) = f \cdot \prod_{\sigma \neq 1} \sigma(f)$, we see that there exists a $g \in A' \otimes_V L$ such that $0 \neq fg \in A' \otimes_V K$. Since $A' \otimes_V K = A'[\frac{1}{\pi}]$ where π is a generator of $\mathfrak{m}_V$, there exists an n such that $0 \neq fg\pi^n \in A'$. Let $h = fg\pi^n$, and let $A'' = A'/(h)$, and denote by B'' the henselization of A'' at the image of $\mathfrak{m}'$. Note that $B'' = B'/(h) = B/(h)$. Then $\dim B'' = n$, hence by the induction assumption $B'' \otimes_V \overline{K}$ is Noetherian. Since f divides h, $(B \otimes_V \overline{K})/(f)$ is a quotient of $B'' \otimes_V \overline{K}$, hence is Noetherian as well. We conclude that the ring $B \otimes_V \overline{K}$ has the property that its quotient by any nonzero principal ideal is Noetherian. By Lemma 4.1.9 below, $B \otimes_V \overline{K}$ is Noetherian, as desired. $\square$

Lemma 4.1.9. *Let R be a ring such that for every nonzero element $f \in R$, the ring $R/(f)$ is Noetherian. Then R is Noetherian.*

Proof. Let $I \subseteq R$ be an ideal. We wish to show that I is finitely generated. If $I = (0)$, there is nothing to show. If $I \neq (0)$, let $f \in I$ be any nonzero element. Then $R/(f)$ is Noetherian, hence the image of I in $R/(f)$ is generated by elements $x_1, \ldots, x_n \in R/(f)$. Let $y_i \in R$ be elements mapping to $x_i \in R/I$. Then $I = (f, y_1, \ldots, y_n)$. $\square$

4.1.3 The completed Milnor fibers

In the situation of the previous paragraph, suppose furthermore that the residue field k of V is algebraically closed, that V is complete, and that $\overline{x} \in X(k)$. We denote the schematic point underlying $\overline{x}$ by x.

Definition 4.1.10. We call the scheme $\hat{M}_x = \mathrm{Spec}(\hat{\mathcal{O}}_{X,x} \otimes_V \overline{K})$ the *completed Milnor fiber* of f at $\overline{x}$.

Note that $\hat{\mathcal{O}}_{X,x}$ is the completion of the local ring $\mathcal{O}_{X,x}$ at its maximal ideal, as opposed to the ideal $m_V \cdot \mathcal{O}_{X,x}$.

4.2 Algebraic vs completed Milnor fibers

The following definition provides an analogue of homotopy equivalence in algebraic geometry.

Definition 4.2.1. A morphism $f : X \longrightarrow Y$ of topoi is a $\natural$-*isomorphism* if for every locally constant sheaf $\mathscr{F}$ of finite sets (resp. finite groups, resp. finite abelian groups) on Y, the pullback map

$$f^* : H^q(Y, \mathscr{F}) \longrightarrow H^q(X, f^*\mathscr{F})$$

is an isomorphism for $q = 0$ (resp. for $q = 0, 1$, resp. for $q \geq 0$).

The goal of this section is to prove the following result.

Theorem 4.2.2. *Let A be a henselian Noetherian local ring over* $\mathbf{Q}$, $X = \mathrm{Spec}\, A$, *let $Z \subseteq X$ be a closed subscheme, $U = X \setminus Z$. Let $\hat{A}$ be the completion of A, $\hat{X} = \mathrm{Spec}\, \hat{A}$, $\hat{U} = U \times_X \hat{X}$. Then the natural map*

$$\hat{U}_{\text{ét}} \longrightarrow U_{\text{ét}}$$

is a $\natural$-isomorphism.

Theorem 4.2.3 (Proper Base Change). *Let X be a proper scheme over* $\mathrm{Spec}\, A$ *where A is a henselian local ring with residue field k and completion $\hat{A}$. Let $X_0 = X \otimes_A k$, $\hat{X} = X \otimes_A \hat{A}$, and let $\mathscr{X}$ be the formal completion of X along X_0. Then the three natural morphisms*

$$(X_0)_{\text{ét}} \longrightarrow \mathscr{X}_{\text{ét}} \longrightarrow \hat{X}_{\text{ét}} \longrightarrow X_{\text{ét}}$$

are $\natural$-isomorphisms.

Theorem 4.2.4 (Cohomological Descent, cf. [SV96, §10], [Bei12, §2])**.** *Let $X_\bullet \longrightarrow X$ be an h-hypercover. Then the induced map*

$$(X_\bullet)_{\text{ét}} \longrightarrow X_{\text{ét}}$$

is a $\natural$-isomorphism.

We will need some logarithmic versions of these statements. The "correct" analog of $X_{\text{ét}}$ for a fs log scheme $(X, \mathscr{M}_X)$ is the Kummer étale topos $(X, \mathscr{M}_X)_{\text{két}}$. See [Ill02a] for a survey of the Kummer étale topology.

Definition 4.2.5. (i) A morphism $f : (Y, \mathscr{M}_Y) \longrightarrow (X, \mathscr{M}_X)$ of fs log schemes is *Kummer étale* if it is étale and the cokernel of $f^*\overline{\mathscr{M}}_X^{\text{gp}} \longrightarrow \overline{\mathscr{M}}_Y^{\text{gp}}$ is torsion.

(ii) A family $\{f_i : (Y_i, \mathscr{M}_{Y_i}) \longrightarrow (X, \mathscr{M}_X)\}$ is a Kummer étale cover if the maps f_i are Kummer étale and jointly surjective.

(iii) The *Kummer étale topos* $(X, \mathscr{M}_X)_{\text{két}}$ is the topos associated to the site of Kummer étale fs log schemes over $(X, \mathscr{M}_X)$, endowed with the topology induced by the Kummer étale covers.

Theorem 4.2.6 (Log Purity [Fuj02], [Nak98, 2.0.1,2.0.5])**.** *Let $(X, \mathscr{M}_X)$ be a regular fs log scheme over $\mathbf{Q}$, and let $U = (X, \mathscr{M}_X)_{\text{tr}}$. Then the natural map*

$$U_{\text{ét}} = (U, \mathscr{M}_X|_U)_{\text{két}} \longrightarrow (X, \mathscr{M}_X)_{\text{két}}$$

is a $\natural$-isomorphism.

Theorem 4.2.7 (a variant of Log Proper Base Change [Ill02a, Proposition 6.3])**.** *Let X be a proper scheme over* Spec A *where A is a henselian local ring with residue field k and completion $\hat{A}$. Let $X_0 = X \otimes_A k$, $\hat{X} = X \otimes_A \hat{A}$, and let $\mathscr{X}$ be the formal completion of X along X_0. Let $\mathscr{M}_X \longrightarrow \mathscr{O}_X$ be a fs log structure on X. Then the three natural morphisms*

$$(X_0, \mathscr{M}_X|_{X_0})_{\text{két}} \longrightarrow (\mathscr{X}, \mathscr{M}_X|_{\mathscr{X}})_{\text{két}} \longrightarrow (\hat{X}, \mathscr{M}_X|_{\hat{X}})_{\text{két}} \longrightarrow (X, \mathscr{M}_X)_{\text{két}}$$

are $\natural$-isomorphisms.

Proof of Theorem 4.2.2. By Hironaka (or de Jong), there exists a proper hypercover $X_\bullet \longrightarrow X$ such that the X_n are regular and $Z_n = Z \times_X X_n$ are divisors with normal crossings. Let $U_n = X_n \setminus Z_n = U \times_X X_n$. Let $\hat{X}_n$, $\hat{Z}_n$, $\hat{U}_n$ denote the base changes $- \times_X \hat{X}$. By Theorem 4.2.4, the maps $(U_\bullet)_{\text{ét}} \longrightarrow U_{\text{ét}}$ and $(\hat{U}_\bullet)_{\text{ét}} \longrightarrow \hat{U}_{\text{ét}}$ are $\natural$-isomorphisms. Thus it suffices to show that the maps $(\hat{U}_n)_{\text{ét}} \longrightarrow (U_n)_{\text{ét}}$ are $\natural$-isomorphisms. Let x be the closed point of X. Endow each X_n with the compactifying log structure $\mathcal{M}_{U_n/X_n}$. We get a diagram

$$\begin{tikzcd}
(\hat{U}_n)_{\text{ét}} \arrow[r] \arrow[d] & (\hat{X}_n, \mathcal{M}_{\hat{X}_n})_{\text{két}} \arrow[d] & ((\hat{X}_n, \mathcal{M}_{\hat{X}_n}) \times_{\hat{X}} x)_{\text{két}} \arrow[l] \arrow[d, equal] \\
(U_n)_{\text{ét}} \arrow[r] & (X_n, \mathcal{M}_{X_n})_{\text{két}} & ((X_n, \mathcal{M}_{X_n}) \times_X x)_{\text{két}} \arrow[l]
\end{tikzcd}$$

in which the horizontal arrows are $\natural$-isomorphisms by Theorems 4.2.6 and 4.2.7. $\square$

For the following corollaries we use the notation of §1.4:

Theorem 4.2.8. *Assume that* $\operatorname{char} k = 0$ *and* k *is algebraically closed. Let* X *be an* S*-scheme of finite type, and let* $\overline{x} \in X(k)$*. Let* $\hat{X}_{(\overline{x})} = \operatorname{Spec} \hat{\mathcal{O}}_{X,x}$*. Then the natural maps*

$$(\hat{X}_{(\overline{x})})_{\eta,\text{ét}} \longrightarrow (X_{(\overline{x})})_{\eta,\text{ét}}$$

and

$$(\hat{M}_{\overline{x}})_{\text{ét}} \longrightarrow (M_{\overline{x}})_{\text{ét}}$$

are $\natural$*-isomorphisms.*

Theorem 4.2.9. *In the situation of Theorem 4.2.8, assume that* $V = \mathbf{C}\{t\}$ *and* $\overline{x} \in X(\mathbf{C})$*. Let* $\mathcal{O}^{\text{hol}}_{X,\overline{x}}$ *be the ring of germs of holomorphic functions at* $\overline{x}$ *and let* $X^{\text{hol}}_{(\overline{x})} = \operatorname{Spec} \mathcal{O}^{\text{hol}}_{X,\overline{x}}$*. Then the natural maps*

$$(\hat{X}_{(\overline{x})})_{\eta,\text{ét}} \longrightarrow (X^{\text{hol}}_{(\overline{x})})_{\eta,\text{ét}}$$

are $\natural$*-isomorphisms, so in particular* $(X^{\text{hol}}_{(\overline{x})})_{\eta,\text{ét}}$ *and* $(X_{(\overline{x})})_{\eta,\text{ét}}$ *are* $\natural$*-isomorphic.*

Proof. Apply Theorem 4.2.2 to both $A = \mathcal{O}^h_{X,\overline{x}}$ and $A = \mathcal{O}^{\text{hol}}_{X,\overline{x}}$, and note that $\hat{\mathcal{O}}^h_{X,\overline{x}} = \hat{\mathcal{O}}^{\text{hol}}_{X,\overline{x}}$. $\square$

4.3 Classical vs algebraic Milnor fibers

Let $S = \mathbf{A}^1_{\mathbf{C}} = \operatorname{Spec}\mathbf{C}[z]$, $S^* = S \setminus \{0\}$. For an integer $n \geq 1$, let $S^*_n = \mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$, considered as an S^*-scheme via the map $z \mapsto z^n$. If m divides n, the map $z \mapsto z^{n/m}$ defines a map of S^*-schemes $S^*_n \longrightarrow S^*_m$. Altogether, these define a diagram $S^*_\bullet : P \longrightarrow (\text{Schemes}/S^*)$ where P is the poset of positive integers ordered by divisibility. We denote by $\overline{\eta}$ its limit $\lim_n S^*_n = \operatorname{Spec}\mathbf{C}[z^{\mathbf{Q}}]$.

Let $\widetilde{S}^* = \mathbf{C}$ considered as a space over $S^*(\mathbf{C})$ via the map $\exp(z)$. For any $n \geq 1$, $z \mapsto \exp(z/n)$ defines a map $\widetilde{S}^* \longrightarrow S^*_n(\mathbf{C})$ of analytic spaces over $S^*(\mathbf{C})$. These maps assemble into a cone $\widetilde{S}^* \longrightarrow S^*_\bullet(\mathbf{C})$ over the diagram $S^*_\bullet(\mathbf{C}) : P \longrightarrow (\text{An. Spaces}/S^*(\mathbf{C}))$.

For an S-scheme (resp. an $S(\mathbf{C})$-analytic space) Y, we denote by Y^* the base change to S^* (resp. $S^*(\mathbf{C})$), and by Y^*_n (resp. $\widetilde{Y}^*$) the base change $Y \times_S S^*_n$ (resp. $Y \times_{S(\mathbf{C})} \widetilde{S}^*$). We denote by Y_0 the preimage of $0 \in S$.

Let X be a locally closed subscheme of $\mathbf{A}^N_S = \mathbf{A}^{N+1}_{\mathbf{C}}$, $x \in X_0(\mathbf{C})$ a point. For $\varepsilon > 0$, let $B(\varepsilon)$ be the intersection of $X(\mathbf{C})$ with the open ball in $\mathbf{C}^{N+1}$ of radius ε centered at x. Note that $F_{x,\varepsilon} = \widetilde{B}^*(\varepsilon)$ $(\varepsilon \ll 1)$ is the classical Milnor fiber (Definition 4.1.4). Similarly, $M_x = (X_{(x)})_{\overline{\eta}}$ is the algebraic Milnor fiber (Definition 4.1.6) of the base change of X to $S_{(0)}$ (the henselization of S at 0).

Our goal is to compare the homotopy types of $F_{x,\varepsilon}$ and its algebraic counterpart M_x. Recall the definition of the Verdier functor [AM69, §9]

$$\Pi : (\text{locally connected sites}) \longrightarrow \text{pro-}\mathscr{H}$$

where $\mathscr{H}$ is the homotopy category of CW complexes. In a locally connected site C, every object X is a coproduct of a well-defined and functorial set of connected objects. This defines the "connected components" functor $\pi : C \longrightarrow \text{Set}$. Thus if $X_\bullet$ is a hypercovering in C, the composition $\pi(X_\bullet)$ is a simplicial set, and we can consider the homotopy type of its geometric realization, which we also denote $\pi(X_\bullet)$. The hypercoverings of C form a co-filtering category $HR(C)$, and the functor Π associates to C the pro-object

$$\Pi(C) = \{\pi(X_\bullet)\}_{X_\bullet \in HR(C)}.$$

This definition is naturally extended to projective systems of locally connected sites:

$$\Pi(\{C_i\}_{i\in I}) := \{\pi(X_\bullet)\}_{i\in I, X_\bullet \in HR(C_i)}.$$

For a topological space (resp. a scheme) Y, we denote also by Y the associated site of local homeomorphisms (resp. étale morphisms) $U \longrightarrow Y$ with the usual topology. It is locally connected if Y is locally connected (resp. locally Noetherian). If Y is a CW complex, we have a canonical isomorphism $Y \simeq \Pi(Y)$.

In order to compare $\Pi(F_{x,\varepsilon})$ to $\Pi(M_x)$, we will consider the following projective systems of sites:

1. $\widetilde{C} = \{\widetilde{U}^*\}_{U \text{ cl}} \simeq \{\widetilde{B}^*(\varepsilon)\}_\varepsilon$,
2. $C_n = \{U_n^*\}_{U \text{ cl}} \simeq \{B_n^*(\varepsilon)\}_\varepsilon$,
3. $\hat{C} = \{C_n\}_n = \{U_n^*\}_{n,U \text{ cl}} \simeq \{B_n^*(\varepsilon)\}_{n,\varepsilon}$,
4. $C_{n,\text{alg}} = \{U_n^*\}_{U \text{ ét}}$,
5. $\hat{C}_{\text{alg}} = \{C_{n,\text{alg}}\} = \{U_n^*\}_{n,U \text{ ét}}$.

Here U cl (resp. U ét) stands for the system of all local homeomorphisms $U \longrightarrow X(\mathbf{C})$ (resp. étale maps $U \longrightarrow X$) with a chosen point mapping to x. The sites $F_{x,\varepsilon}$ and M_x are interconnected by the following maps:

1. $\alpha_\varepsilon : \widetilde{C} \longrightarrow F_{x,\varepsilon}$, the projection $\widetilde{C} \simeq \{tildeB^*(\varepsilon')\}_{\varepsilon'<\varepsilon} \longrightarrow \widetilde{B}^*(\varepsilon) = F_{x,\varepsilon}$,
2. $\beta : \widetilde{C} \longrightarrow \hat{C}$, induced by the maps $U \times_{S(\mathbf{C})} \exp(z/n) : \widetilde{U}^* \longrightarrow U_n^*$,
3. $\gamma : \hat{C} \longrightarrow \hat{C}_{\text{alg}}$, induced by the "change of topology" maps $\varepsilon : U_n^*(\mathbf{C}) \longrightarrow U_n^*$,
4. $\delta : M_x \longrightarrow \hat{C}_{\text{alg}}$ induced by the projection maps $(X_{(x)})_{\overline{\eta}} \longrightarrow U_n^*$.

Applying the Verdier functor Π, we get a chain of maps

$$\begin{tikzcd}
 & & \Pi(M_x) \arrow[d, "\delta"] \\
\Pi(\widetilde{C}) \arrow[r, "\beta"] \arrow[d, "\alpha_\varepsilon"'] & \Pi(\hat{C}) \arrow[r, "\gamma"] & \Pi(\hat{C}_{\text{alg}}) \\
\Pi(F_{x,\varepsilon}) & &
\end{tikzcd}$$

Theorem 4.3.1. *The maps β, γ, δ induce isomorphisms on profinite completions $\hat{\Pi}$. Moreover, the map $\alpha_\varepsilon : \Pi(\widetilde{C}) \longrightarrow \Pi(F_{x,\varepsilon}) \simeq F_{x,\varepsilon}$ is an isomorphism for $\varepsilon \ll 1$. Thus the profinite completions of the homotopy type of $F_{x,\varepsilon}$ and the étale homotopy type of M_x are canonically isomorphic for $\varepsilon \ll 1$. In particular, if X^* is normal in a neighborhood of x, we have*

$$\hat{F}_{x,\varepsilon} \xrightarrow{\sim} \Pi(M_x) \quad \textit{for } \varepsilon \ll 1.$$

We prove Theorem 4.3.1 at the end of this section. The main ingredient is the (relative) comparison theorem from SGA4 (Theorem 4.3.6 below).

Let $C = \{C_i\}_{i \in I}$ be a projective system of sites. If $i \in I$ and $\mathscr{F}$ is a sheaf on C_i the pullbacks of $\mathscr{F}$ to $C_{i'}$ via the maps induced by $i' \in I/i$ induce a projective system of sheaves $\{\mathscr{F}_{i'}\}_{i' \in I/i}$ over the system $\{C_{i'}\}_{i' \in I/i}$. As $I/i \longrightarrow I$ is cofinal, the latter system is naturally isomorphic to C. We define $H^q(C, \mathscr{F}) = \operatorname{colim}_{i' \longrightarrow i} H^q(C_{i'}, \mathscr{F}_{i'})$ for $q = 0$ (resp. $q = 0, 1$, resp. $q \geq 0$) if $\mathscr{F}$ is a sheaf of sets (resp. of groups, resp. of abelian groups).

Lemma 4.3.2. *Let $C = \{C_i\}_{i \in I}$ and $D = \{D_j\}_{j \in J}$ be projective systems of locally connected sites, and let $f : D \longrightarrow C$ be a morphism. Suppose that*

(1) there exists an integer $d \geq 0$ such that for every $i \in I$ (resp. $j \in J$) and every locally constant sheaf $\mathscr{F}$ of finite abelian groups on C_i (resp. D_j), we have $H^q(C_i, \mathscr{F}) = 0$ (resp. $H^q(D_j, \mathscr{F}) = 0$) for $q > d$,

(2) for every finite group G, the pullback map

$$H^1(C, G) \longrightarrow H^1(D, G)$$

is an isomorphism,

(3) for every locally constant sheaf of abelian groups $\mathscr{F}$ on C_i, $j \mapsto i$, inducing systems $\{\mathscr{F}_{i'}\}_{i' \in I/i}$ and $\{f^* \mathscr{F}_{j'}\}_{j' \in J/j}$, the pullback maps

$$H^q(C, \mathscr{F}) \longrightarrow H^q(D, f^* \mathscr{F})$$

are isomorphisms for $q \geq 0$.

Then the induced map $\hat{\Pi}(f) : \hat{\Pi}(D) \longrightarrow \hat{\Pi}(C)$ is an isomorphism.

Proof. This follows from [AM69, 4.3] and [AM69, 12.5]. □

Lemma 4.3.3. *Let Y be a topological space, $\widetilde{Y} \longrightarrow Y$ a $\mathbf{Z}$-covering space,*

$$Y_n = \widetilde{Y} \times_{\mathbf{Z}} (\mathbf{Z}/n\mathbf{Z}) \longrightarrow Y$$

the induced $\mathbf{Z}/n\mathbf{Z}$-covering space. Suppose that $\widetilde{Y}$ has the homotopy type of a finite CW complex. Then for every locally constant sheaf $\mathscr{F}$ of finite abelian groups on Y, the pullback maps

$$\varinjlim_n H^q(Y_n, \mathscr{F}) \longrightarrow H^q(\widetilde{Y}, \mathscr{F})$$

are isomorphisms for $q \geq 0$. Here for a Y space $f : Y' \longrightarrow Y$ we abbreviate $H^q(Y', f^*\mathscr{F})$ to $H^q(Y', \mathscr{F})$. The same statement holds for H^1 of locally constant sheaves of finite groups.

Proof. To avoid confusion, let $G = \mathbf{Z}$ denote the structure group of $\widetilde{Y}$, and let T be a generator of G. Let us treat the abelian case first. Note that in this case we have a spectral sequence

$$E_2^{p,q} = H^p(G, H^q(\widetilde{Y}, \mathscr{F})) \quad \Rightarrow \quad H^{p+q}(Y, \mathscr{F}).$$

But for a G-module M, we have $H^0(G, M) = M^G$, $H^1(G, M) = M_G = M/(1-T)M$, and $H^p(G, M) = 0$ for $p > 1$. Thus the spectral sequence reduces to short exact sequences

$$0 \longrightarrow H^{q-1}(\widetilde{Y}, \mathscr{F})_G \longrightarrow H^q(Y, \mathscr{F}) \longrightarrow H^q(\widetilde{Y}, \mathscr{F})^G \longrightarrow 0.$$

The same is true if we replace Y by Y_n and G by $G_n = n\mathbf{Z} \subseteq G$. Moreover, whenever m divides n, the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^{q-1}(\widetilde{Y}, \mathscr{F})_{G_m} \arrow[r] \arrow[d, "1+T^m+\ldots+T^{n-m}"'] & H^q(Y_m, \mathscr{F}) \arrow[r] \arrow[d] & H^q(\widetilde{Y}, \mathscr{F})^{G_m} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{q-1}(\widetilde{Y}, \mathscr{F})_{G_n} \arrow[r] & H^q(Y_n, \mathscr{F}) \arrow[r] & H^q(\widetilde{Y}, \mathscr{F})^{G_n} \arrow[r] & 0
\end{tikzcd}$$

commutes. Taking inductive limit, we get a short exact sequence

$$0 \longrightarrow \varinjlim_n H^{q-1}(\widetilde{Y}, \mathscr{F})_{G_n} \longrightarrow \varinjlim_n H^q(Y_n, \mathscr{F}) \longrightarrow \bigcup_n H^q(\widetilde{Y}, \mathscr{F})^{G_n} \longrightarrow 0. \tag{4.1}$$

But, since $\widetilde{Y}$ has the homotopy type of a finite CW complex and $\mathscr{F}$ is a locally constant sheaf of finite groups, $H^{q-1}(\widetilde{Y}, \mathscr{F})$ and $H^q(\widetilde{Y}, \mathscr{F})$ are finite groups, and hence there exists an $m \geq 1$ such that G_m acts trivially on them. Therefore $\varinjlim_n H^{q-1}(\widetilde{Y}, \mathscr{F})_{G_n} = 0$ (remember that the transition maps are $1 + T^m + \ldots + T^{n-m}$, which equals n/m if $T^m = id$), and $\bigcup_n H^q(\widetilde{Y}, \mathscr{F})^{G_n} = H^q(\widetilde{Y}, \mathscr{F})$.

Now let $\mathscr{G}$ be a locally constant sheaf of finite groups on Y. In this case, $H^1(Y, \mathscr{G})$ and $H^1(\widetilde{Y}, \mathscr{G})$ classify $\mathscr{G}$-torsors. Moreover, descent theory identifies $H^1(Y_n, \mathscr{G})$ with isomorphism classes of G_n-linearized $\mathscr{G}$-torsors on $\widetilde{Y}$, under which identification the pullback map $H^1(Y_n, \mathscr{G}) \longrightarrow H^1(\widetilde{Y}, \mathscr{G})$ corresponds to forgetting the G_n-structure. Thus $H^1(Y_n, \mathscr{G}) \longrightarrow H^1(\widetilde{Y}, \mathscr{G})^{G_n}$ is surjective, and hence $\varinjlim_n H^1(Y_m, \mathscr{G}) \longrightarrow H^1(\widetilde{Y}, \mathscr{G})$ is surjective as $H^1(\widetilde{Y}, \mathscr{G})$ is a finite set. In fact, in analogy with the abelian case, we have a "short exact sequence"

$$\varinjlim_n H^0(\widetilde{Y}, \mathscr{G}) \longrightarrow \varinjlim_n H^1(Y_n, \mathscr{G}) \longrightarrow \bigcup_n H^1(\widetilde{Y}, \mathscr{G})^{G_n} \longrightarrow *$$

(meaning that the first term acts on the second term — by changing the G_n-structure — and the third term is the orbit space), obtained by taking inductive limit of the system

$$\begin{tikzcd}
H^0(\widetilde{Y}, \mathscr{G}) \arrow[r] \arrow[d, "\beta"'] & H^1(Y_m, \mathscr{G}) \arrow[r] \arrow[d] & H^1(\widetilde{Y}, \mathscr{G})^{G_m} \arrow[r] \arrow[d] & * \\
H^0(\widetilde{Y}, \mathscr{G}) \arrow[r] & H^1(Y_n, \mathscr{G}) \arrow[r] & H^1(\widetilde{Y}, \mathscr{G})^{G_n} \arrow[r] & *,
\end{tikzcd}$$

where $\beta(g) = g \cdot (T^m g) \cdot \ldots \cdot (T^{n-m} g)$. We conclude as before. □

Remark 4.3.4. As the example $Y = S^1 \vee S^2$, $\widetilde{Y} =$ (universal cover of Y) shows (cf. [AM69, Example 6.11]), the finiteness assumption on $\widetilde{Y}$ (rather than Y) in Lemma 4.3.3 is necessary. In general, the proof shows that without the finiteness assumptions on $\widetilde{Y}$ and $\mathscr{F}$ one always has the short exact sequence (4.1). In particular, the image of $\varinjlim_n H^q(Y_n, \mathscr{F})$ in $H^q(\widetilde{Y}, \mathscr{F})$ is the subgroup of elements of $H^q(\widetilde{Y}, \mathscr{F})$ with finite orbit under the action of $\mathbf{Z}$ by deck transformations.

Proposition 4.3.5. *Let W be a compact topological space, $W_0 \subseteq W$ a closed subspace. Let x be the vertex of the open cone $X := C^\circ(W)$. Suppose we are given a $\mathbf{Z}$-covering space*

$$\widetilde{X}^* \longrightarrow X^* := C^\circ(W) \setminus C^\circ(W_0).$$

For any integer $n \geq 1$, denote by $X_n^* = \widetilde{X}^* \times_{\mathbf{Z}} (\mathbf{Z}/n\mathbf{Z}) \longrightarrow X^*$ the induced $\mathbf{Z}/n\mathbf{Z}$-covering space. Suppose that $\widetilde{X}^*$ has the homotopy type of a finite CW complex. Then the pullback maps

$$\varinjlim_{U \ni x, n} H^q(U_n^*, \mathscr{F}) \longrightarrow \varinjlim_{U \ni x} H^q(\widetilde{U}^*, \mathscr{F}) \longrightarrow H^q(\widetilde{X}^*, \mathscr{F})$$

are isomorphisms for every locally constant sheaf of finite abelian groups $\mathscr{F}$ on $\widetilde{X}^*$. The same for H^1 of a locally constant sheaf of finite groups.

Proof. Let $W^* = W \backslash W_0$. As $X^* \simeq W^* \times (0,1)$, there exists a $\mathbf{Z}$-covering space $\widetilde{W}^* \longrightarrow W^*$ and an isomorphism $\widetilde{X}^* \xrightarrow{\sim} \pi^* \widetilde{W}^*$ over X, where

$$\pi : X^* = W^* \times (0,1) \longrightarrow W^*$$

is the natural projection. Moreover, $\mathscr{F}$ is a pullback from W^* (being locally constant). Noting that the sets $U_\varepsilon = \{y \in X : d(y) < \varepsilon\}$ constitute a basis of neighborhoods of x in X, we can replace the limits over $U \ni x$ by limits over U_ε. Note that $\widetilde{X}^*$ and $\widetilde{W}^* \times (0,1)$ are isomorphic over X, and that the preimage of U_ε in $\widetilde{W}^* \times (0,1)$ is $\widetilde{W}^* \times (0,\varepsilon)$. It follows that the transition maps in $\{H^q(U_{\varepsilon,n}^*, \mathscr{F})\}_\varepsilon$ (for a fixed n) and $\{H^q(\widetilde{U}_\varepsilon^*, \mathscr{F})\}$ are isomorphisms, which in particular implies that the second map in the assertion is an isomorphism. By Lemma 4.3.3, for every ε the maps

$$\varinjlim_{n} H^q(U_{\varepsilon,n}^*, \mathscr{F}) \longrightarrow H^q(\widetilde{U}_\varepsilon^*, \mathscr{F})$$

are isomorphisms. Passing to the limit with ε, we get the desired assertion. □

Theorem 4.3.6 (cf. [SGA73b, Exp. XVI Théorème 4.1])**.** *Let $f : X \longrightarrow S$ be a finite type morphism of schemes locally of finite type over* Spec $\mathbf{C}$*, and let $\mathscr{F}$ be a constructible sheaf of sets (resp. of finite groups, resp. of finite abelian groups) on X. Denote by $\varepsilon : X_{cl} \longrightarrow X_{\text{ét}}$ and $\varepsilon : S_{cl} \longrightarrow S_{\text{ét}}$ the comparison maps. Then the base change morphisms*

$$\varphi : \varepsilon^* R^q f_* \mathscr{F} \longrightarrow R^q f_* \varepsilon^* \mathscr{F}$$

are isomorphisms for $q = 0$ (resp. for $q = 0, 1$, resp. for $q \geq 0$).

Proof of Theorem 4.3.1. (α) Let ε and u be as in Theorem 4.1.2, and let $S(\varepsilon)$ denote the sphere of radius ε centered at $x \in \mathbf{C}^{N+1}$. The transition maps in the system $\{\widetilde{B}^*(\varepsilon')\}_{\varepsilon' \leq \varepsilon} \simeq \widetilde{C}$ are isomorphic via u to the inclusions

$$(S_x(\varepsilon) \times_X \widetilde{X}^*) \times (0, \varepsilon''/\varepsilon) \hookrightarrow (S_x(\varepsilon) \times_X \widetilde{X}^*) \times (0, \varepsilon'/\varepsilon) \quad (\varepsilon'' \leq \varepsilon' \leq \varepsilon),$$

and therefore are homotopy equivalences. It follows that α_ε induces an isomorphism $\Pi(\widetilde{C}) \xrightarrow{\sim} F_{x,\varepsilon}$. Of course any smaller ε will do.

For β, γ, and δ, we will use the criterion of Lemma 4.3.2. In each case, assumption (1) is satisfied with $d = 2\dim(X)$, so we only need to check (2) and (3).

(β) By Theorem 4.1.5(2), we know that $F_{x,\varepsilon}$ has the homotopy type of a finite CW complex. Thus passing to the limit with n we obtain the desired claim by Proposition 4.3.5.

(γ) Fix an integer $n \geq 1$, and let $\mathscr{F}$ be a constructible sheaf of finite abelian groups on X_n^*, and denote the map $X_n^* \longrightarrow X$ by j_n. Then Theorem 4.3.6 shows that the base change map

$$\varphi : \varepsilon^* R^q j_{n*} \mathscr{F} \longrightarrow R^q j_{n*} \varepsilon^* \mathscr{F}$$

is an isomorphism. Taking stalks at x, we see that in the commutative square

$$\begin{array}{ccc} (R^q j_{n*}\mathscr{F})_x & \xrightarrow{\varphi} & (R^q j_{n*}\varepsilon^*\mathscr{F})_x \\ \downarrow & & \downarrow \\ H^q(C_{n,\mathrm{alg}}, \mathscr{F}) & \xrightarrow[\varepsilon^*]{} & H^q(C_n, \mathscr{F}) \end{array}$$

the bottom arrow is an isomorphism, and so are the vertical arrows, hence the top arrow has to be an isomorphism as well. The same holds for H^1 of constructible sheaves of groups. Taking the limit over n, we get the desired result.

(δ) This follows from [SGA73a, Exp. XIII Proposition 2.1.4] $\square$

Chapter 5

Nearby cycles and monodromy

5.1 Introduction

Kato–Nakayama spaces

Let us start with a short review of the theory of Kato–Nakayama spaces of log complex analytic spaces, which in the author's opinion provide the most intuitive explanation of many phenomena in logarithmic geometry. To a fs log complex analytic space $(X, \mathscr{M}_X)$ one functorially associates a space $(X, \mathscr{M}_X)_{\log}$ (which we often abbreviate to $X_{\log}$) together with a proper continuous map $\tau : (X, \mathscr{M}_X)_{\log} \longrightarrow X$ [KN99, NO10]. If X is smooth, $(X, \mathscr{M}_X)_{\log}$ is a manifold with boundary. A point of $X_{\log}$ corresponds to a pair (x, h) of a point $x \in X$ and a homomorphism $h : \mathscr{M}^{\mathrm{gp}}_{X,x} \longrightarrow S^1$ making the diagram

$$\begin{tikzcd}
\mathscr{O}^*_{X,x} \arrow[r, "f \mapsto f(x)"] \arrow[d] & \mathbf{C}^* \arrow[d, "\frac{z}{|z|}"] \\
\mathscr{M}^{\mathrm{gp}}_{X,x} \arrow[r, "h"'] & \mathbf{S}^1
\end{tikzcd}$$

commute. In particular, the fiber $\tau^{-1}(x)$ is in a natural way a torsor under the compact abelian Lie group $\mathrm{Hom}(\overline{\mathscr{M}}^{gp}_{X,x}, \mathbf{S}^1)$.

Example 5.1.1. 1. If $(0,\mathcal{M}_0)$ is the standard log point $\mathrm{Spec}(\mathbf{N}\to\mathbf{C})$ then $(0,\mathcal{M}_0)_{\log}=\mathbf{S}^1$.

2. If $X=\mathbf{C}$ with $\mathcal{M}_X$ the compactifying log structure induced by $\mathbf{C}\setminus\{0\}\subseteq\mathbf{C}$, then $X_{\log}=[0,\infty)\times\mathbf{S}^1$, $\tau(r,\theta)=re^{i\theta}$.

3. More generally, if $(X,\mathcal{M}_X)=\mathrm{Hom}(P,\mathbf{C})$ is an affine toric variety, then $(X,\mathcal{M}_X)_{\log}=\mathrm{Hom}(P,\mathbf{S}^1)\times\mathrm{Hom}(P,[0,\infty))$ with $\tau(r,\theta)(p)=r(p)e^{i\theta(p)}$.

The geometry of $X_{\log}$ reflects the log geometry of $(X,\mathcal{M}_X)$. For example, one has $H^*(X_{\log},\mathbf{C})\simeq H^*(X,\Omega^\bullet_{X/\mathbf{C},\log})$ [KN99]. If $(X,\mathcal{M}_X)$ is smooth over $\mathbf{C}$ with trivial log structure, with $U=(X,\mathcal{M}_X)_{\mathrm{tr}}$, then the inclusion $U\hookrightarrow X$ factors as $U\hookrightarrow X_{\log}\xrightarrow{\tau}X$, where $U\to X_{\log}$ is a local homotopy equivalence. We have thus replaced the open immersion $U\hookrightarrow X$ with the proper map τ with the same cohomological properties (compare this with Theorem 4.2.6 for $U_{\text{ét}}\to(X,\mathcal{M}_X)_{\text{két}}\xrightarrow{\varepsilon}X_{\text{ét}}$).

The same is true for maps $f:(X,\mathcal{M}_X)\longrightarrow(Y,\mathcal{M}_Y)$. For example, $f_{\log}$ is proper if f is, and $f_{\log}$ is a submersion (of manifolds with boundary) if f is smooth and exact. We also have a logarithmic analog of Ehresmann's theorem: if f is smooth, exact, and proper, then $f_{\log}$ is a locally trivial fibration whose fibers are compact manifolds with boundary. If f is moreover vertical (i.e., $\mathcal{M}_X$ is generated by $f^*\mathcal{M}_Y$ as a sheaf of faces), the fibers are actual manifolds.

The formation of $(X,\mathcal{M}_X)_{\log}$ commutes with base change with respect to strict morphisms [KN99, Lemma 1.3(3)], that is, the square

$$\begin{tikzcd}(X',\mathcal{M}_{X'})_{\log} \arrow[r, "u_{\log}"] \arrow[d, "\tau"'] & (X,\mathcal{M}_X)_{\log} \arrow[d, "\tau"] \\ X' \arrow[r, "u"'] & X\end{tikzcd} \tag{5.1}$$

is cartesian if $u:(X',\mathcal{M}_{X'})\to(X,\mathcal{M}_X)$ is strict. In particular, as any fs log analytic space locally admits a strict map to an affine toric variety, Example 5.1.13 provides a local description of $(X,\mathcal{M}_X)_{\log}$ in general. Moreover, the square is also cocartesian (i.e., a pushout) if u is a strict closed immersion.

One beautiful application of this construction is the procedure of (topologically) reconstructing a log smooth degeneration from its special fiber together with its log structure. For a map $f : X \longrightarrow Y$ of topological spaces, let us denote by $\mathrm{Cyl}(f)$ the open mapping cylinder $X \times [0,1) \sqcup Y/((x,0) \sim f(x))$.

Theorem 5.1.2. *Let $(S, \mathcal{M}_S)$ be a disc around $0 \in \mathbf{C}$ with the log structure induced by the inclusion $S^* = S \setminus \{0\} \hookrightarrow S$, and let $f : (X, \mathcal{M}_X) \longrightarrow (S, \mathcal{M}_S)$ be a proper and smooth morphism of complex analytic spaces, with $\mathcal{M}_X$ the log structure induced by the inclusion $X^* = X \times_S S^* \hookrightarrow X$.*

Then, after shrinking the radius of S, there exist isomorphisms $X \xrightarrow{\sim} \mathrm{Cyl}(\tau_{X_0})$ and $S \xrightarrow{\sim} \mathrm{Cyl}(\tau_{S_0})$ fitting inside a commutative diagram

$$\begin{tikzcd}
& X_0 \arrow[dl] \arrow[dr] \arrow[d] & \\
X \arrow[rr, "\sim"] \arrow[d] & S_0 \arrow[dl] \arrow[dr] & \mathrm{Cyl}(\tau_{X_0}) \arrow[d] \\
S \arrow[rr, "\sim"] & & \mathrm{Cyl}(\tau_{S_0}).
\end{tikzcd}$$

In other words, the morphism of pairs $f : (X, X_0) \longrightarrow (S, S_0 = 0)$ is isomorphic to the map of mapping cylinders $(\mathrm{Cyl}(\tau_{X_0}), X_0) \longrightarrow (\mathrm{Cyl}(\tau_{S_0}), S_0)$ via an isomorphism inducing the identity on X_0 and S_0.

Proof. Note that the open cylinder $\mathrm{Cyl}(\tau_{X_0})$ is by definition the pushout of the diagram

$$\begin{tikzcd}
(X_0, \mathcal{M}_{X_0})_{\log} \arrow[r, "\mathrm{id}\times 0"] \arrow[d, "\tau_{X_0}"'] & (X_0, \mathcal{M}_{X_0})_{\log} \times [0,1) \\
X_0. &
\end{tikzcd} \tag{5.2}$$

On the other hand, by [NO10, Theorem 5.1], the map $f_{\log} : (X, \mathcal{M}_X)_{\log} \longrightarrow (S, \mathcal{M}_S)_{\log}$ is a locally trivial fibration. The base of this fibration is $(S, \mathcal{M}_S)_{\log} \simeq \mathbf{S}^1 \times [0,1)$. Collapsing $\mathbf{S}^1 \times \{0\} = (0, \mathcal{M}_0)_{\log}$ into a point yields the isomorphism $S \simeq \mathrm{Cyl}(\tau_{S_0})$. The interval being contractible, every fiber bundle over $\mathbf{S}^1 \times [0,1)$ can by obtained by pull-back along the projection to $\mathbb{S}^1$ of the induced fibration over $\mathbf{S}^1 \times \{0\} = \mathbf{S}^1$ (cf. [Hus75, I 4, Theorem 9.6] for the group G of homeomorphisms of the fiber of $f_{\log}$, endowed with the compact-open topology). This shows that there is an isomorphism

$$(X, \mathcal{M}_X)_{\log} \xrightarrow{\sim} (X_0, \mathcal{M}_{X_0})_{\log} \times [0,1)$$

fitting inside a commutative diagram

$$\begin{tikzcd} (X, \mathscr{M}_X)_{\log} \arrow[r, "\sim"] \arrow[d, "f_{\log}"'] & (X_0, \mathscr{M}_{X_0})_{\log} \times [0,1) \arrow[d, "\tau_{X_0} \times \mathrm{id}"] \\ (S, \mathscr{M}_S)_{\log} \arrow[r, "\sim"'] & \mathbf{S}^1 \times [0,1). \end{tikzcd}$$

The diagram (5.2) is thus isomorphic to

$$\begin{tikzcd} (X_0, \mathscr{M}_{X_0})_{\log} \arrow[r, "\mathrm{id} \times 0"] \arrow[d, "\tau_{X_0}"'] & (X, \mathscr{M}_X)_{\log} \\ X_0. \end{tikzcd} \tag{5.3}$$

To finish the proof, it suffices to note that the square

$$\begin{tikzcd} (X_0, \mathscr{M}_{X_0})_{\log} \arrow[r, "\mathrm{id} \times 0"] \arrow[d, "\tau_{X_0}"'] & (X, \mathscr{M}_X)_{\log} \arrow[d, "\tau_X"] \\ X_0 \arrow[r, "i"'] & X \end{tikzcd} \tag{5.4}$$

is a pushout. Since (5.4) clearly commutes, we get an induced map

$$\varphi : \mathrm{Pushout}((5.3)) \longrightarrow X.$$

Since τ_X is surjective, so is φ. To show injectivity, we use the fact that the log structure on X is trivial on $X \setminus X_0$. The question whether φ is a homeomorphism is local on X. Let $x \in X$, and let $S' \subseteq S$ be a smaller *closed* disc containing $f(x)$ in its interior. Then $X' = X \times_S S'$ is compact and Hausdorff, and φ is bijective over X', thus φ is a homeomorphism over X', which contains a neighborhood of the point x. □

The monodromy formula of Ogus

Theorem 5.1.2 implies that the all of the local topological invariants of f around X_0, including nearby cycles and monodromy, are completely determined by $(X_0, \mathscr{M}_{X_0})$. Making this relationship explicit, particularly in terms of the extension

$$0 \longrightarrow f^* \overline{\mathscr{M}}^{\mathrm{gp}}_S \longrightarrow \overline{\mathscr{M}}^{\mathrm{gp}}_X \longrightarrow \overline{\mathscr{M}}^{\mathrm{gp}}_{X/S} \longrightarrow 0 \tag{5.5}$$

was the principal aim of [Ogu13]. The goal of this chapter is to provide several variants of one of Ogus' results, namely the "monodromy formula" explained below.

Let $f:(X,\mathcal{M}_X)\longrightarrow(S,\mathcal{M}_S)$ be a smooth and saturated (Definition 2.2.2) morphism, where $(S,\mathcal{M}_S)$ is the standard log point $\mathrm{Spec}(\mathbf{N}\longrightarrow\mathbf{C})$. Let

$$\widetilde{X}=(X,\mathcal{M}_X)_{\log}\times_{\mathbf{S}^1}\mathbf{R}(1),$$

$\overline{\tau}:\widetilde{X}\longrightarrow X$ the projection. We consider the complexes of sheaves $R\overline{\tau}_*\mathbf{C}$ on X, with the induced action of $\mathbf{Z}(1)=\pi_1(S^1)$, which in the situation where $(X,\mathcal{M}_X)$ is the special fiber of Theorem 5.1.2 are nothing else than nearby cycles $R\Psi\mathbf{C}$. Standard calculations (cf. §5.2 below) show that one can identify the cohomology sheaves $R^q\overline{\tau}_*\mathbf{C}$ with the exterior powers $\bigwedge^q\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}\otimes\mathbf{C}$, and that the action of $\mathbf{Z}(1)$ on $R^q\overline{\tau}_*\mathbf{C}$ is trivial. This implies that for $\gamma\in\mathbf{Z}(1)$, the map $1-\gamma:R\overline{\tau}_*\mathbf{C}\longrightarrow R\overline{\tau}_*\mathbf{C}$ induces homomorphisms

$$L_{1-\gamma}:R^q\overline{\tau}_*\mathbf{C}\longrightarrow R^{q-1}\overline{\tau}_*\mathbf{C}[1]$$

in the derived category. One might think of the collection of these homomorphisms as of a "first order approximation" of the monodromy action on $R\overline{\tau}_*\mathbf{C}$. On the other hand, cup product with the canonical generator of $\overline{\mathcal{M}}^{\mathrm{gp}}_S$, or equivalently the class of the extension (5.5), induces homomorphisms

$$E:\bigwedge^q\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}\otimes\mathbf{C}\longrightarrow\left(\bigwedge^{q-1}\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}\otimes\mathbf{C}\right)[1].$$

Theorem 5.1.3 (Monodromy Formula, [Ogu13]). *For every $q\geq 0$ and for every $\gamma\in\mathbf{Z}(1)$, the diagram*

$$\begin{tikzcd}
\bigwedge^q\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}\otimes\mathbf{C} \arrow[r, "E"] \arrow[d] & \left(\bigwedge^{q-1}\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}\otimes\mathbf{C}\right)[1] \arrow[r, "\cdot\gamma"] & \left(\bigwedge^{q-1}\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}\otimes\mathbf{C}\right)[1] \arrow[d] \\
R^q\overline{\tau}_*\mathbf{C} \arrow[rr, "L_{1-\gamma}"'] & & R^{q-1}\overline{\tau}_*\mathbf{C}[1]
\end{tikzcd}$$

commutes.

In this chapter, we provide ℓ-adic analogues of the monodromy formula for smooth and saturated log schemes over a standard log point (Theorem 5.4.4) and over a trait with the standard log structure (Theorem 5.5.1). Moreover, we provide a version of Theorem 5.1.3 with coefficients in $\mathbf{Z}$. In particular, we obtain a completely different proof of Theorem 5.1.3. Our methods resemble those used by Rapoport and Zink [RZ82] (see also [Ill94, Ill02b]).

5.2 The complex analytic case

Before giving a complete treatment of the several variants of the monodromy formula, we provide an outline of our proof of Theorem 5.1.3 (with coefficients $\mathbf{Z}$ rather than $\mathbf{C}$). The proof has two steps, the first being purely an argument of homological algebra, and the second one relating the homological algebra to the geometry in a very easy way.

Let $(S, \mathcal{M}_S)$ be the standard log point. We will identify $(S, \mathcal{M}_S)_{\log}$ with $\mathbf{S}^1$, and denote the fundamental group $\pi_1(\mathbf{S}^1)$ by Γ. For an abelian group A an an integer q, $A(q)$ denotes the "Tate twist" $A \otimes_{\mathbf{Z}} \Gamma^{\otimes q}$. Let $\theta \in H^1(\Gamma, \mathbf{Z}(1)) = H^1(\mathbf{Z}(1), \mathbf{Z}(1))$ be the canonical generator, corresponding to the identity on Γ. Under the identification $H^1(\mathbf{S}^1, \mathbf{Z}(1)) = H^1(\mathbf{Z}(1), \mathbf{Z}(1))$, θ corresponds to the class of the $\Gamma = \mathbf{Z}(1)$-torsor $\exp : \mathbf{R}(1) \longrightarrow \mathbf{S}^1$.

As in the introduction, we will consider a smooth and saturated log complex analytic space $f : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$. Let $\overline{\tau} : \widetilde{X} = (X, \mathcal{M}_X)_{\log} \times_{\mathbf{S}^1} \mathbf{R}(1) \to X$ be the canonical projection.

Step 1 — Homological Algebra

Let $K = R\overline{\tau}_* \mathbf{Z}$, which we consider as a complex of sheaves of Γ-modules on X. As mentioned above, this step is of a purely homological algebraic nature. The only input from geometry we need is that K is a complex of Γ-modules (on some space — or topos — X) such that the Γ-action on the cohomology sheaves $H^q(K)$ is trivial for all $q \in \mathbf{Z}$. This well-known property of K will be verified in the next step. It implies (cf. Lemma 5.3.1) that for $\gamma \in \Gamma$ the map $1 - \gamma : K \longrightarrow K$ induces canonical maps

$$L_{1-\gamma} : H^q(K) \longrightarrow H^{q-1}(K)[1]$$

which the monodromy formula seeks to describe. Recall that for objects M and N of an abelian category A with enough injectives one has $\mathrm{Hom}(M, N[1]) = \mathrm{Ext}^1(M, N)$. In order to explicate $L_{1-\gamma}$, we will find the corresponding extension of $H^q(K)$ by $H^{q-1}(K)$.

The key point is that $L_{1-\gamma}$ can be expressed using the element $\theta \in H^1(\Gamma, \mathbf{Z}(1))$. Cup product with θ induces maps $\wedge\theta : H^q(\Gamma, K(q)) \to H^{q+1}(\Gamma, K(q+1))$. Here $H^q(\Gamma, -)$ is the q-th derived functor of the functor taking a sheaf of Γ-modules on X to its *subsheaf* of Γ-invariants. Since $H^2(\Gamma, \mathbf{Z}(2)) = 0$, we have $\theta \wedge \theta = 0$, and hence we get a complex

$$\ldots \longrightarrow H^{q-1}(\Gamma, K) \xrightarrow{\wedge\theta} H^q(\Gamma, K) \xrightarrow{\wedge\theta} H^{q+1}(\Gamma, K) \longrightarrow \ldots \tag{5.6}$$

which is exact (cf. Proposition 5.3.3(c)). Furthermore, the canonical projections $H^q(\Gamma, K(q)) \to H^0(\Gamma, H^q(K(q))) = H^q(K)$ are surjective, and $\wedge\theta : H^{q-1}(\Gamma, K(q-1)) \to H^q(\Gamma, K(q))$ induce injections $H^{q-1}(K) \to H^q(\Gamma, K(q))$ (by the same proposition). Thus the exact sequence (5.6) splits into short exact sequences

$$0 \longrightarrow H^{q-1}(K(q-1)) \longrightarrow H^q(\Gamma, K(q)) \longrightarrow H^q(K(q)) \longrightarrow 0. \tag{5.7}$$

Theorem 5.2.1 (cf. Proposition 5.3.5)**.** *Let $\gamma \in \Gamma$. The map $L_{1-\gamma}(q) : H^q(K(q)) \to H^{q-1}(K(q))[1]$ corresponds to the pushout of the extension* (5.7) *along the map*

$$\mathrm{id} \otimes \gamma : H^{q-1}(K(q-1)) \longrightarrow H^{q-1}(K(q-1))(1) = H^{q-1}(K(q)).$$

Interlude — Description of $R^q\tau_*\mathbf{Z}$ and $R^q\overline{\tau}_*\mathbf{Z}$

Recall the definition of the logarithmic exponential sequence [KN99, §1.4] of a fs log complex analytic space $(X, \mathscr{M}_X)$. Points of $X_{\log}$ correspond to pairs $(x \in X, h : \mathscr{M}^{\mathrm{gp}}_{X,x} \to \mathbf{S}^1)$ such that h extends the map $f \mapsto f(x)/|f(x)|$ on $\mathscr{O}^*_{X,x}$, and these homomorphisms h assemble into a map $c : \tau^{-1}\mathscr{M}^{\mathrm{gp}}_X \to \mathbf{S}^1$ (here $\mathbf{S}^1$ stands for the sheaf of *continuous* functions into $\mathbf{S}^1$). Let $\mathscr{L}$ be the fiber product of $\exp : \mathbf{R}(1) \to \mathbf{S}^1$ and c. Thus $\mathscr{L}$ is the "sheaf of logarithms of $\mathscr{M}^{\mathrm{gp}}_X$". We get a pullback diagram of short exact sequences of sheaves on $X_{\log}$

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}(1) \arrow[r] \arrow[d, equal] & \tau^{-1}(\mathscr{O}_X) \arrow[r, "\exp"] \arrow[d] & \tau^{-1}(\mathscr{O}^*_X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{Z}(1) \arrow[r] \arrow[d, equal] & \mathscr{L} \arrow[r, "\exp"] \arrow[d] & \tau^{-1}(\mathscr{M}^{\mathrm{gp}}_X) \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & \mathbf{Z}(1) \arrow[r] & \mathbf{R}(1) \arrow[r, "\exp"] & \mathbf{S}^1 \arrow[r] & 0
\end{tikzcd}$$

whose middle row is called the logarithmic exponential sequence.

Applying τ_* to it yields a connecting homomorphism $\delta : \tau_*\tau^{-1}(\mathcal{M}_X)\to R^1\tau_*\mathbf{Z}(1)$. The composition $\mathcal{M}_X^{\mathrm{gp}}\to\tau_*\tau^{-1}(\mathcal{M}_X^{\mathrm{gp}})\to R^1\tau_*\mathbf{Z}(1)$ with the adjunction map $\mathcal{M}_X^{\mathrm{gp}}\to\tau_*\tau^{-1}(\mathcal{M}_X^{\mathrm{gp}})$ factors through $\overline{\mathcal{M}}_X^{\mathrm{gp}} = \mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^*$ because $\exp : \mathcal{O}_X = \tau_*\tau^{-1}(\mathcal{O}_X)\to\tau_*\tau^{-1}(\mathcal{O}_X^*) = \mathcal{O}_X^*$ is surjective. Let us denote by $\mu : \overline{\mathcal{M}}_X^{\mathrm{gp}}(-1)\to R^1\tau_*\mathbf{Z}$ the homomorphism we just constructed (after a -1 Tate twist), and by $\mu^q : \bigwedge^q\overline{\mathcal{M}}_X^{\mathrm{gp}}(-1)\to R^q\tau_*\mathbf{Z}$ the map induced by μ and cup product.

Theorem 5.2.2 ([KN99, Lemma 1.5])**.** *The maps $\mu^q : \bigwedge^q(\overline{\mathcal{M}}_X^{\mathrm{gp}}(-1))\to R^q\tau_*\mathbf{Z}$ are isomorphisms for all $q\geq 0$.*

This is not a difficult result: since $\tau : X_{\log}\to X$ is proper, the proper base change theorem reduces us immediately to the case where $X=\{x\}$ is a point, in which situation $X_{\log}$ is the real torus $\mathrm{Hom}(\overline{\mathcal{M}}_{X,x}^{\mathrm{gp}},\mathbf{S}^1)$.

Now let us assume, as before, that $(X,\mathcal{M}_X)$ is log smooth and saturated over the standard log point $(S,\mathcal{M}_S)$. Since Γ acts on $\widetilde{X}$ and the map $\overline{\tau}:\widetilde{X}\to X$ is Γ-invariant, the complexes $R\overline{\tau}_*\mathbf{Z}$ are complexes of sheaves of Γ-modules on X.

Similarly to $R^q\tau_*\mathbf{Z}$, the sheaves $R^q\overline{\tau}_*\mathbf{Z}$ admit a description in terms of $\overline{\mathcal{M}}_{X/S}^{\mathrm{gp}} = \overline{\mathcal{M}}_X^{\mathrm{gp}}/f^*\overline{\mathcal{M}}_S^{\mathrm{gp}}$. As $(S,\mathcal{M}_S)$ is the standard log point, there is a canonical section t of $\mathcal{M}_S$ whose image in $\overline{\mathcal{M}}_S\simeq\mathbf{N}$ is a generator. The structure morphism $f:(X,\mathcal{M}_X)\to(S,\mathcal{M}_S)$ corresponds to a choice of a global section f^*t of $\mathcal{M}_X$, which we also denote by t. Thus the points of $\widetilde{X}$ are triples (x,h,r) with $(x,h)\in X_{\log}$ and $r\in\mathbf{R}(1)$ with $\exp(r)=h(t)$. In particular, there is a global section $\log t$ of the pullback of $\mathcal{L}$ to $\widetilde{X}$ with $\exp(\log t)=t$. We consider the pullback of the logarithmic exponential sequence to $\widetilde{X}$ and the associated connecting homomorphism $\overline{\tau}_*\overline{\tau}^{-1}\mathcal{M}_X^{\mathrm{gp}}\to R^1\overline{\tau}_*\mathbf{Z}(1)$. As before, the composition $a : \mathcal{M}_X^{\mathrm{gp}}\to\overline{\tau}_*\overline{\tau}^{-1}(\mathcal{M}_X^{\mathrm{gp}})\to R^1\overline{\tau}_*\mathbf{Z}(1)$ annihilates $\mathcal{O}_X^*$. But because $\log t$ exists on $\widetilde{X}$, $a(f)=0$, and hence a descends further to $\overline{\mathcal{M}}_{X/S}^{\mathrm{gp}}$. We denote by $\overline{\mu}:\overline{\mathcal{M}}_{X/S}^{\mathrm{gp}}(-1)\to R^1\tau_*\mathbf{Z}$ the constructed homomorphism (after a Tate twist), and by

$$\overline{\mu}^q : \bigwedge^q(\overline{\mathcal{M}}_{X/S}^{\mathrm{gp}}(-1))\to R^q\overline{\tau}_*\mathbf{Z}$$

the map induced by $\overline{\mu}$ and cup product. We have the following analogue of Theorem 5.2.2.

Theorem 5.2.3. *The maps* $\overline{\mu}^q : \bigwedge^q(\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}(-1)) \to R^q\overline{\tau}_*\mathbf{Z}$ *are isomorphisms for all* $q \geq 0$. *In particular, the action of* Γ *on* $R^q\overline{\tau}_*\mathbf{Z}$ *is trivial for all* $q \geq 0$.

We will make use of the following explicit description of the maps

$$\mu(1) : \overline{\mathcal{M}}^{\mathrm{gp}}_X \to R^1\tau_*\mathbf{Z}(1) \quad \text{and} \quad \overline{\mu}(1) : \overline{\mathcal{M}}^{\mathrm{gp}}_{X/S} \to R^1\overline{\tau}_*\mathbf{Z}(1).$$

Note that $R^1\tau_*\mathbf{Z}(1)$ is the sheafification of the presheaf on X

$$U \mapsto H^1(\tau^{-1}(U), \mathbf{Z}(1)) = (\text{isomorphism classes of } \mathbf{Z}(1)\text{-torsors on } \tau^{-1}(U)).$$

Let m be a local section of $\overline{\mathcal{M}}^{\mathrm{gp}}_X$. Suppose that m' is a section of $\mathcal{M}^{\mathrm{gp}}_X$ whose image in $\overline{\mathcal{M}}^{\mathrm{gp}}_X$ equals m. The logarithmic exponential sequence makes the preimage of the section $\tau^{-1}(m)$ of $\tau^{-1}(\mathcal{M}^{\mathrm{gp}}_X)$ under the map $\exp : \mathcal{L} \to \tau^{-1}(\mathcal{M}^{\mathrm{gp}}_X)$ into a torsor under $\mathbf{Z}(1)$, and the section $\mu(1)(m)$ of $R^1\tau_*\mathbf{Z}(1)$ corresponds to the isomorphism class of this torsor. The analogous description holds for $\overline{\mu}(1)$. Moreover, the two descriptions being compatible, we see that the square

$$\begin{tikzcd} \overline{\mathcal{M}}^{\mathrm{gp}}_X(-1) \arrow[r] \arrow[d, "\mu"'] & \overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}(-1) \arrow[d, "\overline{\mu}"] \\ R^1\tau_*\mathbf{Z} \arrow[r] & R^1\overline{\tau}_*\mathbf{Z} \end{tikzcd} \tag{5.8}$$

commutes.

Consider the case $(X, \mathcal{M}_X) = (S, \mathcal{M}_S)$. In this situation, $\widetilde{X} = \mathbf{R}(1)$, and the map $\overline{\tau}$ is the map $\exp : \mathbf{R}(1) \longrightarrow \mathbf{S}^1$. This $\mathbf{Z}(1)$-torsor is identified with the torsor on $\mathbf{S}^1$ coming from the logarithmic exponential sequence and the generating section t of $\tau^{-1}(\mathcal{M}^{\mathrm{gp}}_S)$. We deduce that $\mu_S(1)(t) = \theta$. For a general $(X, \mathcal{M}_X)$, pulling this back to X we obtain

$$\mu(1)(f^*t) = f^*\theta. \tag{5.9}$$

Step 2 — The end of the proof

Note that since $\widetilde{X} \to X_{\log}$ is a Γ-torsor, we have $R\tau_*\mathbf{Z} = H^*(\Gamma, R\overline{\tau}_*\mathbf{Z})$, so $R^q\tau_*\mathbf{Z} = H^q(\Gamma, R\overline{\tau}_*\mathbf{Z})$. Using this identification, the map $\wedge\theta$: $H^q(\Gamma, R\overline{\tau}_*\mathbf{Z}(q)) \to H^{q+1}(\Gamma, R\overline{\tau}_*\mathbf{Z}(q+1))$ corresponds to cup product with the (image in $R^1\tau_*\mathbf{Z}(1)$ of the) generator θ of $R^1\tau_{S*}\mathbf{Z}(1) = H^1(\mathbf{S}^1, \mathbf{Z}(1)) = H^1(\Gamma, \mathbf{Z}(1))$. Therefore, the extension (5.7) (which, after Theorem 5.2.1, determines $L_{1-\gamma}$) corresponds to the extension

$$0 \longrightarrow R^{q-1}\overline{\tau}_*\mathbf{Z}(q-1) \xrightarrow{a^q} R^q\tau_*\mathbf{Z}(q) \xrightarrow{b^q} R^q\overline{\tau}_*\mathbf{Z}(q) \longrightarrow 0 \tag{5.10}$$

where the map $a^q : R^{q-1}\overline{\tau}_*\mathbf{Z}(q-1) \longrightarrow R^q\tau_*\mathbf{Z}(q)$ is "lift to $R^{q-1}\tau_*\mathbf{Z}(q-1)$ and take cup product with θ" and the map $b^q : R^q\tau_*\mathbf{Z}(q) \longrightarrow R^q\overline{\tau}_*\mathbf{Z}(q)$ is deduced from pullback along $\widetilde{X} \longrightarrow X_{\log}$.

We will first prove the monodromy formula for $q = 1$. Let t be the canonical section of $\mathcal{M}_S$ (whose image in $\mathcal{M}_X$ equals f by definition). Our goal is to relate the the extension (5.5) to the extension 5.10 for $q = 1$. In other words, we need to check that the diagram

$$\begin{tikzcd}
0 \arrow[r] & f^*\overline{\mathcal{M}}^{\mathrm{gp}}_S(-1) \arrow[r, "t"] \arrow[d] & \overline{\mathcal{M}}^{\mathrm{gp}}_X(-1) \arrow[r] \arrow[d, "\mu"] & \overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}(-1) \arrow[r] \arrow[d, "\overline{\mu}"] & 0 \\
0 \arrow[r] & \mathbf{Z}(-1) \arrow[r, "a^1"'] & R^1\tau_*\mathbf{Z} \arrow[r, "b^1"'] & R^1\overline{\tau}_*\mathbf{Z} \arrow[r] & 0
\end{tikzcd} \tag{5.11}$$

commutes. Here the square on the right is a Tate twist of (5.8), and the commutativity of the square on the left follows directly from (5.9).

Since the maps $\overline{\mu}^q$, μ^q, and a^q are defined by cup product, we easily deduce that the diagrams

$$\begin{tikzcd}
0 \arrow[r] & (\bigwedge^{q-1}\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}(-1))(-1) \arrow[r, "\wedge t"] \arrow[d, "\overline{\mu}^{q-1}"'] & \bigwedge^q\overline{\mathcal{M}}^{\mathrm{gp}}_X(-1) \arrow[r] \arrow[d, "\mu^q"] & \bigwedge^q\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}(-1) \arrow[r] \arrow[d, "\overline{\mu}^q"] & 0 \\
0 \arrow[r] & R^{q-1}\overline{\tau}_*\mathbf{Z}(-1) \arrow[r, "a^q"'] & R^q\tau_*\mathbf{Z} \arrow[r, "b^q"'] & R^q\overline{\tau}_*\mathbf{Z} \arrow[r] & 0,
\end{tikzcd} \tag{5.12}$$

where the bottom row is an appropriate Tate twist of (5.10), commute. Combined with Theorem 5.2.1, this shows that the diagrams

$$\begin{tikzcd}
\bigwedge^q\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}\otimes\mathbf{Z}(-1) \arrow[r, "E"] \arrow[d, "\overline{\mu}^q"'] & (\bigwedge^{q-1}\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}\otimes\mathbf{Z}(-1))(-1)[1] \arrow[r, "\cdot\gamma"] & (\bigwedge^{q-1}\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}\otimes\mathbf{Z}(-1))[1] \arrow[d, "\overline{\mu}^{q-1}"] \\
R^q\overline{\tau}_*\mathbf{Z} \arrow[rr, "L_{1-\gamma}"'] & & R^{q-1}\overline{\tau}_*\mathbf{Z}[1],
\end{tikzcd}$$

where E corresponds to the class of the top row extension (5.12), commute as well. This completes the proof.

5.3 Homological algebra and group cohomology

5.3.1 Homological algebra

Recall the definition of the truncation functors $\tau_{\leq q}$ and $\tau_{\geq q}$ on the category of complexes in an abelian category A:

$$\tau_{\leq q}K = [\ldots \longrightarrow K^{q-1} \longrightarrow \ker(K^q \longrightarrow K^{q+1}) \longrightarrow 0 \longrightarrow \ldots],$$

$$\tau_{\geq q}K = [\ldots \longrightarrow 0 \longrightarrow \operatorname{coker}(K^{q-1} \longrightarrow K^q) \longrightarrow K^{q+1} \longrightarrow \ldots].$$

These functors descend to the derived category $D(A)$. For a pair of integers $a \leq b$, we write $\tau_{[a,b]} = \tau_{\geq a}\tau_{\leq b} = \tau_{\leq b}\tau_{\geq a}$ and $\tau_{[a,b)} = \tau_{[a,b-1]}$. If $a < b < c$, we have an exact triangle of functors

$$\tau_{[a,b)} \longrightarrow \tau_{[a,c)} \longrightarrow \tau_{[b,c)} \longrightarrow .$$

We have $\tau_{[q,q]}K = H^q(K)[-q]$.

Lemma 5.3.1. *Let $f : K \longrightarrow K'$ be a morphism in $D(A)$, and let q be an integer. Suppose that the induced maps $H^{q-1}K \longrightarrow H^{q-1}K'$ and $H^qK \longrightarrow H^qK'$ are zero.*

1. *There exists a unique morphism $L_f : H^q(K) \longrightarrow H^{q-1}(K')[1]$ making the diagram*

$$\begin{tikzcd} \tau_{[q-1,q]}K \arrow[r] \arrow[d, "\tau_{[q,q-1]}(f)"'] & H^q(K)[-q] \arrow[d, "L_f[-q]"] \\ \tau_{[q-1,q]}K' & H^{q-1}(K')[1-q] \arrow[l] \end{tikzcd}$$

   commute.

2. *Let $C \longrightarrow K \xrightarrow{f} K' \longrightarrow C[1]$ be an exact triangle. The corresponding long cohomology exact sequence*

$$\ldots \longrightarrow H^{q-1}K \xrightarrow{f=0} H^{q-1}K' \longrightarrow H^qC \longrightarrow H^qK \xrightarrow{f=0} H^qK' \longrightarrow \ldots$$

   gives a short exact sequence

$$0 \longrightarrow H^{q-1}K' \longrightarrow H^qC \longrightarrow H^qK \longrightarrow 0.$$

   The corresponding map $H^q(K) \longrightarrow H^{q-1}(K')[1]$ equals L_f.

Proof. (1) Consider the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
\mathrm{Hom}(\tau_{[q]}K, \tau_{[q]}K'[-1]) \arrow[r] \arrow[d] & \mathrm{Hom}(\tau_{[q-1,q]}K, \tau_{[q]}K'[-1]) \arrow[r] \arrow[d] & \mathrm{Hom}(\tau_{[q-1]}K, \tau_{[q]}K'[-1]) \arrow[d] \\
\mathrm{Hom}(\tau_{[q]}K, \tau_{[q-1]}K') \arrow[r] \arrow[d] & \mathrm{Hom}(\tau_{[q-1,q]}K, \tau_{[q-1]}K') \arrow[r] \arrow[d] & \mathrm{Hom}(\tau_{[q-1]}K, \tau_{[q-1]}K') \arrow[d] \\
\mathrm{Hom}(\tau_{[q]}K, \tau_{[q-1,q]}K') \arrow[r] \arrow[d] & \mathrm{Hom}(\tau_{[q-1,q]}K, \tau_{[q-1,q]}K') \arrow[r] \arrow[d] & \mathrm{Hom}(\tau_{[q-1]}K, \tau_{[q-1,q]}K') \arrow[d] \\
\mathrm{Hom}(\tau_{[q]}K, \tau_{[q]}K') \arrow[r] & \mathrm{Hom}(\tau_{[q-1,q]}K, \tau_{[q]}K') \arrow[r] & \mathrm{Hom}(\tau_{[q-1]}K, \tau_{[q]}K').
\end{tikzcd}$$

Note that the groups in the top row and the group in the bottom right corner are zero, as $\mathrm{Hom}(A,B) = 0$ if there exists an $n \in \mathbf{Z}$ such that $\tau_{\geq n}A = 0$ and $\tau_{\leq n-1}B = 0$. Similarly, the left horizontal maps are injective. The claim follows then by diagram chasing.

(2) Without loss of generality, we can assume that we have a short exact sequence of complexes $0 \longrightarrow C \longrightarrow K \longrightarrow K' \longrightarrow 0$. Applying truncation $\tau_{[q-1,q]}$ yields a diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H^{q-1}(C) \arrow[r] \arrow[d] & H^{q-1}(K) \arrow[r, "0"] \arrow[d] & H^{q-1}(K') \arrow[d] \arrow[dddll] & \\
 & C^{q-1}/B^{q-1}C \arrow[r, "\alpha"'] \arrow[d] & K^{q-1}/B^{q-1}K \arrow[r] \arrow[d] & (K')^{q-1}/B^{q-1}K' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^qC \arrow[r] \arrow[d] & Z^qK \arrow[r] \arrow[d] & Z^qK' \arrow[d] & \\
 & H^q(C) \arrow[r] \arrow[d] & H^q(K) \arrow[r, "0"] \arrow[d] & H^q(K') \arrow[d] & \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

We need to check that the diagram

$$\begin{tikzcd}
{[K^{q-1}/B^{q-1}K \longrightarrow Z^qK]} \arrow[r] \arrow[d] & {[0 \longrightarrow H^qK]} & {[H^{q-1}K' \longrightarrow H^qC]} \arrow[l, "\sim"'] \arrow[d] \\
{[(K')^{q-1}/B^{q-1}K \longrightarrow Z^qK']} & & {[H^{q-1}K' \longrightarrow 0]} \arrow[ll]
\end{tikzcd}$$

commutes in the derived category. For this, it suffices to find a complex L together with a quasi-isomorphism $L \longrightarrow [K^{q-1}/B^{q-1}K \longrightarrow Z^qK]$ and a map $L \longrightarrow [H^{q-1}K' \longrightarrow H^qC]$ such that the diagrams of complexes

$$\begin{tikzcd}
& L \arrow[dl, "\sim"'] \arrow[dr] & \\
{[K^{q-1}/B^{q-1}K \longrightarrow Z^qK]} \arrow[dr] & & {[H^{q-1}K' \longrightarrow H^qC]} \arrow[dl, "\sim"] \\
& {[0 \longrightarrow H^qK]} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& L \arrow[dl, "\sim"'] \arrow[dr] & \\
{[K^{q-1}/B^{q-1}K \longrightarrow Z^qK]} \arrow[d] & & {[H^{q-1}K' \longrightarrow H^qC]} \arrow[d] \\
{[(K')^{q-1}/B^{q-1}(K') \longrightarrow Z^qK']} & & {[H^{q-1}K' \longrightarrow 0]} \arrow[ll]
\end{tikzcd}$$

commute. Let $Z = \ker(K^{q-1}/B^{q-1}K \longrightarrow Z^qK')$ and let

$$L = [C^{q-1}/B^{q-1}C \xrightarrow{\mathrm{id}\times(-\alpha)} C^{q-1}/B^{q-1}C \oplus Z \xrightarrow{d_C \oplus d_K} Z^qC]$$

with Z^qC in degree 1. Maps in the big diagram induce maps $L \longrightarrow [K^{q-1}/B^{q-1}K \longrightarrow Z^qK]$ and $L \longrightarrow [H^{q-1}K' \longrightarrow H^qC]$ and it is easy to check the required assertions. □

5.3.2 Cohomology of procyclic groups

Let Γ be either a free cyclic group or a torsion free procyclic group. In the first case, we consider Γ as a discrete topological group and set $R = \mathbf{Z}$. In the latter case, $\Gamma \simeq \prod_{p \in S} \mathbf{Z}_p$

for a set of primes S (cf. [RZ10, §2.7]), and we set $R=\prod_{p\in S}\mathbf{Z}_p$. Either way, Γ is a free topological R-module of rank 1. For a topological (e.g. discrete) R-module M, we write $M(n)=M\otimes_R\Gamma^{\otimes n}$ if $n\geq 0$, $M(n)=M\otimes_R(\Gamma^\vee)^{\otimes(-n)}$ if $n<0$, where $\Gamma^\vee=\mathrm{Hom}_{R,\mathrm{cont}}(\Gamma,R)$. We have natural identifications $M(n)(m)\simeq M(n+m)$. Any $\gamma\in\Gamma$ defines a morphism $t_\gamma:M\longrightarrow M(1)$ sending m to $m\otimes\gamma$.

Let Λ be a discrete quotient ring of R, i.e. $\mathbf{Z}/N\mathbf{Z}$ for any $N\geq 0$ if $R=\mathbf{Z}$, or $\Lambda=\mathbf{Z}/N\mathbf{Z}$ with ass $N\subseteq S$ if $R=\prod_{p\in S}\mathbf{Z}_p$. Let $\mathscr{C}_{\Gamma,\Lambda}$ denote the category of discrete $\Lambda[\Gamma]$-modules, i.e., Λ-modules equipped with a Γ-action with open stabilizers. It is an abelian category with enough injectives. For a bounded below complex $M\in D^+(\mathscr{C}_{\Gamma,\Lambda})$, we write $H^n(\Gamma,M)$ for the derived functors of the functor $(-)^\Gamma:\mathscr{C}_{\Gamma,\Lambda}\longrightarrow\Lambda-\mathrm{mod}$ of Γ-invariants.

For any object M of $D^+(\mathscr{C}_{\Gamma,\Lambda})$ and any choice of a topological generator $\gamma\in\Gamma$, the total complex of the double complex $[M\xrightarrow{1-\gamma}M]$ (in columns 0 and 1) is canonically quasi-isomorphic to $H^*(\Gamma,M)$ (total derived functor of Γ-invariants). We follow the sign conventions of [Sai03, p. 585-587]. Concretely, this implies that

$$H^q(\Gamma,M)=\frac{\{(a,b)\in M^q\oplus M^{q-1}:da=0,db=(1-\gamma)a\}}{\{(da',(1-\gamma)a'-db'):(a',b')\in M^{q-1}\oplus M^{q-2}\}}. \tag{5.13}$$

In particular, for M an object of $\mathscr{C}_{\Gamma,\Lambda}$, we have

$$H^0(\Gamma,M)=\ker(1-\gamma:M\longrightarrow M)\quad\text{and}\quad H^1(\Gamma,M)=\mathrm{coker}(1-\gamma:M\longrightarrow M). \tag{5.14}$$

Lemma 5.3.2 (cf. [RZ82, §1]). *(1) Let $\theta\in\mathrm{Ext}^1_{\mathscr{C}_{\Gamma,\Lambda}}(\Lambda,\Lambda(1))=H^1(\Gamma,\Lambda(1))$ be the class of the extension*

$$0\longrightarrow\Lambda(1)\xrightarrow{x\mapsto(x,0)}\Lambda(1)\oplus\Lambda\xrightarrow{(x,y)\mapsto y}\Lambda\longrightarrow 0$$

where $\gamma\cdot(x,y)=(x+t_\gamma(y),y)$. The element θ corresponds to the identity map under the identification $H^1(\Gamma,\Lambda(1))\simeq\mathrm{Hom}(\Gamma,\Lambda(1))=\mathrm{Hom}(\Lambda(1),\Lambda(1))$, and corresponds to the the class of $t_\gamma(1)\in\Lambda(1)=\mathrm{coker}(1-\gamma:\Lambda(1)\longrightarrow\Lambda(1))$ under the identification (5.14).

(2) For any $M\in\mathscr{C}_{\Gamma,\Lambda}$, the map $M^\Gamma(-1)=\mathrm{Hom}(\Lambda(1),M)\longrightarrow H^1(\Gamma,M)$ sending $f:\Lambda(1)\longrightarrow M$ to $f(\theta)$ coincides with the map $H^0(\Gamma,M(-1))\longrightarrow H^1(\Gamma,M)$ defined by cup product with θ. Using the identifications (5.14)*, this map corresponds to the composition*

$$\ker(1-\gamma:M(-1)\longrightarrow M(-1))\xrightarrow{\text{incl.}}M(-1)\xrightarrow{t_\gamma}M\xrightarrow{\text{proj.}}\mathrm{coker}(1-\gamma:M\longrightarrow M).$$

In particular, it is an isomorphism if M is a Λ-module with trivial Γ-action.

(3) *For any complex $M \in D^+(\mathscr{C}_{\Gamma,\Lambda})$, the map $\theta : H^{q-1}(\Gamma, M) \longrightarrow H^q(\Gamma, M(1))$ corresponds to the map $(a, b) \mapsto (0, t_\gamma(a))$ under the identification (5.13).*

Proof. (1) The homomorphism $\varphi : \Gamma \longrightarrow \Lambda(1)$ corresponding to the extension satisfies $\gamma(0,1) = (\varphi(\gamma), 1)$, which equals $(t_\gamma(1), 1)$. Assertion (3) follows from [RZ82, Lemma 1.2], and (2) follows from (3). □

Recall that if $F : A \longrightarrow B$ is a left-exact functor between abelian categories, and if A has enough injectives, then for every $M \in D^+(A)$ there is a spectral sequence

$$E_2^{pq} = R^p F(H^q(M)) \quad \Rightarrow \quad H^{p+q}(R^+F(M)).$$

Applying this to the functor $(-)^\Gamma : \mathscr{C}_{\Gamma,\Lambda} \longrightarrow \Lambda-\mathrm{mod}$, we get a spectral sequence, which we call the *Cartan–Leray spectral sequence*

$$E_2^{pq} = H^p(\Gamma, H^q(M)) \quad \Rightarrow \quad H^{p+q}(\Gamma, M). \tag{5.15}$$

Because $E_2^{pq} = 0$ for $p > 1$, this spectral sequence degenerates, yielding for every $q \in \mathbf{Z}$ a short exact sequence

$$0 \longrightarrow H^1(\Gamma, H^{q-1}(M)) \xrightarrow{\delta_{\mathrm{CL}}} H^q(\Gamma, M) \xrightarrow{\pi_{\mathrm{CL}}} H^0(\Gamma, H^q(M)) \longrightarrow 0. \tag{5.16}$$

Proposition 5.3.3. *Let M be an object of $D^+(\mathscr{C}_{\Gamma,\Lambda})$.*

(1) *Using the identifications of (5.13)–(5.14), $\pi_{\mathrm{CL}}(a, b) = [a]$ (which is invariant because $a - \gamma a = d\,b$ is a coboundary) and $\delta_{\mathrm{CL}}(c) = [(0, \overline{c})]$ where $\overline{c} \in Z^{q-1}(M)$ is any lift of c.*

(2) *The diagram*

$$\begin{tikzcd}
H^{q-1}(\Gamma, M) \arrow[r, "\theta"] \arrow[d, "\pi_{\mathrm{CL}}"'] & H^q(\Gamma, M(1)) \\
H^0(\Gamma, H^{q-1}(M)) \arrow[r, "\theta"'] & H^1(\Gamma, H^{q-1}(M)(1)) \arrow[u, "\delta_{\mathrm{CL}}"']
\end{tikzcd}$$

commutes.

(3) *Suppose that $H^{q-1}(M)$ and $H^q(M)$ are trivial Γ-modules for some integer q. Then the sequence*

$$H^{q-1}(\Gamma, M(q-1)) \xrightarrow{\theta} H^q(\Gamma, M(q)) \xrightarrow{\theta} H^{q+1}(\Gamma, M(q+1))$$

is exact.

Proof. Assertion (1) follows from the fact that the spectral sequence (5.15) is the spectral sequence of the double complex $[M \xrightarrow{1-\gamma} M]$. Then (2) follows from (1) by direct calculation, and (3) follows from (2) and Lemma 5.3.2(2). □

Lemma 5.3.4. *Let $\gamma \in \Gamma$ be a topological generator, and let $\gamma' = \gamma^n$ for some $n \geq 1$. Let Γ' be the closure of the subgroup generated by γ'. Then under the identifications (5.14), the restriction map $H^1(\Gamma, M) \longrightarrow H^1(\Gamma', M)$ corresponds to the map*

$$1+\gamma+\ldots+\gamma^{n-1} : M/(1-\gamma) \longrightarrow M/(1-\gamma').$$

In particular, the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(\Gamma, H^{q-1}(M)) \arrow[r, "\delta_{\mathrm{CL}}"] \arrow[d, "1+\gamma+\ldots+\gamma^{n-1}"'] & H^q(\Gamma, M) \arrow[r, "\pi_{\mathrm{CL}}"] \arrow[d, "\mathrm{res}"'] & H^0(\Gamma, H^q(M)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(\Gamma', H^{q-1}(M)) \arrow[r, "\delta_{\mathrm{CL}}"] & H^q(\Gamma', M) \arrow[r, "\pi_{\mathrm{CL}}"] & H^0(\Gamma', H^q(M)) \arrow[r] & 0
\end{tikzcd}$$

commutes.

Proof. This is clear in view of the identity $1-\gamma^n = (1-\gamma)(1+\gamma+\ldots+\gamma^{n-1})$. □

Proposition 5.3.5. *Let $M \in D^+(\mathscr{C}_{\Gamma,\Lambda})$ and $q \in \mathbf{Z}$. Suppose that Γ acts trivially on $H^{q-1}(M)$ and $H^q(M)$, and hence for every $\gamma \in \Gamma$ there is an induced map $L_{1-\gamma}$: $H^q(M) \longrightarrow H^{q-1}(M)[1]$ as in Lemma 5.3.1. Let*

$$E : H^q(M) = H^0(\Gamma, H^q(M)) \longrightarrow H^1(\Gamma, H^{q-1}(M))[1] = H^{q-1}(M)(-1)[1]$$

be the map corresponding to the extension (5.16). Then the diagram

$$\begin{tikzcd}
H^0(\Gamma, H^q(M)) \arrow[r, "E"] \arrow[d, equal] & H^1(\Gamma, H^{q-1}(M))[1] \arrow[r, equal, "\theta"] & H^0(\Gamma, H^{q-1}(M)(-1))[1] \arrow[d, "t_\gamma"] \\
H^q(M) \arrow[r, "L_{1-\gamma}"'] & H^{q-1}(M)[1] \arrow[r, equal] & H^0(\Gamma, H^{q-1}(M))
\end{tikzcd}$$

commutes.

Proof. Suppose first that γ is a topological generator. We apply Lemma 5.3.1(2) to the exact triangle $C \longrightarrow M \xrightarrow{1-\gamma} M \longrightarrow C[1]$ where $C = [M \xrightarrow{1-\gamma} M]$. Note that by the explicit description of (5.13), the map $H^q(C) = H^q(\Gamma, M) \longrightarrow H^q(M) = H^0(\Gamma, H^q(M))$ coincides with π_{CL}. Similarly, the map $H^{q-1}(M) = H^1(\Gamma, H^{q-1}(M)(1)) \longrightarrow H^q(C) = H^q(\Gamma, M)$ coincides with δ_{CL}. This identifies E with $L_{1-\gamma}$ up to twist as desired.

For a general $\gamma \in \Gamma$, note that we can write $\gamma = \gamma_0^n$ where γ_0 is a topological generator and $n \geq 0$. The assertion follows then from the case $n = 1$ and Lemma 5.3.4. □

Remark 5.3.6. Analogous statements hold if we replace $\mathscr{C}_{\Gamma,\Lambda}$ with the category of sheaves of discrete Γ-modules in a topos X. This will be the case of interest in the following section.

5.4 The algebraic case

The proof of the monodromy formula in the algebraic setting is completely analogous. Let k be an algebraically closed field and let $(S, \mathscr{M}_S) = \mathrm{Spec}(\mathbf{N} \longrightarrow k)$ be the standard log point over k. Let $\mathbf{N} \subseteq P \subseteq \mathbf{Q}$ be the monoid of non-negative rational numbers whose denominators are invertible in k. Let $(\overline{S}, \mathscr{M}_{\overline{S}}) = \mathrm{Spec}(P \longrightarrow k)$, and let $\Gamma = \mathrm{Aut}((\overline{S}, \mathscr{M}_{\overline{S}})/(S, \mathscr{M}_S))$ (the logarithmic inertia group of $(S, \mathscr{M}_S)$). We have a natural identification $\Gamma = \hat{\mathbf{Z}}'(1) = \varprojlim_N \mu_N(k)$ [Ill02a, 4.7(a)]. We can identify the topos $(S, \mathscr{M}_S)_{\text{két}}$ with the classifying topos of Γ. Fix an integer $N \in P$ and let $\Lambda = \mathbf{Z}/N\mathbf{Z}$ with $N \in P$. Thus $\Lambda(1) = \mu_N(k)$. Finally, let $\theta \in \Gamma(S, R^1\varepsilon_{S,*}\Lambda) = H^1(\Gamma, \Lambda(1))$ be the canonical generator, where $\varepsilon_S : (S, \mathscr{M}_S)_{\text{két}} \longrightarrow S_{\text{ét}}$ is the projection.

Let $f : (X, \mathscr{M}_X) \longrightarrow (S, \mathscr{M}_S)$ be a smooth and saturated morphism. Let $(\overline{X}, \mathscr{M}_{\overline{X}}) = (X, \mathscr{M}_X) \times_{(S, \mathscr{M}_S)} (\overline{S}, \mathscr{M}_{\overline{S}})$, and let $\overline{\varepsilon} : (\overline{X}, \mathscr{M}_{\overline{X}})_{\text{két}} \longrightarrow X_{\text{ét}}$ be the projection.

Step 1 — Homological Algebra

Let $K = R\overline{\varepsilon}_*\Lambda$, which is a complex of sheaves of continuous Γ-modules on $X_{\text{ét}}$. As we will see shortly, the Γ-action on the cohomology sheaves $H^q(K)$ is trivial for all q, and hence every $\gamma \in \Gamma$ induces a map

$$L_{1-\gamma} : H^q(K) \longrightarrow H^{q-1}(K)[1].$$

Cup product with θ yields a long exact sequence (5.6) which splits into short exact sequences

$$0 \longrightarrow H^{q-1}(K(q-1)) \longrightarrow H^q(\Gamma, K(q)) \longrightarrow H^q(K(q)) \longrightarrow 0. \tag{5.17}$$

Theorem 5.4.1 (cf. Proposition 5.3.5)**.** *Let $\gamma \in \Gamma$. The map $L_{1-\gamma}(q) : H^q(K(q)) \longrightarrow H^{q-1}(K(q))[1]$ corresponds to the pushout of the extension* (5.7) *along the map* $\mathrm{id} \otimes \gamma : H^{q-1}(K(q-1)) \longrightarrow H^{q-1}(K(q-1))(1) = H^{q-1}(K(q))$.

Interlude — Description of $R^q\varepsilon_*\Lambda$ and $R^q\overline{\varepsilon}_*\Lambda$

Recall the definition of the logarithmic Kummer sequence [KN99, Proposition 2.3] of a fs log scheme $(X, \mathcal{M}_X)$. Let N be an integer invertible on X, and let $\Lambda = \mathbf{Z}/N\mathbf{Z}$. The sheaf $\mathcal{M}_X$ extends naturally to a sheaf $\mathcal{M}_X^{\log}$ on $(X, \mathcal{M}_X)_{\text{két}}$. The logarithmic Kummer sequence is the exact sequence

$$0 \longrightarrow \Lambda(1) \longrightarrow \mathcal{M}_X^{\log,\mathrm{gp}} \xrightarrow{N} \mathcal{M}_X^{\log,\mathrm{gp}} \longrightarrow 0.$$

Consider the projection $\varepsilon : (X, \mathcal{M}_X)_{\text{két}} \longrightarrow X_{\text{ét}}$. Applying $R\varepsilon_*$ to the logarithmic Kummer sequence, we get a connecting homomorphism $\delta : \varepsilon_*\mathcal{M}_X^{\log,\mathrm{gp}} \longrightarrow R^1\varepsilon_*\Lambda(1)$. The composition $\mathcal{M}_X \longrightarrow \varepsilon_*\mathcal{M}_X^{\log,\mathrm{gp}} \longrightarrow R^1\varepsilon_*\Lambda(1)$ with the adjunction map $\mathcal{M}_X \longrightarrow \varepsilon_*\mathcal{M}_X^{\log,\mathrm{gp}}$ factors through $\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes \Lambda$ because the N-th power map $\mathcal{O}_X^* \longrightarrow \mathcal{O}_X^*$ is surjective for the étale topology (as N is invertible on X). Let us denote by $\mu : \overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes \Lambda(-1) \longrightarrow R^1\varepsilon_*\Lambda$ the homomorphism we just constructed (after a -1 Tate twist), and by $\mu^q : \bigwedge^q(\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes \Lambda(-1)) \longrightarrow R^q\varepsilon_*\Lambda$ the map induced by μ and cup product.

Theorem 5.4.2 ([KN99, Theorem 2.4], [Ill02a, Theorem 5.2])**.** *The maps*

$$\mu^q : \bigwedge^q(\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes \Lambda(-1)) \longrightarrow R^q\varepsilon_*\Lambda$$

are isomorphisms for all $q \geq 0$.

Now let as assume, as before, that $(X, \mathcal{M}_X)$ is smooth and saturated over the standard log point $(S, \mathcal{M}_S)$ over k. Since Γ acts on $(\overline{X}, \mathcal{M}_{\overline{X}})$ and the map $\overline{\varepsilon} : (\overline{X}, \mathcal{M}_{\overline{X}})_{\text{két}} \longrightarrow X_{\text{ét}}$ is Γ-invariant, the complexes $R\overline{\varepsilon}_*$ are sheaves of continuous Γ-modules on X.

Note that, since f is saturated, $(\overline{X}, \mathcal{M}_{\overline{X}})$ is a saturated log scheme. The map $\overline{X} \longrightarrow X$ is an isomorphism, and hence we can identify X and $\overline{X}$. Moreover, the sequence

$$0 \longrightarrow f^*\overline{\mathcal{M}}_S^{\mathrm{gp}} \otimes \Lambda \longrightarrow \overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes \Lambda \longrightarrow \overline{\mathcal{M}}_{\overline{X}}^{\mathrm{gp}} \otimes \Lambda \longrightarrow 0$$

is exact, so we can identify $\overline{\mathcal{M}}_{\overline{X}}^{\mathrm{gp}} \otimes \Lambda$ with $\overline{\mathcal{M}}_{X/S}^{\mathrm{gp}} \otimes \Lambda$. Even though $(\overline{X} = X, \mathcal{M}_{\overline{X}})$ is not an fs log scheme, it is a limit of the fs log schemes $(X_N = X, \mathcal{M}_{X_N}) = (X, \mathcal{M}_X) \times_{(S, \mathcal{M}_S)}$

$(S_N, \mathscr{M}_{S_N})$ where $(S, \mathscr{M}_S) = \operatorname{Spec}(\frac{1}{N}\mathbf{N} \longrightarrow k)$ for $N \in P$, and hence statements analogous to those discussed above hold for $(\overline{X}, \mathscr{M}_{\overline{X}})$. More precisely, we have a morphism of topoi

$$\overline{\varepsilon} : (\overline{X}, \mathscr{M}_{\overline{X}})_{\text{két}} \longrightarrow \overline{X}_{\text{ét}} = X_{\text{ét}}.$$

The logarithmic Kummer sequence

$$0 \longrightarrow \Lambda(1) \longrightarrow \mathscr{M}_{\overline{X}}^{\log,\mathrm{gp}} \xrightarrow{N} \mathscr{M}_{\overline{X}}^{\log,\mathrm{gp}} \longrightarrow 0$$

is exact, giving a connecting homomorphism $\overline{\varepsilon}_* \mathscr{M}_{\overline{X}}^{\log,\mathrm{gp}} \to R^1\overline{\varepsilon}^*\Lambda(1)$. As before, the composition $\mathscr{M}_{\overline{X}}^{\mathrm{gp}} \to \overline{\varepsilon}_* \mathscr{M}_{\overline{X}}^{\log,\mathrm{gp}} \longrightarrow R^1\overline{\varepsilon}^*\Lambda(1)$ factors through $\overline{\mathscr{M}}_{\overline{X}}^{\mathrm{gp}} \otimes \Lambda = \overline{\mathscr{M}}_{X/S}^{\mathrm{gp}} \otimes \Lambda$. We denote by $\overline{\mu} : \overline{\mathscr{M}}_{X/S}^{\mathrm{gp}} \otimes \Lambda(-1) \to R^1\overline{\varepsilon}_*\Lambda$ the constructed homomorphism (after a -1 Tate twist), and by $\overline{\mu}^q : \bigwedge^q \overline{\mathscr{M}}_{X/S}^{\mathrm{gp}} \to R^q\overline{\varepsilon}_*\Lambda$ the map induced by $\overline{\mu}$ and cup product. Again we have

Theorem 5.4.3. *The maps $\overline{\mu}^q : \bigwedge^q \overline{\mathscr{M}}_{X/S}^{\mathrm{gp}} \to R^q\overline{\varepsilon}_*\Lambda$ are isomorphisms for all $q \geq 0$. In particular, the action of Γ on $R^q\overline{\varepsilon}_*\Lambda$ is trivial for all $q \geq 0$.*

Step 2 — The end of the proof

We have $R\varepsilon_*\Lambda = H^*(\Gamma, R\overline{\varepsilon}_*\Lambda)$, so $R^q\varepsilon_*\Lambda = H^q(\Gamma, R\overline{\varepsilon}_*\Lambda)$. Using this identification, the map $\wedge\theta : H^q(\Gamma, R\overline{\varepsilon}_*\Lambda(q)) \to H^{q+1}(\Gamma, R\overline{\varepsilon}_*\Lambda(q+1))$ corresponds to cup product with the (image in $R^1\varepsilon_*\Lambda(1)$ of the) generator θ of $R^q\varepsilon_{S*}\Lambda(1) = H^1(\Gamma, \Lambda(1))$. Therefore, the extension (5.7) (which, after Theorem 5.2.1, determines $L_{1-\gamma}$) corresponds to the extension

$$0 \longrightarrow R^{q-1}\overline{\varepsilon}_*\Lambda(q-1) \xrightarrow{a^q} R^q\varepsilon_*\Lambda(q) \xrightarrow{b^q} R^q\overline{\varepsilon}_*\Lambda(q) \longrightarrow 0 \tag{5.18}$$

where the map $a^q : R^{q-1}\overline{\varepsilon}_*\Lambda(q-1) \longrightarrow R^q\varepsilon_*\Lambda(q)$ is "lift to $R^{q-1}\varepsilon_*\Lambda(q-1)$ and take cup product with θ" and the map $b^q : R^q\varepsilon_*\Lambda(q) \longrightarrow R^q\overline{\varepsilon}_*\Lambda(q)$ is deduced from pullback along $(\overline{X}, \mathscr{M}_{\overline{X}})_{\text{két}} \to (X, \mathscr{M}_X)_{\text{két}}$.

Let t be the canonical section of $\mathscr{M}_S$ (whose image in $\mathscr{M}_X$ equals f by definition). Let $\mu_S : \overline{\mathscr{M}}_S^{\mathrm{gp}} \otimes \Lambda(-1) \to R^1\varepsilon_{S*}\Lambda$ be the map in Theorem 5.4.2 for S in place of X. Then the Tate twist $\mu_S(1) : \overline{\mathscr{M}}_S^{\mathrm{gp}} \otimes \Lambda \to R^1\varepsilon_{S*}\Lambda$ sends t to θ. Since the maps $\overline{\mu}^q$, μ^q, and a^q are defined by cup product, we easily deduce that the diagrams

$$\begin{tikzcd} 0 \arrow[r] & (\bigwedge^{q-1}\overline{\mathscr{M}}^{\mathrm{gp}}_{X/S}\otimes\Lambda(-1))(-1) \arrow[r, "\wedge t"] \arrow[d, "\overline{\mu}^{q-1}"'] & \bigwedge^{q}\overline{\mathscr{M}}^{\mathrm{gp}}_{X}\otimes\Lambda(-1) \arrow[r] \arrow[d, "\mu^q"] & \bigwedge^{q}\overline{\mathscr{M}}^{\mathrm{gp}}_{X/S}\otimes\Lambda(-1) \arrow[r] \arrow[d, "\overline{\mu}^q"] & 0 \\ 0 \arrow[r] & R^{q-1}\overline{\varepsilon}_*\Lambda(-1) \arrow[r, "a^q"'] & R^q\varepsilon_*\Lambda \arrow[r, "b^q"'] & R^q\overline{\varepsilon}_*\Lambda \arrow[r] & 0, \end{tikzcd} \tag{5.19}$$

where the bottom row is an appropriate Tate twist of (5.18), commute. Let $E : \bigwedge^{q}\overline{\mathscr{M}}^{\mathrm{gp}}_{X/S}\otimes\Lambda(-1)\longrightarrow(\bigwedge^{q-1}\overline{\mathscr{M}}^{\mathrm{gp}}_{X/S}\otimes\Lambda(-1))(-1)[1]$ be the map corresponding to the class of the top row extension (5.19). We have proved:

Theorem 5.4.4. *For every $q\geq 0$ and for every $\gamma\in\Gamma$, the diagram*

$$\begin{tikzcd} \bigwedge^{q}(\overline{\mathscr{M}}^{\mathrm{gp}}_{X/S}\otimes\Lambda(-1)) \arrow[r, "E"] \arrow[d, "\overline{\mu}^q"'] & \bigwedge^{q-1}(\overline{\mathscr{M}}^{\mathrm{gp}}_{X/S}\otimes\Lambda(-1))(-1)[1] \arrow[r, "\gamma"] & \bigwedge^{q-1}(\overline{\mathscr{M}}^{\mathrm{gp}}_{X/S}\otimes\Lambda(-1))[1] \arrow[d, "\overline{\mu}^{q-1}"] \\ R^q\overline{\varepsilon}_*\Lambda \arrow[rr, "L_{1-\gamma}"'] & & R^{q-1}\overline{\varepsilon}_*\Lambda[1]. \end{tikzcd}$$

commutes.

5.5 Variants and applications

5.5.1 The case over a trait

Let $(S,\mathscr{M}_S)$ be a henselian trait as in §1.4. We do not have to assume that the fraction field K has characteristic zero[1]. Let $\overline{S}=\operatorname{Spec}\overline{V}$ where $\overline{V}$ is the integral closure of V in $\overline{K}$, and let $\mathscr{M}_{\overline{S}}$ be the log structure on $\overline{S}$ induced by the open immersion $\overline{\eta}\hookrightarrow\overline{S}$. Let $G=\mathrm{Gal}(\overline{\eta}/\eta)$, let $I\subseteq G$ be the inertia group, $P\subseteq I$ the wild inertia subgroup. Let $G^t=G/P$ be the tame quotient of G. As explained in [Ill02a, 8.1], G acts on the log point $(\overline{s},\mathscr{M}_{\overline{s}})$ (where $\mathscr{M}_{\overline{s}}=\mathscr{M}_{\overline{S}}|_{\overline{s}}$) through G^t, and the tame inertia $I^t=I/P$ via the tame character $t : I^t\longrightarrow\hat{\mathbf{Z}}'(1)(\overline{k})$. Thus I^t is identified with the log inertia group Γ of $(s,\mathscr{M}_s)$ using the log geometric point $(\overline{s},\mathscr{M}_{\overline{s}})\longrightarrow(s,\mathscr{M}_s)$.

Let $f : (X,\mathscr{M}_X)\longrightarrow(S,\mathscr{M}_S)$ be a smooth and saturated morphism. Let $X^\circ=(X,\mathscr{M}_X)_{\mathrm{tr}}$, let $\overline{X}=X\times_S\overline{S}$, and let $\overline{\imath} : X_{\overline{s}}\longrightarrow\overline{X}$, $\overline{\jmath} : X^\circ_{\overline{\eta}}\longrightarrow\overline{X}$ be the natural maps. Let

[1] In fact, the setup and notation here is (almost) taken from [Ill02a, §8]

N be an integer invertible on S, and let $\Lambda = \mathbf{Z}/N\mathbf{Z}$. Our goal is to describe the action of I on the nearby cycle sheaves $R\Psi^\circ\Lambda := \bar{\imath}^* R\bar{\jmath}_*\Lambda$. We use the notation Ψ° to indicate that the log structure on X might not be vertical.

Let $(\overline{X}, \mathcal{M}_{\overline{X}}) = (X, \mathcal{M}_X) \times_{(S,\mathcal{M}_S)} (\overline{S}, \mathcal{M}_{\overline{S}})$. Because f is saturated, we have $\overline{X} = X \times_S \overline{S}$. Consider the following commutative diagram of topoi

$$\begin{tikzcd}
(X_{\bar\eta}, \mathcal{M}_{X_{\bar\eta}})_{\text{két}} \arrow[r, "\bar{\jmath}^{\log}"] & (\overline{X}, \mathcal{M}_{\overline{X}}) \arrow[d, "\bar\varepsilon"] & (X_{\bar s}, \mathcal{M}_{X_{\bar s}}) \arrow[l, "\bar{\imath}^{\log}"'] \arrow[d, "\bar\varepsilon"] \\
(X^\circ_{\bar\eta})_{\text{ét}} \arrow[u, "u"] \arrow[r, "\bar\jmath"'] & \overline{X}_{\text{ét}} & (X_{\bar s})_{\text{ét}}. \arrow[l, "\bar\imath"]
\end{tikzcd}$$

Here $u : (X^\circ_{\bar\eta})_{\text{ét}} = (X^\circ_{\bar\eta}, \mathcal{M}_{X^\circ_{\bar\eta}})_{\text{két}} \to (X_{\bar\eta}, \mathcal{M}_{X_{\bar\eta}})$ is induced by the inclusion j.

The purity theorem [Fuj02] (cf. [Nak98, 2.0.1, 2.0.5]) implies that $Ru_*\Lambda = \Lambda$. Nakayama's theorem [Nak98, 3.2] implies that $R\Psi^{\log}\Lambda := \bar{\imath}^{\log *} R\bar{\jmath}^{\log}_*\Lambda = \Lambda$. Furthermore, we have $R\Psi^\circ\Lambda = \bar{\imath}^* R\bar{\jmath}_*\Lambda = R\bar\varepsilon_* R\Psi^{\log}\Lambda = R\bar\varepsilon_*\Lambda$ [Ill02a, 8.2.3]. In particular, the wild inertia P acts trivially on $R\Psi^\circ\Lambda$, and I acts trivially on $R^q\Psi^\circ\Lambda$. We thus get for $g \in I$ maps $L_{g-1} : R^q\Psi^\circ\Lambda \to R^{q-1}\Psi^\circ\Lambda[1]$, which equal the maps $L_{1-\gamma} : R^q\bar\varepsilon_*\Lambda \to R^{q-1}\bar\varepsilon_*\Lambda[1]$ (described by Theorem 5.4.4 applied to $(X_{\bar s}, \mathcal{M}_{X_{\bar s}})/(\bar s, \mathcal{M}_{\bar s})$) under the above identification, where $\gamma = t(g) \in \Gamma$ is the image of g under the tame character $t : I \longrightarrow \hat{\mathbf{Z}}'(1)(\bar k) = \Gamma$.

Since $(X, \mathcal{M}_X)$ is regular, we have $\mathcal{M}_X = \mathcal{M}_{X^\circ/X}$, and hence $\mathcal{M}_X^{\text{gp}} = j_*\mathcal{O}^*_{X^\circ}$ where $j : X^\circ \hookrightarrow X$ is the inclusion. Similarly, $\mathcal{M}^{\text{gp}}_{\overline{X}} = \bar\jmath_*\mathcal{O}^*_{X^\circ_{\bar\eta}}$. The Kummer sequence

$$0 \longrightarrow \Lambda(1) \longrightarrow \mathcal{O}^*_{X^\circ_{\bar\eta}} \xrightarrow{N} \mathcal{O}^*_{X^\circ_{\bar\eta}} \longrightarrow 0$$

induces, upon applying $\bar\jmath_*$, a connecting homomorphism $\overline{\mathcal{M}}^{\text{gp}}_{\overline{X}} \to R^1\bar\jmath_*\Lambda$, and therefore maps $\bigwedge^q \overline{\mathcal{M}}^{\text{gp}}_{X/S} \otimes \Lambda(-1) \longrightarrow R^q\bar\jmath_*\Lambda$. Restricting to the special fiber, we get maps $\bar\mu^q : \bigwedge^q(\overline{\mathcal{M}}^{\text{gp}}_{X_{\bar s}/\bar s} \otimes \Lambda(-1) \to R^q\Psi^\circ\Lambda$, which agree with $\bar\mu^q : \bigwedge^q \overline{\mathcal{M}}^{\text{gp}}_{X_{\bar s}/\bar s} \otimes \Lambda(-1) \longrightarrow R^q\bar\varepsilon_*\Lambda$ as defined in the previous section. Using Theorem 5.4.4, we deduce

Theorem 5.5.1. *For every $q \geq 0$ and for every $g \in I$, the diagram*

$$\begin{tikzcd}
\bigwedge^q(\overline{\mathcal{M}}^{\text{gp}}_{X_{\bar s}/\bar s} \otimes \Lambda(-1)) \arrow[r, "E"] \arrow[d, "\bar\mu^q"'] & \bigwedge^{q-1}(\overline{\mathcal{M}}^{\text{gp}}_{X_{\bar s}/\bar s} \otimes \Lambda(-1))(-1)[1] \arrow[r, "t(g)"] & \bigwedge^{q-1}(\overline{\mathcal{M}}^{\text{gp}}_{X_{\bar s}/\bar s} \otimes \Lambda(-1))[1] \arrow[d, "\bar\mu^{q-1}"] \\
R^q\Psi^\circ\Lambda \arrow[rr, "L_{g-1}"'] & & R^{q-1}\Psi^\circ\Lambda[1].
\end{tikzcd}$$

commutes, where $t : G \longrightarrow \hat{\mathbf{Z}}'(1)(\overline{k})$ *denotes the tame character.*

5.5.2 The case over a disc

Let $(S, \mathcal{M}_S)$ be a disc around $0 \in \mathbf{C}$ of radius $\delta > 0$, with the log structure induced by the open immersion $S^* = S \setminus \{0\} \hookrightarrow S$. Let $\widetilde{S} = \{\mathrm{Re}(z) > \log \delta\}$, $\exp : \widetilde{S} \longrightarrow S^*$, a universal cover of S^* (identifying $\pi_1(S^*)$ with $\mathbf{Z}(1)$). Let $(0, \mathcal{M}_0)$ be the center of S, considered as a standard log point

Let $f : (X, \mathcal{M}_X) \longrightarrow (S, \mathcal{M}_S)$ be a log smooth and saturated morphism of log complex analytic spaces. Let $(X_0, \mathcal{M}_{X_0})$ the log special fiber $(f^{-1}(0), \mathcal{M}_X|_{X_0})$ with exact closed immersion $i : (X_0, \mathcal{M}_{X_0}) \longrightarrow (X, \mathcal{M}_X)$, and let $X^\circ \subseteq X$ be the biggest subset where $\mathcal{M}_X = \mathcal{O}_X^*$. Finally, define $\widetilde{X}^\circ = X^\circ \times_S \widetilde{S}$ with $\overline{j} : \widetilde{X}^\circ \longrightarrow X$ the natural map. We will be interested in the complex $R\Psi^\circ \mathbf{Z} = i^* R\overline{j}_* \mathbf{Z}$ of sheaves of $\mathbf{Z}(1)$-modules on X_0. As before, there are isomorphisms $\overline{\mu}^q : \bigwedge^q \mathcal{M}^{\mathrm{gp}}_{X_0/0} \otimes \mathbf{Z}(-1) \longrightarrow R^q \Psi^\circ \mathbf{Z}$, whose construction implies that the action of $\mathbf{Z}(1)$ on $R^q \Psi^\circ \mathbf{Z}$ is trivial, and hence once again we get for every $\gamma \in \mathbf{Z}(1)$ maps $L_{1-\gamma} : R^q \Psi^\circ \mathbf{Z} \longrightarrow R^{q-1} \Psi^\circ \mathbf{Z}[1]$.

In the same spirit as before, using Theorem 5.1.2, we obtain

Theorem 5.5.2. *For every* $q \geq 0$ *and for every* $\gamma \in \mathbf{Z}(1)$*, the diagram*

$$\begin{tikzcd}
\bigwedge^q(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0/0} \otimes \mathbf{Z}(-1))(-1) \arrow[r, "E"] \arrow[d, "\overline{\mu}^q"'] & \bigwedge^{q-1}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0/0} \otimes \mathbf{Z}(-1))(-1)[1] \arrow[r, "\gamma"] & \bigwedge^{q-1}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0/0} \otimes \mathbf{Z}(-1))[1] \arrow[d, "\overline{\mu}^{q-1}"] \\
R^q \Psi^\circ \mathbf{Z} \arrow[rr, "L_{1-\gamma}"'] & & R^{q-1} \Psi^\circ \mathbf{Z}[1].
\end{tikzcd}$$

commutes.

Example 5.5.3. This completes our treatment of the Dwork family of elliptic curves in Example 1.1.9. The map $\overline{\alpha}$ appearing there equals the map L_{1-T} up to sign.

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