Problem Set 9

due Jan 10, 2021

In the following exercises, $\mathcal{O}$ is a complete discrete valuation ring with uniformizer t, residue field k, and fraction field K.

Problem 1. Give an example of a diagram $\mathscr{X} \to \mathscr{S} \leftarrow \mathscr{Y}$ of admissible $\mathcal{O}$-formal schemes such that the fiber product $\mathscr{Z} = \mathscr{X} \times_{\mathscr{S}} \mathscr{Y}$ in the category of $\mathcal{O}$-formal schemes is not admissible. Compute $\mathscr{Z}_{\text{ad}}$.

Hint: An example was featured at the beginning of Lecture 17. To simplify it further, you can try to make $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{S}$ affine.

Problem 2. Let X be (a) the affine line $\mathbf{A}^1_{\mathcal{O}}$ with doubled "zero section" $V(x)$, or (b) the affine line $\mathbf{A}^1_{\mathcal{O}}$ with doubled "origin in the special fiber" $V(x,t)$, where x is the coordinate on $\mathbf{A}^1_{\mathcal{O}}$. In both cases, compute the canonical map of rigid-analytic spaces

$$(\widehat{X})_{\text{rig}} \to (X_K)^{\text{an}}$$

and check that it is not an open immersion.

Problem 3. Let $\mathscr{X}$ be a formal scheme locally of finite type over $\mathcal{O}$, let X_0 be its special fiber (a scheme locally of finite type over k), let $X = \mathscr{X}_{\text{rig}}$ be its rigid-analytic generic fiber, and let $\text{sp}\colon X \to \mathscr{X}$ be the specialization map. Let Z_i ($i \in I$) be the irreducible components of $|X_0|$. Show that the tubes

$$]Z_i[= \text{sp}^{-1}(Z_i) \subseteq X \quad (i \in I)$$

(where we identify $|\mathscr{X}| = |X_0|$) form an admissible cover of X.

Problem 4. Construct a flat lifting X_1 of $X_0 = \mathbf{A}^2_k \setminus 0$ over $k[[t]]/(t^2)$ for which the restriction map $\Gamma(X_1, \mathcal{O}_{X_1}) \to \Gamma(X_0, \mathcal{O}_{X_0}) = k[x,y]$ is not surjective.

Hint: Use the standard covering by two affine opens.

Problem 5. Let $X = \mathbf{A}^1_{\mathcal{O}}$ with coordinate x and let $X' \to X$ be the blowup at the "origin of the special fiber," defined by the ideal (t,x). Show that the induced morphism of rigid-analytic generic fibers of formal completions

$$\widehat{X'}_{\text{rig}} \to \widehat{X}_{\text{rig}}$$

is an isomorphism. (This is a basic example of an admissible blowup.)