Problem Set 8

due Dec 20, 2020

Problem 1. Recall *Jacobi triple product formula* in the form from the lecture:

$$\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{n(n+1)}{2}} w^n = (1-w^{-1})\prod_{m\geq 1}(1+q^m)(1-wq^m)(1-w^{-1}q^m).$$

Here $w, q \in K^\times$ and $|q| < 1$ in some non-Archimedean field K. Prove the weaker version: there exists a constant $C(q)$ depending on q such that

$$\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{n(n+1)}{2}} w^n = C(q)\cdot(1-w^{-1})\prod_{m\geq 1}(1-wq^m)(1-w^{-1}q^m)$$

for every $w \in K^\times$.

Problem 2. Let $f(q) = q^{-1} + \sum_{n\geq 0} a_n q^n \in K((t))$ be a Laurent series with $|a_n| \leq 1$. Show that f defines a bijection between the sets $\{0 < |q| < 1\}$ and $\{|w| > 1\}$.

Problem 3. Let $Y = \mathbf{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}$ be a Tate curve. Prove that every endomorphism of Y lifts to an endomorphism $\mathbf{G}_m^{\mathrm{an}}$. Conclude that $\mathrm{End}(Y) \simeq \mathbf{Z}$.

Problem 4. Let $Y = \mathbf{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}$ be a Tate curve. For every $n \geq 1$, compute the order of the n-torsion subgroup $Y(\overline{K})[n]$.

Problem 5. Let k be an algebraically closed field and let $\mathscr{B}$ be the category of finitely generated field extensions K of k. Let $\mathscr{P}$ denote the category of projective varieties over k and dominant maps, and let $W \subseteq \mathscr{P}$ be the subcategory consisting of all non-trivial blow-up maps $\pi\colon X' \to X \in \mathscr{P}$. Prove that W admits calculus of right fractions and that the association $X \mapsto K(X)$ induces an equivalence of categories

$$\mathscr{P}[W^{-1}] \xrightarrow{\sim} \mathscr{B}^{\mathrm{op}}.$$