Problem Set 6

due Dec 6, 2020

Problem 1. Let $D = \mathrm{Sp}\, K\langle X\rangle$ and let $o \in D$ be the origin. Prove that the local ring $\mathscr{O}_{D,o}$ is henselian.

Problem 2. Suppose that K is algebraically closed but not spherically complete. Show that there exists a descending sequence of open discs $D_n^\circ \subseteq D = \mathrm{Sp}\, K\langle X\rangle$ with empty intersection. Let $U_n = D \setminus D_n^\circ$, which is an increasing sequence of affinoid subdomains of D covering D. Show that $\{U_n\}$ is not an admissible cover of D, and that there is a unique structure of a rigid-analytic space on D with $\{U_n\}$ an admissible open cover. If D' is the resulting space, show that the identity map $D' \to D$ is a morphism of rigid-analytic spaces.

Hint: Since this example is quite puzzling, here is a toy example analog. One can define a natural "admissible topology" on $\mathbf{Q}\setminus\{\sqrt{2}\} = \mathbf{Q}$ by considering only closed rational intervals $[a,b]_{\mathbf{Q}}$ with $\sqrt{2} \notin [a,b]$. The resulting map of G-topological spaces $\mathbf{Q}\setminus\{\sqrt{2}\} \to \mathbf{Q}$ is a continuous bijection but not an isomorphism.

Problem 3. A rigid-analytic space X is called *quasi-compact* if every admissible cover admits a finite subcover, and *quasi-separated* if the intersection of two quasi-compact admissible opens is quasi-compact. Prove that X is quasi-compact if and only if it admits an finite admissible cover by affinoids, and that it is quasi-separated if and only if the intersection of two affinoid opens admits a finite admissible cover by affinoids.

Problem 4. Let $D = \mathrm{Sp}\, K\langle X\rangle$. Rigorously construct a rigid-analytic space "D with doubled W," where (1) $W =$ the origin, (2) $W = \{|X| < 1\}$, (3) $W = \{|X| \le |t|\}$. Which of those spaces are quasi-separated?

Problem 5. Let X be a G-topological space satisfying (G_0), (G_1), and (G_2), and let Γ be a group acting freely and continuously on X (meaning that the maps $\gamma\colon X \to X$ are continuous maps of G-topological spaces for all $\gamma \in \Gamma$). We call this action *properly discontinuous* if X admits an admissible cover of the form $\{\gamma\cdot U_i\}_{i\in I,\gamma\in\Gamma}$ with $\gamma\cdot U_i \cap U_i = \emptyset$ for $\gamma \neq e$ and such that the sets $\bigcup_{\gamma\in\Gamma}\gamma\cdot U_i$ are admissible for all $i \in I$.

Hint: Prove the case of topological spaces first.

(a) Show that if the action of Γ on X as above is properly discontinuous, then there exists a natural structure of a G-topological space on the orbit space $Y = X/\Gamma$ satisfying (G_0), (G_1), and (G_2) and such that $\pi\colon X \to Y$ is continuous.

(b) A Γ-*equivariant sheaf* on X is a sheaf $\mathscr{F}$ endowed with isomorphisms $u_\gamma\colon \gamma^*\mathscr{F} \to \mathscr{F}$ for which the following diagrams commute for all $\gamma, \delta \in \Gamma$:

$$\begin{array}{ccc} (\gamma\delta)^*\mathscr{F} & \xrightarrow{u_{\gamma\delta}} & \mathscr{F} \\ \Vert & & \uparrow{\scriptstyle u_\delta} \\ \delta^*(\gamma^*\mathscr{F}) & \xrightarrow[\delta^*(u_\gamma)]{} & \delta^*\mathscr{F}. \end{array}$$

With assumptions and notation as in (a), construct an equivalence of categories between sheaves on Y and Γ-equivariant sheaves on X.

(c) Let $q \in K$ with $0 < |q| < 1$. Show that the action of $q^{\mathbf{Z}}$ on $\mathbf{A}_K^{n,\mathrm{an}} \setminus 0$ by rescaling the coordinates is properly discontinuous.