Problem Set 5

due November 29, 2020

Problem 1. Let $\varphi\colon A \to B$ be a K-algebra homomorphism between affinoid K-algebras. Suppose that there exists a surjection

$$\beta\colon K\langle X_1,\dots,X_r\rangle \to B, \quad \beta(X_i) = b_i$$

and powerbounded elements $a_1,\dots,a_r \in A$ such that

$$|b_i - \varphi(a_i)|_\beta < 1 \quad \text{for } i = 1,\dots,r.$$

Prove that φ is surjective. Give an example showing that the assumption that the a_i are powerbounded is necessary.

Problem 2. Prove that every covering of $\mathrm{Sp} A$ by Zariski open subsets is admissible.

Problem 3 (Inadmissible open)**.** Consider the following open subset of $X = \mathrm{Sp} K\langle X, Y\rangle$:

Hint: Use Theorem 6.4.1.

$$U = \{|y| = 1\} \cup \bigcup_{n\geq 1} \{|x| \leq |t|^n,\, |y| \leq |t|^{1/n}\}.$$

Prove that U is not an admissible open.

Problem 4 (Admissible sheaf with zero stalks)**.** Let $D = \mathrm{Sp} K\langle X\rangle$ be the unit disc over an algebraically closed non-Archimedean field K. For an affinoid subdomain $U \subseteq D$, we say that U is *huge* if U contains the complement of finitely many open discs of radii ≤ 1. We set

Hint: Use the classification of affinoid subdomains of D given in Theorem 9.7.2/2 in BGR.

$$\mathscr{F}(U) = \begin{cases} \mathbf{Z} & \text{if } U \text{ is huge} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\mathscr{F}$ is a sheaf for the admissible topology. Note: $\mathscr{F} \neq 0$ but $\mathscr{F}_x = 0$ for every $x \in D$.

Problem 5 (The Riemann–Zariski space is quasi-compact)**.** Let $k \subseteq K$ be a field extension and let $\mathbf{ZR}(K/k)$ be the space of all valuation subrings of K containing k, with topology generated by the sets

Hint: Embed the set $\mathbf{ZR}(K/k)$ into $2^{K^\times}$. Show that with the induced topology $\mathbf{ZR}(K/k)$ becomes compact Hausdorff. Compare this topology with the given topology on $\mathbf{ZR}(K/k)$.

$$X(f) = \{\mathscr{O} : f \in \mathscr{O}\}, \quad f \in K^\times.$$

Show that $\mathbf{ZR}(K/k)$ is quasi-compact.

Here is the relevant excerpt of the notes needed for Problem 3 in case I do not manage to upload the new version in time.

6.4 Affinoid neighborhoods of Zariski closed subsets

The following result is easier to prove using other approaches to rigid geometry, such as Berkovich theory or formal schemes (see [1, §5.2]). We will need it in one of the homework problems.

A subset $Y \subseteq X$ of $X = \mathrm{Sp} A$ is *Zariski closed* if it is closed in the topology induced by the inclusion $X \subseteq \mathrm{Spec} A$. Equivalently, there exists an ideal $I \subseteq A$ such that Y is the image of $\mathrm{Sp} A/I \to X$. As in algebraic geometry, there is an inclusion-reversing bijection between Zariski closed subsets of X and radical ideals of A.

Theorem 6.4.1. *Let $X = \mathrm{Sp} A$ for an affinoid algebra A and let $Y \subseteq X$ be a Zariski closed subset cut out by an ideal $I = (f_1, \dots, f_r) \subseteq A$. Let $U \subseteq X$ be an affinoid subdomain containing Y. Then there exists a $\varepsilon > 0$ such that U contains the open subset*

$$\{|f_i(x)| \leq \varepsilon,\ i = 1, \dots, r\}.$$

Note that if $Y = \{f_i(x) = 0,\ i = 1, \dots, r\}$ is a closed subset of a compact Hausdorff space X cut out by continuous functions $f_1, \dots, f_r : X \to \mathbf{R}$, then the sets $\{|f_i(x)| < \varepsilon,\ i = 1, \dots, r\}$ form a basis of open neighborhoods of Y in X.

See [2, Exercise 4.1.8] for a complicated proof without using adic spaces.

Proof. Write $U = \mathrm{Sp} A_U$. We will work with the associated affinoid adic spaces X^{ad}, U^{ad}, and Y^{ad}. We may assume that the f_i are powerbounded. If

$$f : (f_1, \dots, f_r) : X^{\mathrm{ad}} \to (\mathbf{D}_K^r)^{\mathrm{ad}},$$

then $Y^{\mathrm{ad}} = f^{-1}(0)$. The map $U^{\mathrm{ad}} \to X^{\mathrm{ad}}$ is an open immersion, and hence we treat U^{ad} as an open subset of X^{ad}. Let $W = X^{\mathrm{ad}} \setminus U^{\mathrm{ad}}$, which is a closed subset of X^{ad} and hence it is quasi-compact because X^{ad} is. Consider $Z = f(W) \subseteq (\mathbf{D}_K^r)^{\mathrm{ad}}$, which is again quasi-compact and does not contain the classical point 0. Since $(\mathbf{D}_K^r)^{\mathrm{ad}}$ is a coherent valuative space, every two points x, y without a common generization admit disjoint open neighborhoods [3, 0 2.3.18(2)]. Moreover, the rational opens

$$U_n = \{|X_i| \leq |t^n| \, i = 1, \dots, r\}$$

form a basis of open neighborhoods of 0. For every $z \in Z$ we find an open neighborhood V_z of z in $(\mathbf{D}_K^r)^{\mathrm{ad}}$ and an integer n_z such that $U_{n_z} \cap V_z = \emptyset$. Since Z is quasi-compact, finitely many of the V_z cover Z, and then the intersection of the corresponding U_{n_z} produces an n such that $U_n \cap Z = \emptyset$. Then $f^{-1}(U_n) \subseteq U^{\mathrm{ad}}$ and hence $\{x \in X : |f_i(x)| \leq |t^n|\} \subseteq U$. $\square$

References

[1] Brian Conrad. Several approaches to non-Archimedean geometry. In *p-adic geometry*, volume 45 of *Univ. Lecture Ser.*, pages 9–63. Amer. Math. Soc., Providence, RI, 2008.

[2] Jean Fresnel and Marius van der Put. *Rigid analytic geometry and its applications*, volume 218 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2004.

[3] Kazuhiro Fujiwara and Fumiharu Kato. *Foundations of rigid geometry. I.* EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2018.