Problem Set 4

due November 18, 2020

Affinoid algebras

Problem 1. Let A be an affinoid K-algebra and let $a \in A$. Show that $|a|_{\sup} < 1$ if and only if $\lim a^n = 0$. (The latter means that $\lim |a^n|_\alpha = 0$ for some/every residue norm $|\cdot|_\alpha$.)

Hint: Use the characterization of a such that $|a|_{\sup} \leq 1$ proved in the lecture. Try to reduce to this by rescaling.

Sites

Problem 2. Construct an equivalence of categories $\mathrm{Sh}(\mathbf{R}) \simeq \mathrm{Sh}^{\mathrm{adm}}(\mathbf{Q})$.

Problem 3 (Sheaves on a base, site version)**.** Let $\mathscr{C}$ be a site and let $\mathscr{C}_0 \subseteq \mathscr{C}$ be a full subcategory closed under fiber products. Suppose that every object $c \in \operatorname{ob}\mathscr{C}$ admits a covering family $\{c_\alpha \to c\}_{\alpha\in I}$ with every $c_\alpha \in \operatorname{ob}\mathscr{C}_0$. Endow $\mathscr{C}_0$ with the induced topology: a family $\{c_\alpha \to c\}$ is a covering family if it is a covering family in $\mathscr{C}$. Prove that the inclusion functor induces an equivalence $\mathrm{Sh}(\mathscr{C}) \simeq \mathrm{Sh}(\mathscr{C}_0)$.

Hint: Compare with [Vakil, Theorem 2.5.1]. Its proof uses stalks at points, which you cannot do here, so you need a different argument.

If you really get stuck, try [SGA4 Vol. I Exposé III, Théorème 4.1]. You may also find the Stacks Project helpful.

Blowing up

For the following exercises, it will be helpful to brush up on blow-ups, e.g. [Hartshorne, Chapter II 7, pp. 160–169] and on the valuative criteria of separatedness and properness [Chapter II 4].

Problem 4. Let X be a Noetherian scheme.

(a) Let $Y, Z \subseteq X$ be closed subschemes and let $X' = \mathrm{Bl}_{Y\cap Z} X$ be the blow-up of their intersection. Prove that the strict transforms $\tilde{Y}$, $\tilde{Z}$ of Y and Z in X' are disjoint.

= [Hartshorne, Exercise II 7.12]

(b) Suppose that X is integral, and let $f \in K(X)$ be a nonzero rational function on X. Prove that there exists a blow-up $X' = \mathrm{Bl}_W \to X$ which admits an open cover $X' = X'_+ \cup X'_-$ such that f is a regular function on X'_+ and f^{-1} is a regular function on X'_-.

Problem 5 (Riemann–Zariski space)**.** Let X be a separated integral scheme of finite type over a field k, and let K be the field of rational functions on X.

Hint: In part (b), use Problem 4 to construct $\mathscr{O}$, and the valuative criterion of properness to construct a point of $\mathbf{ZR}(X)$.

cofiltering = Every two objects are dominated by a third one, and every two parallel arrows can be equalized.

(a) Show that nontrivial blow-ups $X' \to X$ of X form a cofiltering subcategory $\mathscr{B}_X$ of the slice category $\mathbf{Sch}_{/X}$.

(b) Consider the topological space (called the Riemann–Zariski space)

$$\mathbf{ZR}(X) = \varprojlim_{X'\to X\in\mathscr{B}_X} |X'| \quad \in \quad \mathbf{Top}.$$

Construct a bijection between points of $\mathbf{ZR}(X)$ and the set of valuation subrings $\mathcal{O} \subseteq K$ such that there exists a dotted arrow making the triangle below commute

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & \nearrow & \\ \operatorname{Spec} \mathcal{O} & & \end{array}$$

(the dotted arrow is unique if it exists, thanks to the valuative criterion of properness). We say that the valuation subring $\mathcal{O} \subseteq K$ *has center* on X.

(c) Endow the set of valuation subrings $\mathcal{O} \subseteq K$ with center on X with the topology generated by the subsets

$$X(f) = \{\mathcal{O} : f \in \mathcal{O}\} \quad \text{for} \quad f \in K.$$

Prove that the bijection constructed in (b) is a homeomorphism.