Problem Set 3

due November 8, 2020

Problem 1. Let $A = K\langle X_1, \dots, X_r\rangle$ be the Tate algebra. Show that the functor

Hint: $\Omega^1_{A/K}$ is what you guess it should be.

$$\left\{\begin{array}{c}\text{Banach } A\text{-modules} \\ + \text{ continuous} \\ A\text{-module homomorphisms}\end{array}\right\} \to \text{Sets}, \quad M \mapsto \mathrm{Der}^{\mathrm{cont}}_K(A, M)$$

sending a Banach A-module M to the set of all continuous K-linear derivations $\delta \colon A \to M$ is representable. The representing object is denoted by $d \colon A \to \Omega^1_{A/K}$ (by abuse of notation) and called the module of *continuous (Kähler) differentials*. Compute $\Omega^1_{A/K}$. Is it the same as the module of Kähler differentials of A/K?

Problem 2. Prove that $f \in K\langle X_1, \dots, X_r\rangle$ is a unit if and only if $|f(0)| > |f - f(0)|$.

Problem 3. A nonarchimedean field K is *spherically complete* if every descending sequence of closed balls

Hint: See Example 2.3.5 and Remark 2.3.6.

$$B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots$$

has a non-empty intersection. Prove that the completed algebraic closure of $\mathbf{C}((t))$ (PS1 Problem 4) is not spherically complete.

Problem 4. Consider the ring (see notes, §2.1, p. 8)

$$K\left\langle X, \frac{t}{X}\right\rangle := K\langle X, Y\rangle/(XY - t).$$

Prove that it is isomorphic to the following ring of Laurent series

$$\left\{ f = \sum_{n\in\mathbf{Z}} a_n X^n \,:\, \lim_{n\to+\infty} a_n = 0,\ \lim_{n\to-\infty} a_n t^n = 0 \right\}.$$

Show that

$$|f| := \sup\left(\{|a_n| : n \geq 0\} \cup \{|a_n| \cdot |t|^n : n \leq 0\}\right)$$

is a Banach algebra norm which is not multiplicative.

Problem 5. Let $K = \mathbf{C}_p$, with the absolute value normalized so that $|p| = 1/p$. Compute the radius of convergence of

$$\exp z = \sum_{n\geq 0} \frac{z^n}{n!} \quad \in \quad K[[z]].$$