Problem Set 2

due Nov 1, 2020

Problem 1. Prove that $\overline{\mathbf{Q}}_p$ is not complete with respect to the unique extension of the p-adic norm $|\cdot|_p$ on $\mathbf{Q}_p$.

Hint: Consider $\mathbf{Q}_p(x)$ with $x = \sum_{n=1}^{\infty} \zeta_n p^n \in \mathbf{C}_p$ where ζ_n is a primitive root of unity for $(n,p) = 1$ and $\zeta_n = 1$ otherwise.

Problem 2. Let A be a ring and let $A^\circ \subseteq A$ be a subring endowed with a topology making it into a topological ring. Show that there exists at most one topology on A making it into a topological ring and such that $A^\circ \subseteq A$ is an open subring. Show that such a topology need not exist in general.

Hint: For the second statement, consider $A^\circ = k[[t,x]]$ and $A = A^\circ[1/t]$ (Thanks to Alex for this example!).

Problem 3. Show that a valuation ring is Noetherian if and only if it is a discrete valuation ring.

Hint: Use Example 2.3.3 and Figure 2.1 as an inspiration.

Problem 4. Let $\mathcal{O} = k[[t]]$, and let $\mathfrak{X}$ be the inductive limit of the system of locally ringed spaces

$$\mathfrak{X} = \varinjlim_n X_n, \quad X_n = \mathrm{Spec}(\mathcal{O}/t^n)[x],$$

which is the ringed space $(|X_0|, \varprojlim_n \mathcal{O}_{X_n})$. Prove that $\mathfrak{X}$ is not a scheme.

Hint: Consider the open subset $D(x)$.

Problem 5 (Corrected). Prove Lemma 2.5.2 in its corrected weak form:

> **Lemma** *Let $f \in K[X]$ be a polynomial whose Newton polygon has segments both of negative and non-negative slope. Then f is reducible.*

Hint: Use Hensel's lemma in the form as in Proposition 2.A.1(b) with $h = 1$.

(Optional, additional credit) Find a counterexample to the earlier statement: *if* $\mathrm{NP}(f)$ *has an inner point of the form* $(m,\gamma) \in \mathbf{Z} \times \nu(K^\times)$ *then* f *is reducible.*

Note: The weak form of the lemma is sufficient for the proof of Proposition 2.5.3. In turn, Theorem 2.5.1 implies a stronger form of Lemma 2.5.2: *the Newton polygon of an irreducible polynomial is a single segment*. The lecture notes will soon be updated with both forms of the lemma.