Problem Set 10, part I

due Jan 31, 2021

Problem 1. Let $X = \mathbf{A}^2_k = \operatorname{Spec} k[x,y]$, let $0 = V(x,y)$ be the origin, and let $X' = \operatorname{Bl}_0 X$. Let $p \in X$ be the closed point in the exceptional divisor which lies on the strict transform of the line $V(x) \subseteq X$, and let $X'' = \operatorname{Bl}_p X'$. Find an ideal $I \subseteq k[x,y]$ for which $X'' = \operatorname{Bl}_{V(I)} X$. Perform a sanity check by computing the exceptional divisor.

Hint: This was partially solved during the lecture. Verify all the details.

Problem 2. Let X be a Noetherian scheme, let $U \subseteq X$ be an open subset with open immersion $j \colon U \hookrightarrow X$, and let $Y = X \setminus U$ be the complementary closed subset. Let $\operatorname{Coh}_Y X$ denote the full subcategory of $\operatorname{Coh} X$ consisting of coherent sheaves $\mathscr{F}$ which are set-theoretically supported on Y. (This is equivalent to saying that $\mathscr{I}_Y^n \cdot \mathscr{F} = 0$ for $n \gg 0$, or to $j^*\mathscr{F} = 0$.) Let $\mathscr{W}$ be the class of morphisms $f \colon \mathscr{F} \to \mathscr{F}'$ in $\operatorname{Coh} X$ such that both $\ker(f)$ and $\operatorname{cok}(f)$ belong to $\operatorname{Coh}_Y X$. Prove that j^* induces an equivalence of categories

$$j^* \colon (\operatorname{Coh} X)[\mathscr{W}^{-1}] \xrightarrow{\sim} \operatorname{Coh} U.$$

Hint: Use [Hartshorne, Ex. II 5.15]. You do not have to solve that exercise. See also Stacks Project, Tag 05Q0.

Problem 3 (Integral surface of infinite type)**.** I learned the following example from Z. Jelonek.

Confession: Until I saw this example, I used to believe that if a separated scheme locally of finite type over k is not of finite type, then it must have infinitely many irreducible components.

(a) Construct a morphism $u \colon \mathbf{A}^2_k \to \mathbf{A}^2_k$ which is quasi-finite but not finite.

(b) Use (a) combined with Noether Normalization and Zariski's Main Theorem to show that for every normal integral affine surface S of finite type over k there exists an open immersion $S \hookrightarrow S'$ where S' is a normal integral affine surface of finite type over k and $S' \neq S$.

(c) Use (b) to construct an infinite sequence of non-trivial open immersions

$$S_0 \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots$$

of normal integral affine surfaces of finite type over k. Let $S_\infty = \bigcup_{n \geq 0} S_n$, which is a normal surface locally of finite type over k which is separated but not quasi-compact. Show that S_∞ is not quasi-paracompact.