Problem Sets for Non-Archimedean Geometry

Fall 2020

Problem Set 1

Problem 1.1. A *norm* on a field K is a map $|\cdot|\colon K \to [0,\infty)$ such that $|x| = 0$ iff $x = 0$, $|xy| = |x|\cdot|y|$, and $|x+y| \leq |x|+|y|$ for all $x, y \in K$. Show that the following are equivalent:

(a) $|\cdot|$ is non-Archimedean, i.e. $|x+y| \leq \max\{|x|,|y|\}$ for $x, y \in K$,

(b) $|n| \leq 1$ for all $n \in \mathbf{Z}$,

(c) $\mathscr{O} = \{x \in K : |x| \leq 1\} \subset K$ is a subring of K.

Problem 1.2. Show that $\mathbf{Z}_p \simeq \mathbf{Z}[[X]]/(X-p)$.

Problem 1.3. Let $|\cdot|_1, |\cdot|_2\colon K \to [0,\infty)$ be two *nontrivial* non-Archimedean norms on a field K. Show that the following are equivalent:

(a) $|\cdot|_1$ and $|\cdot|_2$ define the same topology on K,

(b) There exists a $c > 0$ such that $|\cdot|_1 = |\cdot|_2^c$,

(c) $\{x \in K : |x|_1 \leq 1\} \subseteq \{x \in K : |x|_2 \leq 1\}$.

Problem 1.4. Describe the algebraic closure of $\mathbf{C}((t))$ and its completion in terms of Laurent series in fractional powers of t (Puiseux series).

Hint: Show that every finite extension of $\mathbf{C}((t))$ is of the form $\mathbf{C}((s))$ with $s^n = t$.

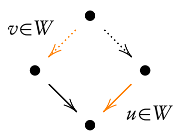
Problem 1.5 (Calculus of fractions)**.** Let $\mathscr{C}$ be a category and let $W \subseteq \mathscr{C}$ be a subcategory containing all isomorphisms and satisfying the two-out-of-three property: if f and g are composable arrows in $\mathscr{C}$ and two of f, g, gf are in W then so is the third. There exists a functor

$$\mathscr{C} \to \mathscr{C}[W^{-1}]$$

which is initial among all functors $\mathscr{C} \to \mathscr{D}$ sending morphisms in W to isomorphisms; the category $\mathscr{C}[W^{-1}]$ is called the *localization* of $\mathscr{C}$ in W.

We say that $(\mathscr{C}, W)$ admits a *calculus of right fractions* if:

a) every pair of solid arrows as below with $u \in W$ can be completed to a commutative square

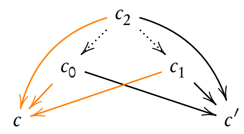


with $v \in W$, and

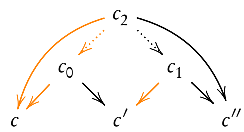
b) for every pair of parallel morphisms $f, g\colon X \to Y$ in $\mathscr{C}$ and every map $u\colon Y \to Z$ in W such that $uf = ug$ there exists a map $v\colon W \to X$ in W such that $fv = gv$.

Prove that if W admits a calculus of right fractions, then the localization $\mathscr{C}[W^{-1}]$ admits the following explicit description: the objects of $\mathscr{C}[W^{-1}]$ are the objects of $\mathscr{C}$, and the morphisms $c \to c'$ in $\mathscr{C}[W^{-1}]$ are equivalence classes of "roofs" $c \leftarrow c_0 \to c'$ with the backwards map in W, where $c \leftarrow c_0 \to c'$ and $c \leftarrow c_1 \to c'$ are equivalent if there exists a third $c \leftarrow c_2 \to c'$ and maps $c_0 \leftarrow c_2 \to c_1$ making the resulting diagram commute. Given $c \leftarrow c_0 \to c'$ and $c' \leftarrow c_1 \to c''$, applying axiom a) to $c_0 \to c' \leftarrow c_1$ gives $c_0 \leftarrow c_2 \to c_1$, and the composition is $c \leftarrow c_0 \leftarrow c_2 \to c_1 \to c''$.

Equivalence:



Composition:



Problem Set 2

Problem 2.1. Prove that $\overline{\mathbf{Q}}_p$ is not complete with respect to the unique extension of the p-adic norm $|\cdot|_p$ on $\mathbf{Q}_p$.

Hint: Consider $\mathbf{Q}_p(x)$ with $x = \sum_{n=1}^{\infty} \zeta_n p^n \in \mathbf{C}_p$ where ζ_n is a primitive root of unity for $(n,p)=1$ and $\zeta_n = 1$ otherwise.

Problem 2.2. Let A be a ring and let $A^\circ \subseteq A$ be a subring endowed with a topology making it into a topological ring. Show that there exists at most one topology on A making it into a topological ring and such that $A^\circ \subseteq A$ is an open subring. Show that such a topology need not exist in general.

Hint: For the second statement, consider $A^\circ = k[[t,x]]$ and $A = A^\circ[1/t]$ (Thanks to Alex for this example!).

Problem 2.3. Show that a valuation ring is Noetherian if and only if it is a discrete valuation ring.

Hint: Use Example 2.3.3 and Figure 2.1 as an inspiration.

Problem 2.4. Let $\mathscr{O} = k[[t]]$, and let $\mathfrak{X}$ be the inductive limit of the system of locally ringed spaces

$$\mathfrak{X} = \varinjlim_n X_n, \quad X_n = \mathrm{Spec}(\mathscr{O}/t^n)[x],$$

which is the ringed space $(|X_0|, \varprojlim_n \mathscr{O}_{X_n})$. Prove that $\mathfrak{X}$ is not a scheme.

Hint: Consider the open subset $D(x)$.

Problem 2.5 (Corrected). Prove Lemma 2.5.2 in its corrected weak form:

> **Lemma** *Let $f \in K[X]$ be a polynomial whose Newton polygon has segments both of negative and non-negative slope. Then f is reducible.*

Hint: Use Hensel's lemma in the form as in Proposition 2.A.1(b) with $h = 1$.

(Optional, additional credit) Find a counterexample to the earlier statement: *if* NP(f) *has an inner point of the form* $(m,\gamma) \in \mathbf{Z} \times v(K^\times)$ *then* f *is reducible.*

Note: The weak form of the lemma is sufficient for the proof of Proposition 2.5.3. In turn, Theorem 2.5.1 implies a stronger form of Lemma 2.5.2: *the Newton polygon of an irreducible polynomial is a single segment*. The lecture notes will soon be updated with both forms of the lemma.

Problem Set 3

Problem 3.1. Let $A = K\langle X_1, \ldots, X_r \rangle$ be the Tate algebra. Show that the functor

$$\left\{ \begin{array}{c} \text{Banach } A\text{-modules} \\ + \text{ continuous} \\ A\text{-module homomorphisms} \end{array} \right\} \to \text{Sets}, \quad M \mapsto \mathrm{Der}_K^{\mathrm{cont}}(A, M)$$

Hint: $\Omega^1_{A/K}$ is what you guess it should be.

sending a Banach A-module M to the set of all continuous K-linear derivations $\delta : A \to M$ is representable. The representing object is denoted by $d : A \to \Omega^1_{A/K}$ (by abuse of notation) and called the module of *continuous (Kähler) differentials*. Compute $\Omega^1_{A/K}$. Is it the same as the module of Kähler differentials of A/K?

Problem 3.2. Prove that $f \in K\langle X_1, \dots, X_r\rangle$ is a unit if and only if $|f(0)| > |f - f(0)|$.

Problem 3.3. A nonarchimedean field K is *spherically complete* if every descending sequence of closed balls

$$B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots$$

has a non-empty intersection. Prove that the completed algebraic closure of $\mathbf{C}((t))$ (PS1 Problem 4) is not spherically complete.

Hint: See Example 2.3.5 and Remark 2.3.6.

Problem 3.4. Consider the ring (see notes, §2.1, p. 8)

$$K\left\langle X, \frac{t}{X} \right\rangle := K\langle X, Y\rangle / (XY - t).$$

Prove that it is isomorphic to the following ring of Laurent series

$$\left\{ f = \sum_{n \in \mathbf{Z}} a_n X^n \ : \ \lim_{n \to +\infty} a_n = 0, \ \lim_{n \to -\infty} a_n t^n = 0 \right\}.$$

Show that

$$|f| := \sup\left(\{|a_n| \ : \ n \geq 0\} \cup \{|a_n| \cdot |t|^n \ : \ n \leq 0\}\right)$$

is a Banach algebra norm which is not multiplicative.

Problem 3.5. Let $K = \mathbf{C}_p$, with the absolute value normalized so that $|p| = 1/p$. Compute the radius of convergence of

$$\exp z = \sum_{n \geq 0} \frac{z^n}{n!} \quad \in \quad K[[z]].$$

Problem Set 4

Affinoid algebras

Problem 4.1. Let A be an affinoid K-algebra and let $a \in A$. Show that $|a|_{\sup} < 1$ if and only if $\lim a^n = 0$. (The latter means that $\lim |a^n|_\alpha = 0$ for some/every residue norm $|\cdot|_\alpha$.)

Hint: Use the characterization of a such that $|a|_{\sup} \leq 1$ proved in the lecture. Try to reduce to this by rescaling.

Sites

Problem 4.2. Construct an equivalence of categories $\mathrm{Sh}(\mathbf{R}) \simeq \mathrm{Sh}^{\mathrm{adm}}(\mathbf{Q})$.

Problem 4.3 (Sheaves on a base, site version)**.** Let $\mathscr{C}$ be a site and let $\mathscr{C}_0 \subseteq \mathscr{C}$ be a full subcategory closed under fiber products. Suppose that every object $c \in \operatorname{ob}\mathscr{C}$ admits a covering family $\{c_\alpha \to c\}_{\alpha \in I}$ with every $c_\alpha \in \operatorname{ob}\mathscr{C}_0$. Endow $\mathscr{C}_0$ with the induced topology: a family $\{c_\alpha \to c\}$ is a covering family if it is a covering family in $\mathscr{C}$. Prove that the inclusion functor induces an equivalence $\mathrm{Sh}(\mathscr{C}) \simeq \mathrm{Sh}(\mathscr{C}_0)$.

Hint: Compare with [Vakil, Theorem 2.5.1]. Its proof uses stalks at points, which you cannot do here, so you need a different argument. If you really get stuck, try [SGA4 Vol. I Exposé III, Théorème 4.1]. You may also find the Stacks Project helpful.

Blowing up

For the following exercises, it will be helpful to brush up on blow-ups, e.g. [Hartshorne, Chapter II 7, pp. 160–169] and on the valuative criteria of separatedness and properness [Chapter II 4].

Problem 4.4. Let X be a Noetherian scheme.

(a) Let $Y, Z \subseteq X$ be closed subschemes and let $X' = \mathrm{Bl}_{Y \cap Z} X$ be the blow-up of their intersection. Prove that the strict transforms $\tilde{Y}$, $\tilde{Z}$ of Y and Z in X' are disjoint.

= [Hartshorne, Exercise II 7.12]

(b) Suppose that X is integral, and let $f \in K(X)$ be a nonzero rational function on X. Prove that there exists a blow-up $X' = \mathrm{Bl}_W \to X$ which admits an open cover $X' = X'_+ \cup X'_-$ such that f is a regular function on X'_+ and f^{-1} is a regular function on X'_-.

Problem 4.5 (Riemann–Zariski space)**.** Let X be a separated integral scheme of finite type over a field k, and let K be the field of rational functions on X.

Hint: In part (b), use Problem 4 to construct $\mathcal{O}$, and the valuative criterion of properness to construct a point of $\mathbf{ZR}(X)$.

(a) Show that nontrivial blow-ups $X' \to X$ of X form a cofiltering subcategory $\mathscr{B}_X$ of the slice category $\mathbf{Sch}_{/X}$.

cofiltering = Every two objects are dominated by a third one, and every two parallel arrows can be equalized.

(b) Consider the topological space (called the Riemann–Zariski space)

$$\mathbf{ZR}(X) = \varprojlim_{X' \to X \in \mathscr{B}_X} |X'| \quad \in \quad \mathbf{Top}.$$

Construct a bijection between points of $\mathbf{ZR}(X)$ and the set of valuation subrings $\mathcal{O} \subseteq K$ such that there exists a dotted arrow making the triangle below commute

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \operatorname{Spec} \mathcal{O} & & \end{array}$$

(the dotted arrow is unique if it exists, thanks to the valuative criterion of properness). We say that the valuation subring $\mathcal{O} \subseteq K$ *has center* on X.

(c) Endow the set of valuation subrings $\mathcal{O} \subseteq K$ with center on X with the topology generated by the subsets

$$X(f) = \{\mathcal{O} : f \in \mathcal{O}\} \quad \text{for} \quad f \in K.$$

Prove that the bijection constructed in (b) is a homeomorphism.

Problem Set 5

Problem 5.1. Let $\varphi : A \to B$ be a K-algebra homomorphism between affinoid K-algebras. Suppose that there exists a surjection

$$\beta : K\langle X_1, \ldots, X_r \rangle \to B, \quad \beta(X_i) = b_i$$

and powerbounded elements $a_1, \dots, a_r \in A$ such that

$$|b_i - \varphi(a_i)|_\beta < 1 \quad \text{for } i = 1, \dots, r.$$

Prove that φ is surjective. Give an example showing that the assumption that the a_i are powerbounded is necessary.

Problem 5.2. Prove that every covering of $\mathrm{Sp} A$ by Zariski open subsets is admissible.

Problem 5.3 (Inadmissible open)**.** Consider the following open subset of $X = \mathrm{Sp} K\langle X, Y\rangle$:

Hint: Use Theorem 6.4.1.

$$U = \{|y| = 1\} \cup \bigcup_{n \geq 1} \{|x| \leq |t|^n,\ |y| \leq |t|^{1/n}\}.$$

Prove that U is not an admissible open.

Problem 5.4 (Admissible sheaf with zero stalks)**.** Let $D = \mathrm{Sp} K\langle X\rangle$ be the unit disc over an algebraically closed non-Archimedean field K. For an affinoid subdomain $U \subseteq D$, we say that U is *huge* if U contains the complement of finitely many open discs of radii ≤ 1. We set

Hint: Use the classification of affinoid subdomains of D given in Theorem 9.7.2/2 in BGR.

$$\mathscr{F}(U) = \begin{cases} \mathbf{Z} & \text{if } U \text{ is huge} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\mathscr{F}$ is a sheaf for the admissible topology. Note: $\mathscr{F} \neq 0$ but $\mathscr{F}_x = 0$ for every $x \in D$.

Problem 5.5 (The Riemann–Zariski space is quasi-compact)**.** Let $k \subseteq K$ be a field extension and let $\mathbf{ZR}(K/k)$ be the space of all valuation subrings of K containing k, with topology generated by the sets

Hint: Embed the set $\mathbf{ZR}(K/k)$ into $2^{K^\times}$. Show that with the induced topology $\mathbf{ZR}(K/k)$ becomes compact Hausdorff. Compare this topology with the given topology on $\mathbf{ZR}(K/k)$.

$$X(f) = \{\mathscr{O} : f \in \mathscr{O}\}, \quad f \in K^\times.$$

Show that $\mathbf{ZR}(K/k)$ is quasi-compact.

Problem Set 6

Problem 6.1. Let $D = \mathrm{Sp} K\langle X\rangle$ and let $o \in D$ be the origin. Prove that the local ring $\mathscr{O}_{D,o}$ is henselian.

Problem 6.2. Suppose that K is algebraically closed but not spherically complete. Show that there exists a descending sequence of open discs $D_n^\circ \subseteq D = \mathrm{Sp} K\langle X\rangle$ with empty intersection. Let $U_n = D \setminus D_n^\circ$, which is an increasing sequence of affinoid subdomains of D covering D. Show that $\{U_n\}$ is not an admissible cover of D, and that there is a unique structure of a rigid-analytic space on D with $\{U_n\}$ an admissible open cover. If D' is the resulting space, show that the identity map $D' \to D$ is a morphism of rigid-analytic spaces.

Hint: Since this example is quite puzzling, here is a toy example analog. One can define a natural "admissible topology" on $\mathbf{Q} \setminus \{\sqrt{2}\} = \mathbf{Q}$ by considering only closed rational intervals $[a,b]_\mathbf{Q}$ with $\sqrt{2} \notin [a,b]$. The resulting map of G-topological spaces $\mathbf{Q} \setminus \{\sqrt{2}\} \to \mathbf{Q}$ is a continuous bijection but not an isomorphism.

Problem 6.3. A rigid-analytic space X is called *quasi-compact* if every admissible cover admits a finite subcover, and *quasi-separated* if the intersection of two quasi-compact admissible opens is quasi-compact. Prove that X is quasi-compact if and only if it admits an finite admissible cover by affinoids, and that it is quasi-separated if and only if the intersection of two affinoid opens admits a finite admissible cover by affinoids.

Problem 6.4. Let $D = \operatorname{Sp} K\langle X\rangle$. Rigorously construct a rigid-analytic space "D with doubled W," where (1) $W =$ the origin, (2) $W = \{|X| < 1\}$, (3) $W = \{|X| \leq |t|\}$. Which of those spaces are quasi-separated?

Problem 6.5. Let X be a G-topological space satisfying (G_0), (G_1), and (G_2), and let Γ be a group acting freely and continuously on X (meaning that the maps $\gamma\colon X \to X$ are continuous maps of G-topological spaces for all $\gamma \in \Gamma$). We call this action *properly discontinuous* if X admits an admissible cover of the form $\{\gamma \cdot U_i\}_{i\in I, \gamma\in\Gamma}$ with $\gamma \cdot U_i \cap U_i = \emptyset$ for $\gamma \neq e$ and such that the sets $\bigcup_{\gamma\in\Gamma} \gamma \cdot U_i$ are admissible for all $i \in I$.

Hint: Prove the case of topological spaces first.

(a) Show that if the action of Γ on X as above is properly discontinuous, then there exists a natural structure of a G-topological space on the orbit space $Y = X/\Gamma$ satisfying (G_0), (G_1), and (G_2) and such that $\pi\colon X \to Y$ is continuous.

(b) A Γ-*equivariant sheaf* on X is a sheaf $\mathscr{F}$ endowed with isomorphisms $u_\gamma\colon \gamma^*\mathscr{F} \to \mathscr{F}$ for which the following diagrams commute for all $\gamma, \delta \in \Gamma$:

$$\begin{array}{ccc} (\gamma\delta)^*\mathscr{F} & \xrightarrow{u_{\gamma\delta}} & \mathscr{F} \\ \| & & \uparrow{\scriptstyle u_\delta} \\ \delta^*(\gamma^*\mathscr{F}) & \xrightarrow[\delta^*(u_\gamma)]{} & \delta^*\mathscr{F}. \end{array}$$

With assumptions and notation as in (a), construct an equivalence of categories between sheaves on Y and Γ-equivariant sheaves on X.

(c) Let $q \in K$ with $0 < |q| < 1$. Show that the action of $q^{\mathbf{Z}}$ on $\mathbf{A}^{n,\mathrm{an}}_K \setminus 0$ by rescaling the coordinates is properly discontinuous.

Problem Set 7

Problem 7.1. Show that finite maps between rigid-analytic spaces are proper.

Problem 7.2. Let X be a proper scheme over K. Prove that the associated rigid-analytic space X^{an} is proper.

Hint: Use Chow's lemma to show that X^{an} is quasi-compact.

Problem 7.3. Let $q \in K$ be such that $0 < |q| < 1$, and consider the action of the cyclic group $q^{\mathbf{Z}}$ on the punctured plane $X = (\mathbf{A}^2_K \setminus 0)^{\mathrm{an}}$ by rescaling. Show that the action is properly discontinuous and describe the quotient $Y = X/q^{\mathbf{Z}}$ (called the non-Archimedean Hopf surface) as a union of four affinoids glued along affinoid subdomains.

Problem 7.4. Let $Y = (\mathbf{A}^2_K \setminus 0)^{\mathrm{an}}/q^{\mathbf{Z}}$ be the non-Archimedean Hopf surface. Compute $H^1(Y, \mathscr{O}_Y)$ and $H^0(Y, \Omega^1_{Y/K})$.

Hint: $H^0(Y, \Omega^1_{Y/K})$ are just the $q^{\mathbf{Z}}$-invariant differentials on X.

Problem 7.5. Let $K \subseteq K'$ be an extension of non-Archimedean fields (that is, the norm on K' restricts to the norm on K). Define a base change functor

$$(-)_{K'}\colon \mathrm{Rig}_K^{\mathrm{qs}} \to \mathrm{Rig}_{K'}^{\mathrm{qs}}.$$

Hint: Define the functor first on affinoid algebras, and then pin down $(-)_{K'}$ by a universal property.

Problem Set 8

Problem 8.1. Recall *Jacobi triple product formula* in the form from the lecture:

$$\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{n(n+1)}{2}} w^n = (1-w^{-1})\prod_{m\geq 1}(1+q^m)(1-wq^m)(1-w^{-1}q^m).$$

Here $w, q \in K^\times$ and $|q| < 1$ in some non-Archimedean field K. Prove the weaker version: there exists a constant $C(q)$ depending on q such that

$$\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{n(n+1)}{2}} w^n = C(q)\cdot(1-w^{-1})\prod_{m\geq 1}(1-wq^m)(1-w^{-1}q^m)$$

for every $w \in K^\times$.

Problem 8.2. Let $f(q) = q^{-1} + \sum_{n\geq 0} a_n q^n \in K((t))$ be a Laurent series with $|a_n| \leq 1$. Show that f defines a bijection between the sets $\{0 < |q| < 1\}$ and $\{|w| > 1\}$.

Problem 8.3. Let $Y = \mathbf{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}$ be a Tate curve. Prove that every endomorphism of Y lifts to an endomorphism $\mathbf{G}_m^{\mathrm{an}}$. Conclude that $\mathrm{End}(Y) \simeq \mathbf{Z}$.

Problem 8.4. Let $Y = \mathbf{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}$ be a Tate curve. For every $n \geq 1$, compute the order of the n-torsion subgroup $Y(\overline{K})[n]$.

Problem 8.5. Let k be an algebraically closed field and let $\mathscr{B}$ be the category of finitely generated field extensions K of k. Let $\mathscr{P}$ denote the category of projective varieties over k and dominant maps, and let $W \subseteq \mathscr{P}$ be the subcategory consisting of all non-trivial blow-up maps $\pi\colon X' \to X \in \mathscr{P}$. Prove that W admits calculus of right fractions and that the association $X \mapsto K(X)$ induces an equivalence of categories

$$\mathscr{P}[W^{-1}] \xrightarrow{\sim} \mathscr{B}^{\mathrm{op}}.$$

Problem Set 9

In the following exercises, $\mathscr{O}$ is a complete discrete valuation ring with uniformizer t, residue field k, and fraction field K.

Problem 9.1. Give an example of a diagram $\mathscr{X} \to \mathscr{S} \leftarrow \mathscr{Y}$ of admissible $\mathscr{O}$-formal schemes such that the fiber product $\mathscr{Z} = \mathscr{X} \times_{\mathscr{S}} \mathscr{Y}$ in the category of $\mathscr{O}$-formal schemes is not admissible. Compute $\mathscr{Z}_{\mathrm{ad}}$.

Hint: An example was featured at the beginning of Lecture 17. To simplify it further, you can try to make $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{S}$ affine.

Problem 9.2. Let X be (a) the affine line $\mathbf{A}^1_{\mathscr{O}}$ with doubled "zero section" $V(x)$, or (b) the affine line $\mathbf{A}^1_{\mathscr{O}}$ with doubled "origin in the special fiber" $V(x,t)$, where x is the coordinate on $\mathbf{A}^1_{\mathscr{O}}$. In both cases, compute the canonical map of rigid-analytic spaces

$$(\widehat{X})_{\mathrm{rig}} \to (X_K)^{\mathrm{an}}$$

and check that it is not an open immersion.

Problem 9.3. Let $\mathscr{X}$ be a formal scheme locally of finite type over $\mathscr{O}$, let X_0 be its special fiber (a scheme locally of finite type over k), let $X = \mathscr{X}_{\mathrm{rig}}$ be its rigid-analytic generic fiber, and let $\mathrm{sp}\colon X \to \mathscr{X}$ be the specialization map. Let Z_i ($i \in I$) be the irreducible components of $|X_0|$. Show that the tubes

$$]Z_i[= \mathrm{sp}^{-1}(Z_i) \subseteq X \quad (i \in I)$$

(where we identify $|\mathscr{X}| = |X_0|$) form an admissible cover of X.

Problem 9.4. Construct a flat lifting X_1 of $X_0 = \mathbf{A}^2_k \setminus 0$ over $k[[t]]/(t^2)$ for which the restriction map $\Gamma(X_1, \mathscr{O}_{X_1}) \to \Gamma(X_0, \mathscr{O}_{X_0}) = k[x,y]$ is not surjective.

Hint: Use the standard covering by two affine opens.

Problem 9.5. Let $X = \mathbf{A}^1_{\mathscr{O}}$ with coordinate x and let $X' \to X$ be the blowup at the "origin of the special fiber," defined by the ideal (t,x). Show that the induced morphism of rigid-analytic generic fibers of formal completions

$$\widehat{X'}_{\mathrm{rig}} \to \widehat{X}_{\mathrm{rig}}$$

is an isomorphism. (This is a basic example of an admissible blowup.)

Problem Set 10

Problem 10.1. Let $X = \mathbf{A}^2_k = \operatorname{Spec} k[x,y]$, let $0 = V(x,y)$ be the origin, and let $X' = \mathrm{Bl}_0 X$. Let $p \in X$ be the closed point in the exceptional divisor which lies on the strict transform of the line $V(x) \subseteq X$, and let $X'' = \mathrm{Bl}_p X'$. Find an ideal $I \subseteq k[x,y]$ for which $X'' = \mathrm{Bl}_{V(I)} X$. Perform a sanity check by computing the exceptional divisor.

Hint: This was partially solved during the lecture. Verify all the details.

Problem 10.2. Let X be a Noetherian scheme, let $U \subseteq X$ be an open subset with open immersion $j\colon U \hookrightarrow X$, and let $Y = X \setminus U$ be the complementary closed subset. Let $\mathrm{Coh}_Y X$ denote the full subcategory of $\mathrm{Coh}\, X$ consisting of coherent sheaves $\mathscr{F}$ which are set-theoretically supported on Y. (This is equivalent to saying that $\mathscr{I}_Y^n \cdot \mathscr{F} = 0$ for $n \gg 0$, or to $j^*\mathscr{F} = 0$.) Let $\mathscr{W}$ be the class of morphisms $f\colon \mathscr{F} \to \mathscr{F}'$ in $\mathrm{Coh}\, X$ such that both $\ker(f)$ and $\mathrm{cok}(f)$ belong to $\mathrm{Coh}_Y X$. Prove that j^* induces an equivalence of categories

$$j^*\colon (\mathrm{Coh}\, X)[\mathscr{W}^{-1}] \xrightarrow{\sim} \mathrm{Coh}\, U.$$

Hint: Use [Hartshorne, Ex. II 5.15]. You do not have to solve that exercise. See also Stacks Project, Tag 05Q0.

Problem 10.3 (Integral surface of infinite type)**.** I learned the following example from Z. Jelonek.

Confession: Until I saw this example, I used to believe that if a separated scheme locally of finite type over k is not of finite type, then it must have infinitely many irreducible components.

(a) Construct a morphism $u\colon \mathbf{A}^2_k \to \mathbf{A}^2_k$ which is quasi-finite but not finite.

(b) Use (a) combined with Noether Normalization and Zariski's Main Theorem to show that for every normal integral affine surface S of finite type over k there exists an open immersion $S \hookrightarrow S'$ where S' is a normal integral affine surface of finite type over k and $S' \neq S$.

(c) Use (b) to construct an infinite sequence of non-trivial open immersions

$$S_0 \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots$$

of normal integral affine surfaces of finite type over k. Let $S_\infty = \bigcup_{n\geq 0} S_n$, which is a normal surface locally of finite type over k which is separated but not quasi-compact. Show that S_∞ is not quasi-paracompact.

Problem 10.4. Let $U = \mathbf{A}^{1,\mathrm{an}}_K$, $X = \mathbf{P}^{1,\mathrm{an}}_K$, and let $j\colon U \to X$ be the open immersion. Prove that there does not exist a formal model $\mathfrak{U} \to \mathfrak{X}$ of j which is an open immersion.

Hint: The morphism j is not quasi-compact.

Problem 10.5. Construct a formal model of the open unit disc $\mathbf{D}^\circ_K = \{|x| < 1\} \subseteq \mathbf{D}^1_K$ over $K = k((t))$. What does the special fiber look like?

Hint: Use the finite type covering

$$X = U_0 \cup \bigcup_{n>0} U_n$$

by

$$U_0 = \{|x| \leq |t|\}$$

and

$$U_n = \{|t|^{\frac{1}{n}} \leq |x| \leq |t|^{\frac{1}{n+1}}\} \quad (n > 1).$$