Piotr Achinger

Introduction to non-Archimedean Geometry

Lecture course, Fall 2020

Version: January 28, 2021

NON-ARCHIMEDEAN or rigid-analytic geometry is an analog of complex analytic geometry over non-Archimedean fields, such as the field of p-adic numbers $\mathbf{Q}_p$ or the field of formal Laurent series $k((t))$. It was introduced and formalized by Tate in the 1960s, whose goal was to understand elliptic curves over a p-adic field by means of a uniformization similar to the familiar description of an elliptic curve over $\mathbf{C}$ as quotient of the complex plane by a lattice. It has since gained status of a foundational tool in algebraic and arithmetic geometry, and several other approaches have been found by Raynaud, Berkovich, and Huber. In recent years, it has become even more prominent with the work of Scholze and Kedlaya in p-adic Hodge theory, as well as the non-Archimedean approach to mirror symmetry proposed by Kontsevich. That said, we still do not know much about rigid-analytic varieties, and many foundational questions remain unanswered.

The goal of this lecture course is to introduce the basic notions of rigid-analytic geometry. We will assume familiarity with schemes.

Problem sets and other materials related to the course are available at

http://achinger.impan.pl/lecture20f.html

Our basic reference is the book *Lectures on Formal and Rigid Geometry* by Siegried Bosch. More references are found in the text.

This document uses the *tufte-book* LaTeX document class based on the design of Edward Tufte's books; some typesetting tricks were shamelessly stolen from Eric Peterson's lecture notes.

These notes owe a lot to the generous help of Alex Youcis. I thank Paweł Poczobut and Piotr Rudnicki for valuable comments and corrections.

Contents

1
Two interpretations of non-Archimedean geometry

THE p-ADIC NUMBERS $\mathbf{Q}_p$ are usually defined either as the completion of the rational numbers $\mathbf{Q}$ with respect to the p-adic absolute value

$$\left|\frac{a}{b}\right|_p = p^{\operatorname{ord}_p b - \operatorname{ord}_p a}, \tag{1.1}$$

or as the fraction field of the p-adic integers $\mathbf{Z}_p$ defined as the inverse limit

$$\mathbf{Z}_p = \varprojlim_n \mathbf{Z}/p^n\mathbf{Z}. \tag{1.2}$$

We can refer to (1.1) as the "metric" or "analytic" point of view, while (1.2) represents a more "algebraic" (or "formal") perspective. [1]

Both interpretations have their advantages and drawbacks. The metric approach is admittedly closer to one's intuition, and allows one to employ right away the powerful tools of topology and analysis. However, the topology of the p-adic numbers is quite pathological: $\mathbf{Q}_p$ is a totally disconnected topological space. This makes it difficult to proceed by analogy with real or complex analysis.

The algebraic approach allows us to reduce questions about $\mathbf{Q}_p$ to pure algebra over the rather simple rings $\mathbf{Z}/p^n\mathbf{Z}$. One therefore has commutative algebra and algebraic geometry at their disposal, which, in turn, allows one to more easily make sound and precise arguments. The downside: the relationship between objects over $\mathbf{Q}_p$ and over $\mathbf{Z}/p^n\mathbf{Z}$ can often be extremely convoluted.

TO ACHIEVE p-ADIC ENLIGHTENMENT, one needs a good grasp of both[2], as well as a means of switching between the two with ease. The goal of these lectures is to explain how to do p-adic geometry (or, more generally, non-Archimedean geometry[3]) by combining the analytic and the algebraic approaches. Roughly speaking, the first will be represented by Tate's notion of rigid analytic varieties, and the second by Raynaud's approach using formal schemes.

WE WILL NOW GO BEYOND p-adic numbers and fix the notation which we will use most of the time. By a *non-Archimedean field* we mean a field K equipped with a non-Archimedean norm, which by definition is a function

$$|\cdot| : K \to [0, \infty)$$

such that

1. $|x| = 0$ if and only if $x = 0$,

[1] We choose to ignore here the (rather useless) definition of p-adic numbers in terms of base-p digit expansions.

[2] *It seems as though we must use sometimes the one theory and sometimes the other, while at times we may use either. We are faced with a new kind of difficulty. We have two contradictory pictures of reality; separately neither of them fully explains the phenomena of light, but together they do.*

A. Einstein, L. Infeld *The Evolution of Physics*

[3] More precisely, *rigid (or rigid-analytic) geometry*, whose strange name we will justify later on.

2. $|xy| = |x| \cdot |y|$,
3. $|x+y| \leq \max(|x|, |y|)$.

We also assume that $|x| \neq 1$ for some $x \neq 0$ (i.e. that $|\cdot|$ is *nontrivial*), and that K is *complete* with respect to (the metric defined by) the norm. [4]

The third axiom, stronger than the triangle inequality $|x+y| \leq |x| + |y|$, is what makes the field non-Archimedean. It implies that the subset

$$\mathcal{O} = \{x \in K \text{ such that } |x| \leq 1\}$$

is a subring of K, called the *valuation ring*. It is local with maximal ideal

$$\mathfrak{m} = \{x \in K \text{ such that } |x| < 1\}.$$

We denote the residue field $\mathcal{O}/\mathfrak{m}$ by k.

Let $t \in m$ be a nonzero element.[5] Completeness of K is equivalent to the fact that the natural map

$$\mathcal{O} \to \varprojlim_n \mathcal{O}/t^n\mathcal{O}$$

is an isomorphism. The field K can be recovered as the fraction field of $\mathcal{O}$, in fact it is the localization $K = \mathcal{O}[\frac{1}{t}]$. The inverse limit above carries the inverse limit topology (with the $\mathcal{O}/t^n\mathcal{O}$ being equipped with the discrete topology), and the isomorphism is an isomorphism of topological rings if $\mathcal{O}$ has the metric topology induced by the norm $|\cdot|$. The topology on K is the unique one with respect to which $\mathcal{O}$ is an *open* subring. This implies that K is encoded *as a topological field* by the inverse system above.

The basic examples are complete discrete valuation fields (cdvf), which can be characterized as those K as above for which the maximal ideal $\mathfrak{m}$ is principal, so that $\mathcal{O}$ is a complete discrete valuation ring (cdvr) with maximal ideal $\mathfrak{m}$, residue field $k = \mathcal{O}/\mathfrak{m}$, and fraction field K. Naturally, our main example is

$$\mathcal{O} = \mathbf{Z}_p, \quad K = \mathbf{Q}_p, \quad \mathfrak{m} = (p), \quad k = \mathbf{F}_p,$$

and another one is the *Laurent series field* (over a base field k)[6]

$$\mathcal{O} = k[[t]] := \varprojlim_n k[t]/(t^n), \quad K = k((t)) := \mathcal{O}\left[\frac{1}{t}\right].$$

The characteristic of k is called the *residue characteristic* of K. If it is equal to the characteristic to K, we say that K is of *equal characteristic*, otherwise it is of *mixed characteristic*. In the latter case, K has characteristic zero. Thus $\mathbf{Q}_p$ and its normed extensions are of mixed characteristic, and the fields $k((t))$ have equal characteristic. In fact, every cdvf of equal characteristic is of the form $k((t))$.

In general, we will have to work with non-Archimedean fields K which are not cdvf's, in which case the valuation ring $\mathcal{O}$ is non-Noetherian. Indeed, it is often useful to consider K algebraically closed, while a complete discrete valuation field is never algebraically closed.[7]

[4] In some sources, non-Archimedean fields are not assumed to be complete and/or nontrivially valued.

[5] We call such a t a *pseudouniformizer*.

[6] Intuition: $k((t))$ is the field of functions on the "infinitesimal punctured disc"

$$\operatorname{Spec} k((t)) = \operatorname{Spec} k[[t]] \setminus \{t = 0\}.$$

[7] Consider a generator of $\mathfrak{m}$, i.e. an element of valuation one. Does it have a square root in K?

1.1 First example: the unit disc

The study of schemes begins with the case of the affine line over a base field k

$$\mathbf{A}^1_k = \operatorname{Spec} k[x],$$

from which one obtains $\mathbf{A}^n_k$ by direct product, then affine schemes of finite type over k by taking closed subschemes $X \subseteq \mathbf{A}^n_k$, and finally schemes locally of finite type over k by gluing. If k is algebraically closed, then by Hilbert's Nullstellensatz, closed points of $\mathbf{A}^1_k$ are in bijection with k.

In non-Archimedean geometry over an algebraically closed[8] non-Archimedean field K, similar role is played by the closed unit disc

[8] We make this assumption only for simplicity and only in this introduction.

$$\mathbf{D}^1_K = \{x \in K : |x| \leq 1\}.$$

Proceeding by analogy with scheme theory, we start with the algebra of functions on $\mathbf{D}^1_K$, which should consist of power series $f = \sum_{n\geq 0} a_n x^n$ which converge for $|x| \leq 1$. An easy check shows that a series in K converges if and only if its terms tend to zero. We conclude that we want the ring of "holomorphic functions" on $\mathbf{D}^1_K$ to be

$$K\langle X\rangle = \left\{ \sum_{n\geq 0} a_n X^n \in K[[X]] \text{ with } a_n \to 0 \text{ as } n \to \infty \right\}.$$

Next, we would like to equip $\mathbf{D}^1_K$ with a *sheaf* of functions whose global sections is the above algebra $K\langle X\rangle$. The naive idea is to define, for an open subset $U \subseteq \mathbf{D}^1_K$, the ring $\mathcal{O}^{\text{wobbly}}(U)$ as the set of functions $U \to K$ which can be represented locally as a power series.

Indeed, this is trivially a sheaf, and we do obtain an injection

$$K\langle X\rangle \to \mathcal{O}^{\text{wobbly}}(\mathbf{D}^1_K).$$

However, this map is very far from being surjective. Indeed, $\mathbf{D}^1_K$ is highly disconnected, for example

$$\mathbf{D}^1_K = \{|x| = 1\} \cup \{|x| < 1\} \tag{1.3}$$

expresses $\mathbf{D}^1_K$ as a union of two disjoint open (!) subsets. The function $f \in \mathcal{O}(\mathbf{D}^1_K)$ equal to 1 on the first open and 0 on the second is not in the image of $K\langle X\rangle$. (This example justifies the adjective *wobbly*.) Clearly, something goes terribly wrong with analytic continuation in the nonarchimedean setting!

1.2 Tate's admissible topology on the unit disc

The first attempt at fixing this issue is due to Krasner, and is based on a non-Archimedean analog of Runge's theorem in complex analysis. A *Krasner analytic function* on $\mathbf{D}^1_K$ is a uniform limit of rational functions with no poles inside $\mathbf{D}^1_K$. This leads to a presheaf $\mathcal{O}$ for which $\mathcal{O}(\mathbf{D}^1_K) = K\langle X\rangle$, and which has the property that $\mathcal{O}(U)$ is a domain if U "should be" connected. Still, it is not a sheaf.

Let us explain, in a simple case, Tate's idea of fixing the issue. Consider the following covering of $\mathbf{D}^1_K$:

$$\mathbf{D}^1_K = \underbrace{\{|x| \leq \rho\}}_{U} \cup \underbrace{\{\rho \leq |x| \leq 1\}}_{V} \tag{1.4}$$

with $0 < \rho < 1$, $\rho = |t|$ for some $t \in K$. The algebra of (Krasner analytic) functions $\mathcal{O}(U)$ on the smaller disc $U = \{|x| \leq \rho\}$ consists of power series converging on this disc, i.e.

$$K\left\langle \frac{X}{t} \right\rangle = \left\{ f = \sum_{n\geq 0} a_n X^n \in K[[X]] : \lim_{n\to\infty} |a_n|\rho^n = 0 \right\}.$$

Similarly, for the annulus $V = \{\rho \leq |x| \leq 1\}$, $\mathcal{O}(V)$ consists of convergent Laurent series

$$K\left\langle X, \frac{t}{X} \right\rangle = \left\{ f = \sum_{n\in\mathbf{Z}} a_n X^n : \lim_{n\to\infty} |a_n| = 0, \lim_{n\to-\infty} |a_n|\rho^n = 0 \right\},$$

and functions $\mathcal{O}(U \cap V)$ on the intersection $U \cap V = \{|x| = \rho\}$ are

$$K\left\langle \frac{X}{t}, \frac{t}{X} \right\rangle = \left\{ f = \sum_{n \in \mathbf{Z}} a_n X^n \,:\, \lim_{|n| \to \infty} |a_n| \rho^n = 0 \right\}.$$

It turns out that we are lucky: the sequence

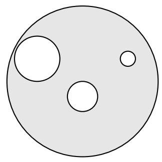
$$0 \to K\langle X \rangle \to K\left\langle \frac{X}{t} \right\rangle \times K\left\langle X, \frac{t}{X} \right\rangle \to K\left\langle \frac{X}{t}, \frac{t}{X} \right\rangle \tag{1.5}$$

is exact.[9] Thus $\mathcal{O}$ satisfies the sheaf condition with respect to the covering $U \cup V$.

[9] Check this!

TATE'S SOLUTION is now to identify a class of *admissible coverings* $U = \bigcup U_i$ of opens $U \subseteq \mathbf{D}^1_K$. For $U = \mathbf{D}^1_K$, these are the coverings admitting a *finite* refinement by subsets of the form

$$\{|x - a| \leq |t|, |x - a_i| \geq |t_i|\}.$$



The covering (1.3) is not admissible in this sense, while (1.4) is. *Tate's acyclicity theorem* says that the presheaf $\mathcal{O}$ satisfies the sheaf condition for all admissible coverings. Exactness of (1.5) is a basic special case.

In particular, this implies that $\mathbf{D}^1_K$ is *quasi-compact* with respect to the admissible topology: every *admissible* cover admits a finite subcover. Moreover, it becomes *connected* in the sense that there is no admissible cover

$$U = \bigcup_{i \in I} U_i \cup \bigcup_{j \in J} V_j,$$

with both summands nonempty, such that $U_i \cap V_j = \emptyset$ for $(i, j) \in I \times J$, as reflected by the fact that $\mathcal{O}(\mathbf{D}^1_K) = K\langle X \rangle$ is a domain.

Formalizing the above requires the notion of a *G-topology* on a topological space X, which is the data of a class of *admissible* open subsets[10] and of *admissible* coverings of admissible open subsets satisfying some axioms. One has a natural notion of a sheaf with respect to a G-topology, which is a presheaf on the category of admissible opens which satisfies the sheaf condition with respect to admissible coverings. Thus $\mathcal{O}$ is a sheaf with respect to the admissible topology on $\mathbf{D}^1_K$.

[10] For $\mathbf{D}^1_K$, we declare all open subsets admissible. The condition will however not be empty for $\mathbf{D}^n_K$ with $n > 1$.

In Tate's formalism, which we shall work out in the first part of the course, the basic geometric objects are *rigid-analytic varieties*. One uses as building blocks the *affinoid algebras*, which are quotients of the *Tate algebras*

$$K\langle X_1, \ldots, X_r \rangle = \left\{ \sum_{n_1, \ldots, n_r \geq 0} a_{n_1 \ldots n_r} X_1^{n_1} \ldots X_r^{n_r} \,:\, a_{n_1 \ldots n_r} \to 0 \text{ as } n_1 + \ldots + n_r \to 0 \right\}.$$

To an affinoid algebra $A = K\langle X_1, \ldots, X_r \rangle / I$ one associates the *affinoid* $\mathrm{Sp}A$. Its underlying topological space is the corresponding closed subset of

$$\mathbf{D}^r_K = \{(x_1, \ldots, x_r) \in K^r \,:\, |x_i| \leq 1 \text{ for } i = 1, \ldots, r\}$$

cut out by the ideal I. One equips it with a G-topology (the admissible topology), and a sheaf of rings $\mathcal{O}$, similarly to the case of $\mathbf{D}^1_K$. A rigid-analytic variety is a topological space with a G-topology and a sheaf of rings with respect to that topology, which is locally (as a G-topologized space!) isomorphic to $\mathrm{Sp}A$ for some affinoid algebra A.

1.3 Raynaud's approach

The main drawbacks of Tate's theory are

- the admissible topology is counterintuitive and complicated to work with,
- and the underlying spaces do not have enough points (e.g. there exist nonzero abelian sheaves for the admissible topology whose stalk at every point is zero),
- one is bound to work over a fixed field; for a non-algebraic extension of nonarchimedean fields K'/K (e.g. $\mathbf{C}_p/\mathbf{Q}_p$) there is no map $\mathbf{D}^1_{K'} \to \mathbf{D}^1_K$,
- (why should there have to be a base field at all?)
- it is quite far from algebraic geometry (e.g. the opens are not defined by non-vanishing loci, but also be inequalities—not algebraic opens, but semi-algebraic opens).

There are several frameworks which address these issues in different ways, notably Huber's theory of *adic spaces*, Berkovich's theory of analytic spaces (usually called *Berkovich spaces*), and Raynaud's approach via *formal schemes*, worked out by Bosch and Lütkebohmert and recently developed further by Fujiwara–Kato and Abbes. In the second half of this course, we will become acquainted with all of these, mostly focusing on Raynaud's theory, as it is the closest to algebraic geometry.

THE STARTING POINT of Raynaud's theory is the following isomorphism (where $t \in K$ is a pseudouniformizer)

$$K\langle X\rangle = \left(\varprojlim_m \mathcal{O}[X]/(t^m)\right)\left[\frac{1}{t}\right]. \tag{1.6}$$

We will prove this later, but you are welcome to try and check it yourself.

The isomorphism (1.6) expresses $K\langle X\rangle$ in terms of (0) the polynomial algebra $\mathcal{O}[X]$ through the algebraic operations of (1) t-adic completion, and (2) localization with respect to t. So, for example, if $\mathcal{O}$ is a discrete valuation ring, we immediately see that $K\langle X\rangle$ is Noetherian, because (0) the polynomial algebra $\mathcal{O}[X]$ is Noetherian, (1) the completion of a Noetherian ring with respect to an ideal is Noetherian, and (2) the localization of a Noetherian ring is Noetherian. (Unfortunately, our $\mathcal{O}$ will not always be Noetherian, so one needs to work harder.)

TO HAVE A GEOMETRIC PICTURE, we replace $\mathcal{O}[X]$ with its spectrum $X = \mathbf{A}^1_{\mathcal{O}}$. The projective system $\mathcal{O}/t^n\mathcal{O}[X]$ corresponds to a system of closed immersions

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots, \quad X_n = \mathbf{A}^1_{\mathcal{O}/t^{n+1}\mathcal{O}}.$$

Each of these immersions is defined a nilpotent ideal, and hence is a homeomorphism on the underlying spaces.

The above inductive system does not have a limit in the category of schemes. Instead, one can take its limit in the larger category of locally ringed spaces:

$$\mathfrak{X} = (|\mathfrak{X}|, \mathcal{O}_{\mathfrak{X}}) = \varinjlim_n X_n.$$

Since $|X_n| \hookrightarrow |X_{n+1}|$ are homeomorphisms, we can identify $|\mathfrak{X}|$ with $|X_0|$. Treating $\mathcal{O}_{X_n}$ as a sheaf on $|X_0| = |\mathfrak{X}|$, we have

$$\mathcal{O}_{\mathfrak{X}} = \varprojlim_n \mathcal{O}_{X_n} = \varprojlim_n \mathcal{O}_X/(t^{n+1}).$$

The locally ringed space $\mathfrak{X}$ is an example of a *formal scheme*, the *formal completion* of $X = \mathbf{A}^1_K$ with respect to the ideal $t\mathcal{O}_X$. In fact, in this context we could *define* formal schemes over $\mathcal{O}$ as systems of closed immersions $X_0 \hookrightarrow X_1 \hookrightarrow \cdots$ between $\mathcal{O}$-schemes, with X_n defined by the ideal $t^{n+1}\mathcal{O}_{X_{n+1}}$.

The final step, inverting t, is the hardest: in Raynaud's approach, one wants to define a rigid-analytic variety over $\mathcal{O}$ as the "generic fiber" of a formal scheme over $\mathcal{O}$. This is done purely formally by localizing the *category* of formal schemes over $\mathcal{O}$ with respect to *admissible blow-ups*, i.e. blowups along an ideal containing a power of t. In the words of Fujiwara and Kato, *rigid geometry is the birational geometry of formal schemes*.

1.4 *Why study rigid geometry?*

The goal of the course is not only to introduce the basic definitions and facts surrounding rigid-analytic varieties—we will see some important applications of the theory as well. I will now try to give a short preview without spoilers.

Disclaimer: There are many possible answers to the question above. The following is heavily influenced by my own perspective and expertise as an algebraic geometer interested in the topology of algebraic varieties.

The broad answer is:

Rigid geometry allows us to use methods of topology and analysis in an otherwise purely algebraic context.

For an explicit example, consider a complex algebraic curve, say a smooth plane curve X in $\mathbf{P}^2$ of degree d. As one learns in the basic algebraic geometry course, this curve has genus

$$g = \frac{(d-1)(d-2)}{2}.$$

Over the complex numbers, the underlying manifold (the *complex analytification*) of X is an oriented surface with g many handles. Can we make sense of the last sentence algebraically? The question sounds crazy at first: to begin with, the underlying topological space of X (with the Zariski topology) does not see the genus at all, so how can we try to decompose it into handles?

Rigid geometry allows us to break varieties into pieces and perform surgery.

The answer is to *degenerate* the curve until it breaks and becomes easier to manage.[11] Thus, let $\ell_1, \ldots, \ell_d$ be generically chosen linear forms on $\mathbf{P}^2$. If $\{f = 0\}$ is the homogeneous equation of our curve X, we consider the equation with an additional parameter t

$$X_t = \{tf + (1-t)\ell_1 \cdot \ldots \cdot \ell_d = 0\} \quad \subseteq \quad \mathbf{P}^2_{k[t]}.$$

[11] Can we study algebraic curves by putting them inside the Large Hadron Collider?

Thus $X_1 = X$, while X_0 is the union of d lines in $\mathbf{P}^2$ in general position.

The curve X_0, while much easier to understand than X, is singular. Its topology differs from that of X. The idea, made possible by rigid geometry, is to study the smooth fibers X_t which "infinitesimally close" to X_0. To make this precise, we first base change the above family to the field $K = k((t))$, obtaining a smooth algebraic curve X_K over K. Next, we turn it into a rigid-analytic variety $\mathscr{X} = (X_K)_{\mathrm{an}}$, its *rigid analytification*. It is cut out by the same equation in a rigid-analytic version of $\mathbf{P}^2_K$.

It turns out that $\mathscr{X}$ is "close enough" to X_0 that there exists a natural morphism of topological spaces (the *specialization map*)

$$\mathrm{sp}\colon |\mathscr{X}| \to |X_0|.$$

The preimage $U_i = \mathrm{sp}^{-1}(L_i)$ of the line $L_i = \{\ell_i = 0\} \subseteq |X_0|$ happens to be an *open* rigid subvariety of $\mathscr{X}$ which closely resembles a sphere with $d-1$ discs removed (the discs are the preimages of the points $L_i \cap L_j$ for $j \neq i$ under sp). This gives a combinatorial decomposition of $\mathscr{X}$ which one can use in place of the triangulation or handlebody decomposition on the complex analytification. For cubic curves (elliptic curves) one has the following picture:

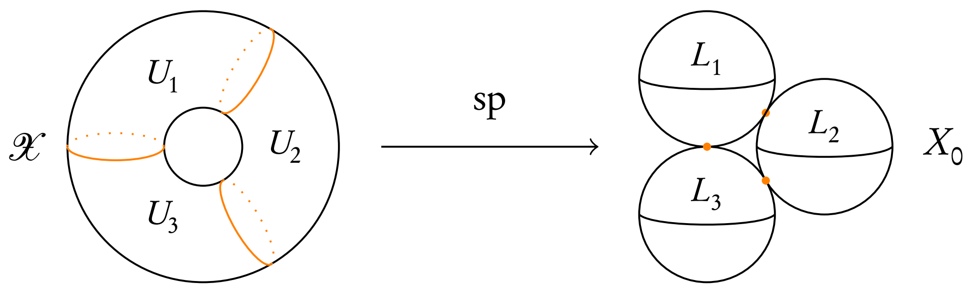


Figure 1.1: Intuitive picture of the specialization map ($d = 3$, so $g = 1$).

HERE ARE SOME EXAMPLES of serious applications of rigid geometry roughly along the above lines:

- Uniformization of curves and abelian varieties. (In fact, constructing a p-adic analytic analog of the expression of a complex elliptic curve as $\mathbf{C}$ modulo a lattice was Tate's original motivation for defining rigid-analytic varieties. We will see Tate's uniformization later in the course.)
- The approach to SYZ mirror symmetry proposed by Kontsevich.
- Raynaud's solution to Abhyankar's conjecture (constructing finite étale covers of $\mathbf{A}^1_{\mathbf{F}_p}$ with given Galois group).
- Study of moduli of curves (often done using tropical methods, which is philosophically similar).
- Semistable reduction.

Other extremely important applications belong to p-adic Hodge theory.

2

Non-archimedean fields

In this chapter, we learn some fundamentals about the kind of base fields we will work with — fields complete with respect to a nontrivial non-archimedean norm. We start with basic facts about general valuation rings; the extra generality is not needed for Tate's theory, but will prove useful later on.

In the appendix to this chapter, we review henselian local rings and Hensel's lemma.

2.1 Valuation rings and valuations

Definition 2.1.1. A subring $\mathcal{O}$ of a field K is a *valuation (sub)ring* of K if for every $x \in K^\times$, either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$.

The above condition implies that $K = \operatorname{Frac} \mathcal{O}$. This motivates the terminology: we will call a ring $\mathcal{O}$ a *valuation ring* if $\mathcal{O}$ is a domain and if it is a valuation ring of $K = \operatorname{Frac} \mathcal{O}$.

Lemma 2.1.2. *Every valuation ring is a local ring.*

Proof. It suffices to check that the set of non-units is closed under addition. If $x, y \in \mathcal{O}$ are nonzero non-units, then either $xy^{-1} \in \mathcal{O}$, in which case $x + y = y(xy^{-1} + 1)$ is a non-unit because y is a non-unit, or $yx^{-1} \in \mathcal{O}$, and we swap x and y. □

Lemma 2.1.3. *The relation*

$$x \le y \quad \textit{if} \quad yx^{-1} \in \mathcal{O} \tag{2.1}$$

induces a linear order on $\Gamma = K^\times/\mathcal{O}^\times$*, making* Γ *into a linearly ordered group.* [1]

[1] An *ordered abelian group* is an abelian group Γ with an order relation $\le$ such that $a \le b$ implies $a + c \le b + c$. It is *linearly* or *totally* ordered if $\le$ is a linear order.

Proof. First, if $x' = ux$ and $y' = vx$ with $u, v \in R^\times$, then $x \le y \iff x' \le y'$, so that $\le$ induces a relation on $K^\times/\mathcal{O}^\times$. The fact that either $x \le y$ or $y \le x$ is the definition of a valuation ring. The rest is straightforward. □

The quotient homomorphism

$$K^\times \to K^\times/\mathcal{O}^\times$$

is a "valuation" on the field K, as we shall now define. First, we introduce the following convention: for an ordered abelian group Γ (written additively), we shall write $\Gamma \cup \{\infty\}$ for the ordered commutative monoid obtained by adding an element ∞ and declaring

$$\gamma \le \infty \quad \text{and} \quad \gamma + \infty = \infty + \infty = \infty \quad (\gamma \in \Gamma).$$

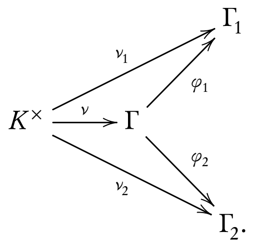
Definition 2.1.4. A *valuation* on a field K is a group homomorphism

$$v\colon K^\times \to \Gamma$$

into a linearly ordered group Γ (written additively, so that $v(xy) = v(x) + v(y)$), which, when extended to a map of monoids $v\colon K \to \Gamma \cup \{\infty\}$ by $v(0) = \infty$, satisfies

$$v(x+y) \geq \min\{v(x), v(y)\}.$$

The *value group* of a valuation $v\colon K^\times \to \Gamma$ is the image $v(K^\times)$. Thus v trivially induces a surjective valuation $v'\colon K^\times \to v(K^\times)$, and it is useful to identify v and v'. More generally, we will call two valuations $v_i\colon K^\times \to \Gamma_i$ $(i = 1,2)$ *equivalent* if there exists a third valuation $v\colon K^\times \to \Gamma$ and monotone homomorphisms $\varphi_i\colon \Gamma \to \Gamma_i$ $(i = 1,2)$ such that $v_i = \varphi_i \circ v$:



A valuation is *trivial* if it has trivial value group, i.e. $v(x) = 0$ for all $x \in K^\times$.

Proposition 2.1.5. *Let K be a field.*

(a) If $\mathcal{O} \subseteq K$ is a valuation ring and $\Gamma = K^\times/\mathcal{O}^\times$ is equipped with the linear order (2.1), *then the projection map $v\colon K^\times \to \Gamma$ is a valuation on K.*

(b) Conversely, if $v\colon K^\times \to \Gamma$ is a valuation, then

$$\mathcal{O} = \{x \in K \mid v(x) \geq 0\}$$

is a valuation ring of K, and its maximal ideal is $\mathfrak{m} = \{x \in K \mid v(x) > 0\}$.

(c) Constructions in (a) and (b) produce mutually inverse bijections

$$\{\text{valuation rings of } K\} \simeq \{\text{valuations on } K\}/\text{equivalence}.$$

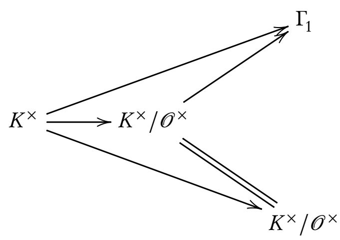
Proof. (a) We check the property $v(x+y) \geq \min\{v(x), v(y)\}$, which resembles the proof that a valuation ring is local. Let $x, y \in K^\times$, and suppose $xy^{-1} \in \mathcal{O}$, then

$$v(x+y) = v(y(xy^{-1}+1)) = v(y) + \underbrace{v(xy^{-1}+1)}_{\geq 0 \text{ since } xy^{-1}+1 \in \mathcal{O}} \geq v(y),$$

and similarly if $yx^{-1} \in \mathcal{O}$.

(b) Clearly for $x \in K$ either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$ and $\mathcal{O}$ is closed under multiplication. The fact that it is also closed under addition follows from $v(x+y) \geq \min\{v(x), v(y)\}$.

(c) Clearly, equivalent valuations define the same valuation ring. The only non-obvious assertion is that if $v_2\colon K^\times \to \Gamma_2 = K^\times/\mathcal{O}^\times$ is the valuation associated via (b) to the valuation ring $\mathcal{O}$ associated to a valuation $v_1\colon K^\times \to \Gamma_1$ via (a), then v_1 and v_2 are equivalent. We let $\Gamma = \Gamma_2 = K^\times/\mathcal{O}^\times$, φ_2 the identity, and $\varphi_2\colon \Gamma = K^\times/\mathcal{O}^\times \to \Gamma_1$ the map induced by v_1. □



2.2 Valuations and norms

If the value group is a subgroup of $\mathbf{R}$, one can turn a valuation into a "norm."

Definition 2.2.1. A *valuation of height one*[2] is a valuation $v\colon K^\times \to \mathbf{R}$.

[2] This terminology is slightly non-standard: what is usually meant by a valuation of height one is a nontrivial valuation whose value group *embeds* in $\mathbf{R}$.

More generally, the *height* (or *rank*) of a valuation is the order type of the set of all convex subgroups of the value group, (lin-

Note that two valuations of height one $v_i : K^\times \to \mathbf{R}$ $(i = 1, 2)$ are equivalent if and only if $v_2(x) = cv_1(x)$ for some positive real c.[3]

Definition 2.2.2. A *nonarchimedean norm* on a field K is a map

$$|\cdot| : K \to [0, \infty)$$

such that

i. $|xy| = |x| \cdot |y|$,

ii. $|x| = 0$ if and only if $x = 0$,

iii. $|x + y| \leq \max\{|x|, |y|\}$.

Proposition 2.2.3. *Let K be a field.*

(a) If $v : K \to \mathbf{R}$ is valuation of height one, then[4]

$$|x| = \exp(-v(x))$$

(where $\exp(-\infty) = 0$) defines a nonarchimedean norm on K.

(b) Conversely, if $|\cdot|$ is a norm on K, then

$$v(x) = -\log|x|$$

(where $\log 0 = -\infty$) defines a valuation of height one. The corresponding valuation ring is the "closed ball" $\mathcal{O} = \{x \mid |x| \leq 1\}$.

(c) The constructions in (a) and (b) produce mutually inverse bijections

$$\{\textit{height one valuations on } K\} \simeq \{\textit{nonarchimedean norms on } K\}.$$

Proof. Clear. □

Proposition 2.2.4. *Let $|\cdot|$ be a nonarchimedean norm on a field K. Then*

$$d(x, y) = |x - y|$$

defines a metric on K, making K into a topological field. This metric and the induced topology have the following properties:

(a) Every triangle is isosceles, every point of an open ball is its center, and every two (open or closed) balls are either disjoint or one contains the other,

(b) The open ball $\{|x - a| < \rho\}$, the closed ball $\{|x - a| \leq \rho\}$, and the sphere $\{|x - a| = \rho\}$ are both open and closed for $\rho > 0$. In particular, the valuation ring $\mathcal{O} = \{|x| \leq 1\} \subseteq K$ is an open subring.

(c) The topological space K is totally disconnected,

(d) Suppose that K is complete (every Cauchy sequence converges). A series $\sum_{n=0}^{\infty} a_n$ with $a_n \in K$ converges if and only if $\lim a_n = 0$.

[3] Exercise 3 on Problem Set 1.

[4] The base e of the exponential is of course an arbitrary choice. Sometimes there exists a more natural one. For example, if K is p-adic, i.e. $|p| < 1$ for a prime p, then one usually considers the norm

$$|x| = p^{-v(x)}.$$

Proof. Continuity of addition, multiplication, and inverse is clear and left to the reader.

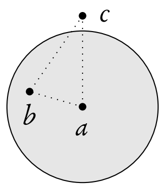
(a) The key observation is that if $|x| > |y|$, then $|x-y| = \max\{|x|,|y|\} = |x|$. Indeed, we have

$$|x| = |y + (x-y)| \leq \max\{|y|, |x-y|\} \leq \max\{|y|,|x|,|y|\} = |x|,$$

so the inequalities are equalities, implying $|x-y| = |x|$. Similarly, if $|y| > |x|$ then $|x-y| = |y|$, thus in general two of the numbers $|x|, |y|, |x-y|$ have to be equal.

If a triangle has vertices a, b, c, apply the above to $x = c - a$, $y = c - b$ to see that it is isosceles, with two longest sides being equal.

Now consider an open ball $B(a,\rho) = \{|x-a| < \rho\}$ and let $b \in B$, i.e. $|b - a| < \rho$. If $c \in K$, then consider the triangle with vertices a, b, c. The above observation shows that $|c-a| \geq \rho$ if and only if $|c - b| \geq \rho$, showing $B(a,\rho) = B(b,\rho)$.



If two open balls B and B' intersect at a point b, then taking b as the center of both balls shows that one is contained in the other.

(b) The open ball is of course open, and the closed ball is the union of the open ball and the sphere. It suffices to treat the sphere $S = \{|x| = \rho\}$ (centered at zero for simplicity). Let $a \in S$; we claim that the open ball $\{|x-a| < \rho\}$ is contained in S. Indeed, if $|x-a| < \rho$ then $|x| = |a + (x-a)|$ and since $|x-a| < \rho = |a|$, we have $|x| = |a| = \rho$, so $x \in S$.

(c) Let $S \subseteq K$ be a subset and let $a, b \in S$ be two distinct points, $\rho = |a - b| > 0$. Then

$$S = (S \cap \{|x-a| < \rho/2\}) \cup (S \cap \{|x-a| \geq \rho/2\})$$

expresses S as a sum of two disjoint and non-empty open subsets. Thus S cannot be connected if it has more than one point.

(d) Clearly if $\sum a_n$ converges then $\lim a_n = 0$. Conversely, suppose $\lim a_n = 0$; we check that $b_n = a_1 + \cdots + a_n$ forms a Cauchy sequence. Let $\varepsilon > 0$, and let N be such that $|a_n| < \varepsilon$ for $n \geq N$. Then for $m > n > N$

$$|b_m - b_n| = |a_{n+1} + \cdots + a_m| < \max\{|a_{n+1}|, \ldots, |a_m|\} < \varepsilon. \qquad \square$$

2.3 *Geometric examples of valuations*

This section is a bit of a digression, but will become important later in the course.

Long long time ago, before schemes were invented by Grothendieck, varieties were studied (or even defined) using valuations on their function fields. E.g. Zariski's proof of resolution of singularities on surfaces heavily relied on the classification of valuations on their function fields. We will see some of these below.

Example 2.3.1. Let R be a Dedekind domain with field of fractions K, and let $\mathfrak{m} \subseteq R$ be a maximal ideal. Standard examples:

- $R = \Gamma(X, \mathcal{O}_X)$ for X a smooth affine algebraic curve, with $\mathfrak{m}$ corresponding to a closed point $x \in X$,
- $R = \mathcal{O}_K$ the ring of integers in a number field K, e.g. $R = \mathbf{Z}[i]$.

The local ring $\mathcal{O} = R_{\mathfrak{m}}$ is a discrete valuation subring of K. The corresponding valuation on K is $\nu(x) = \max\{k : x \in \mathfrak{m}^k\}$. Every valuation on K which is trivial on k is equivalent to exactly one of these. [5]

[5] Sound familiar? [10, Chapter I 6]

The remaining examples deal valuations on function fields of surfaces over a base field k, where the situation is much more complicated, essentially due to the existence of non-trivial blowups. [6] We only consider valuations whose restriction to k is trivial.

[6] See [10, Exercise II 4.12].

Example 2.3.2 (Divisorial valuation)**.** Let X be a normal surface with field of rational functions K and let $D \subseteq S$ be a prime divisor. Then [10, II 6] D defines a function "order of zero along D"

$$\nu_D : K = k(S) \to \mathbf{Z} \cup \{\infty\}$$

which is a valuation. The corresponding valuation ring is $\mathcal{O}_{X,\xi}$ where ξ is the generic point of D. Its residue field is $k(D)$, the function field of D.

Example 2.3.3 (Valuation of height two)**.** In the situation of Example 2.3.2, let $p \in D$ be a closed point at which D is regular. Then x defines a valuation ν_p on $k(D)$ as in Example 2.3.1. We can combine the valuations ν_D on $K = k(S)$ and ν_p on $k(D)$ into a height two valuation

$$\nu_{D,p} : K \to \mathbf{Z}^2_{\mathrm{lex}} \cup \{\infty\},$$

where $\mathbf{Z}^2_{\mathrm{lex}}$ is $\mathbf{Z}^2$ with the lexicographic order ($(x,y) \geq (x',y')$ if $x > x'$ or $x = x'$ and $y \geq y'$). To define $\nu_{D,p}$, we pick a uniformizer (generator of the maximal ideal) $\pi \in \mathcal{O}_{X,\xi} = \mathcal{O}_{\nu_D}$ without zero or pole at p and set

$$\nu_{D,p}(f) = (\nu_D(f), \nu_p(g)), \quad g = (\pi^{-\nu_D(f)} f)|_\xi,$$

where the restriction makes sense because $\nu_D(\pi) = 1$, so $\pi^{-\nu_D(f)} f \in \mathcal{O}_{\nu_D}$.

The valuation ring $\mathcal{O}_{\nu_{D,p}}$ consists of rational functions with no pole along D and whose restriction to D has no pole at p. It has three prime ideals, is of Krull dimension two, and is non-Noetherian. Its residue field is k. See Figure 2.1 for the monoid of monomials in $\mathcal{O}_{\nu_{D,p}}$ for $S = \mathbf{A}^2$.

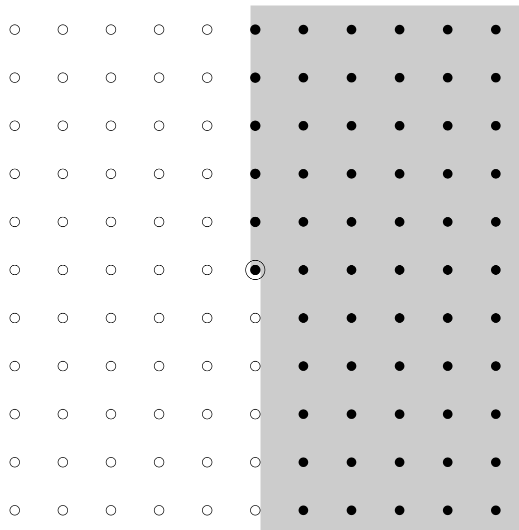


Figure 2.1: In Example 2.3.3, consider $S = \mathbf{A}^2$ with coordinates x, y, the divisor $D = \{x = 0\} \subseteq S$, and the point $p = \{y = 0\} \subseteq D$. The figure shows the monoid consisting of all $(m,n) \in \mathbf{Z}^2$ for which $\nu(x^m y^n) \geq 0$. Can you see why this monoid is not finitely generated? This is related to the fact that the valuation ring is non-Noetherian.

Example 2.3.4 (Valuations from formal curve germs)**.** Let again S be a normal surface with function field K, and let

$$\gamma \colon \operatorname{Spec} k[[t]] \to S$$

be a morphism of schemes (a "formal curve germ"). We say that γ is *nonalgebraic* if its image is not contained in a proper closed subscheme of S, equivalently if γ maps the generic point $\operatorname{Spec} k((t))$ of $\operatorname{Spec} k[[t]]$ to the generic point $\eta = \operatorname{Spec} K$ of S.[7] The composition of γ^* with the standard valuation on $k((t))$ gives a height one valuation

$$\nu_\gamma : K \to k((t)) \to \mathbf{Z} \cup \{\infty\}$$

with residue field k.

[7] There is plenty of nonalgebraic curve germs on an algebraic surface. For example, consider $S = \operatorname{Spec} \mathbf{C}[x,y]$ the affine plane and γ defined by

$$\gamma^*(x) = t, \quad \gamma^*(y) = \exp t = \sum_{n \geq 0} \frac{t^n}{n!}.$$

Example 2.3.5 (Height one valuation with dense value group)**.** Suppose that $K = k(x,y)$. Let λ be an irrational real number. Define the weight function on monomials in x and y by

$$\mathrm{weight}_\lambda(x^m y^n) = m + \lambda n \quad \in \quad \mathbf{R}.$$

Define the valuation $v_\lambda : K \to \mathbf{R} \cup \{\infty\}$ by first defining it on polynomials:

$$v_\lambda\left(\sum_{m,n\geq 0} a_{mn} x^m y^n\right) = \min\{\mathrm{weight}_\lambda(x^m y^n) : a_{mn} \neq 0\}$$

and extending to $k(x,y)$ by $v_\lambda(f/g) = v_\lambda(f) - v_\lambda(g)$. This gives a valuation on K which has height one but whose value group $\mathbf{Z} \oplus \lambda\mathbf{Z} \simeq \mathbf{Z}^2$ is dense in $\mathbf{R}$. See Figure 2.2 for the monoid of monomials in the valuation ring.

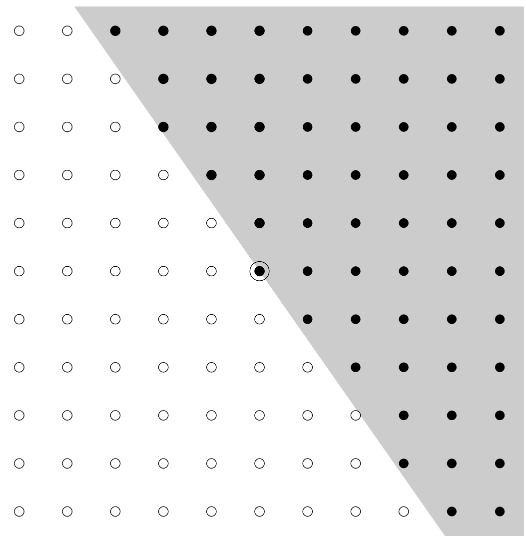


Figure 2.2: The monoid of all $(m,n) \in \mathbf{Z}^2$ for which $v(x^m y^n) \geq 0$ (Example 2.3.5). The boundary of the gray area is the line with slope $-1/\lambda$

$$x + \lambda y = 0.$$

Since $\lambda \notin \mathbf{Q}$, this line contains no nonzero lattice points.

Remark 2.3.6. The valuation v_λ in Example 2.3.5 can be thought of as the valuation of the type considered in Example 2.3.4 induced by the "formal curve germ"

$$t \mapsto (t, t^\lambda).$$

In fact, for $\lambda' = a/b$ rational with $(a,b) = 1$, we can define the curve germ

$$\gamma_{a,b} : \operatorname{Spec} \mathbf{C}[[t]] \to \mathbf{A}^2_{x,y}, \quad \gamma^*_{a,b}(x) = t^b, \quad \gamma^*_{a,b}(x) = t^a.$$

Let $v_{a,b} = \frac{1}{b} v_{\gamma_{a,b}}$ where $\gamma_{a,b}$ is the valuation associated to the curve germ as in Example 2.3.4. If $a_n/b_n \to \lambda$, then the corresponding valuations v_{a_n,b_n} converge pointwise to v_λ.

2.4 *Nonarchimedean fields*

Definition 2.4.1. A *nonarchimedean field*[8] is a field K equipped with a nontrivial nonarchimedean norm $|\cdot|$ with respect to which it is complete.

[8] For many authors, "nonarchimedean field" is simply a field with a nonarchimedean norm.

Proposition 2.4.2. *Let K be a field endowed with a nontrivial nonarchimedean norm $|\cdot|$. The ring operations on K extend uniquely to the completion $\widehat{K}$ of K with respect to $d(x,y) = |x - y|$, making $\widehat{K}$ into a nonarchimedean field.*

Definition 2.4.3. Let K be a field endowed with a nonarchimedean norm $|\cdot|$. A *pseudouniformizer* is an element $t \in K$ with $0 < |t| < 1$.[9]

[9] In other words, t is a *topologically nilpotent unit*, where topologically nilpotent means that $|t^n| \to 0$.

Thus $|\cdot|$ is nontrivial if and only if K admits a pseudouniformizer.

Proposition 2.4.4. *Let K be a field endowed with a nontrivial nonarchimedean norm $|\cdot|$, and let $t \in K$ be a pseudouniformizer. Let $\mathcal{O} = \{x \in K \,|\, |x| \leq 1\}$ be the valuation ring. Then K is complete (i.e. K is a nonarchimedean field) if and only if $\mathcal{O}$ is t-adically complete and separated, i.e. if the natural map*

$$\pi\colon \mathcal{O} \to \varprojlim_n \mathcal{O}/t^n\mathcal{O}$$

is an isomorphism. In this case, the map π is a homeomorphism, where the target is endowed with the inverse limit topology where each $\mathcal{O}/t^n\mathcal{O}$ is given the discrete topology.

Warning: if K is not discretely valued, then $\mathcal{O}$ will not be a complete local ring! In that case, the maximal ideal of $\mathcal{O}$ satisfies $\mathfrak{m}^2 = \mathfrak{m}$, and hence $\mathcal{O}/\mathfrak{m}^n = k$ for all n, so that $\widehat{\mathcal{O}} \simeq k$. This is why we need to work with pseudouniformizers.

Proof. Set $\rho = |t|$; we have $0 < \rho < 1$. First, we note that

$$t^n\mathcal{O} = \{x \in K : |x| \leq \rho^n\}.$$

The kernel of π is $\bigcap_{n\geq 0} t^n\mathcal{O} = \{|x| \leq 0\} = \{0\}$. Thus π is always injective.

An element $\bar{f}$ of the inverse limit is a compatible system $(\bar{f}_n \in \mathcal{O}/t^n\mathcal{O})$. Let $f_n \in \mathcal{O}$ be elements mapping to $\bar{f}_n \in \mathcal{O}/t^n\mathcal{O}$. We claim that (f_n) is a Cauchy sequence. Indeed, we have $f_n - f_m \in t^n\mathcal{O}$ for $m > n$, so $|f_n - f_m| \leq \rho^n$ for $m > n$. Thus if K is complete, then (f_n) has a limit $f \in \mathcal{O}$. Now for every n, we have

$$|f - f_n| = |f_n - f_m| \leq \rho^n \quad \text{for} \quad m \gg 0,$$

which shows that $f - f_n \in t^n\mathcal{O}$. Thus $\pi(f) = \bar{f}$, i.e. π is surjective if K is complete.

Conversely, suppose that π is surjective. We will show that $\mathcal{O}$ is complete with respect to $|\cdot|$ (this easily implies that K is complete). Let $(f_n) \in \mathcal{O}$ be a Cauchy sequence. For every m, the images of f_n in $\mathcal{O}/t^m\mathcal{O}$ have to stabilize for $n \gg 0$. Let $\bar{f}_m \in \mathcal{O}/t^m\mathcal{O}$ be the stable value (i.e. $\bar{f}_m = \lim_n(f_n \bmod t^m)$ for the discrete topology on $\mathcal{O}/t^n\mathcal{O}$). It is easy to see that $\bar{f} = (\bar{f}_m)$ is an element of the inverse limit of $\mathcal{O}/t^n\mathcal{O}$. Let $f \in \mathcal{O}$ be an element with $\pi(f) = \bar{f}$, then $f = \lim f_n$.

The claim about the topologies follows from the fact that $t^n\mathcal{O} = \{|x| \leq \rho^n\}$ is a basis of neighborhoods of zero in $\mathcal{O}$. □

2.5 *Extensions of nonarchimedean fields*

The treatment here follows [5, Appendix A] and [12, II §4 and §6].

Theorem 2.5.1. *Let K be a nonarchimedean field and let L/K be a finite extension. Then there exists a unique norm $|\cdot|$ on L extending K. The field L endowed with this norm is a nonarchimedean field.*

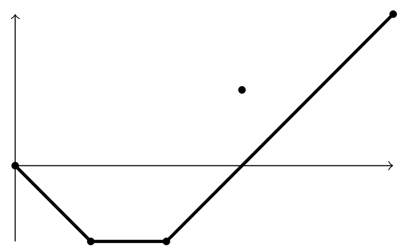


Figure 2.3: Newton polygon of the polynomial

$1+\pi^{-1}X-\pi^{-1}X^2+\pi X^3+\pi^2X^5$

For $f = \sum_{i=0}^n a_i x^i \in K[X]$, we define its *Newton polygon* $\mathrm{NP}(f)$ as the lower convex envelope of the set $\{(0, v(a_0)), \ldots, (n, v(a_n))\}$ in $\mathbf{R}^2$. Its basic property is that $\mathrm{NP}(fg) = \mathrm{NP}(f) + \mathrm{NP}(g)$ (Minkowski sum, i.e. sort the segments of both polygons by slope and concatenate). In particular, if f is reducible, then $\mathrm{NP}(f)$ contains a point of the form (m, γ) with $0 < m < \deg f$ an integer and γ an element of the value group. One form of Hensel's lemma[10] states a partial converse:

Lemma 2.5.2 (Irreducibility and Newton polygons). *Let $f \in K[X]$ be a nonzero polynomial with $f(0) \neq 0$. Then f is irreducible if $\mathrm{NP}(f)$ is a single segment without interior points of the form (m, γ) with $m \in \mathbf{Z}$ and $\gamma \in v(K^\times)$. Conversely:*

(a) (Weak form) If $\mathrm{NP}(f)$ has segments both of negative and of non-negative slope, then f is reducible.

[10] In the appendix to this lecture, we shall discuss different formulations of Hensel's lemma.

(b) (Strong form) If f is irreducible, then $\mathrm{NP}(f)$ *is a single segment.*

We shall prove the weak form now. It will be sufficient for the proof of Theorem 2.5.1, which in turn will be used to prove the strong form.

Proof (of the weak form). The first assertion has already been explained in the discussion preceding the statement of the lemma. To show (a), let (m,γ) be a vertex of $\mathrm{NP}(f)$ with smallest γ, and with smallest m among those. Then $0 < m < \deg f$, otherwise all slopes of $\mathrm{NP}(f)$ have the same sign (see Figure 2.5). Replacing f with $a_m^{-1} f$, we may assume that $\gamma = 0$, and consequently $f \in \mathcal{O}[X]$. The image $\bar{f}$ of f in $k[X]$ decomposes as

$$\bar{f} = X^m h(X) \quad \text{with} \quad h(0) \neq 0.$$

By Hensel's lemma (Proposition 2.A.5) using the formulation as in Proposition 2.A.1(b), the above factorization lifts to a factorization $f = \tilde{g}\tilde{h}$ with $\deg \tilde{g} = m$. Therefore f is reducible. □

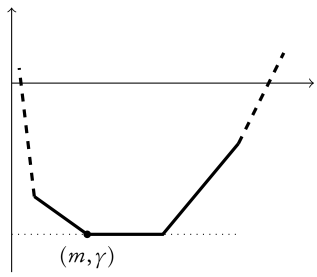


Figure 2.4: Proof of Lemma 2.5.2(a)

Proposition 2.5.3. *In the situation of Theorem 2.5.1, let $\mathcal{O} = \{|x| \leq 1\}$ be the valuation ring of K. An element $x \in L$ is integral over $\mathcal{O}$ if and only if* $\mathrm{Nm}_{L/K}(x) \in \mathcal{O}$.

Proof. Let $f \in K[X]$ be the minimal polynomial of x. Since f is irreducible, by Lemma 2.5.2 its Newton polygon has to be the line segment with endpoints $(\deg f, 0)$ and $(0, c)$ where $c = v(a_0)$ is the valuation of the constant term of f (Figure 2.5). But $c = (-1)^n \mathrm{Nm}_{L/K}(x)$, so if $\mathrm{Nm}_{L/K}(x) \in \mathcal{O}_K$ then $\mathrm{NP}(f)$ lies entirely above the line $y = 0$, which implies that $f \in \mathcal{O}[X]$, so that x is integral over $\mathcal{O}$.

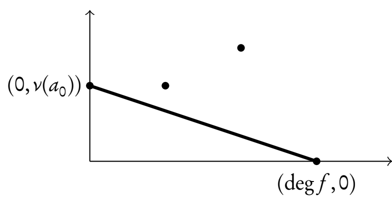


Figure 2.5: Newton polygon of an irreducible monic polynomial f (Proof of Proposition 2.5.3)

Conversely, if x is integral, then in fact its minimal polynomial f belongs to $\mathcal{O}[X]$; in particular, $\mathrm{Nm}_{L/K}(x) = (-1)^{\deg f} f(0) \in \mathcal{O}$. To see this, let $g \in \mathcal{O}[X]$ be monic with $g(x) = 0$. We have $g = fh$ for some (also monic) $h \in K[X]$. Then $\mathrm{NP}(g) = \mathrm{NP}(f) + \mathrm{NP}(h)$ lies above the line $y = 0$ and ends on it (because it is monic), and hence all of its slopes are non-positive. However, $\mathrm{NP}(f)$ is a single segment (connecting $(0, c)$ and $(\deg f, 0)$), and its slope is one of the slopes of $\mathrm{NP}(g)$ and hence is non-positive. Thus $c \geq 0$, i.e. $f \in \mathcal{O}[X]$. □

Proof of Theorem 2.5.1. Let $\mathcal{O} = \{|x| \leq 1\} \subseteq K$ be the valuation ring of K and let $\mathcal{O}' \subseteq L$ be the integral closure of $\mathcal{O}$ inside L. By Proposition 2.5.3, $x \in \mathcal{O}'$ if and only if $|\mathrm{Nm}_{L/K}(x)| \leq 1$. Since the norm is multiplicative, this shows that $\mathcal{O}'$ is a valuation ring of L. Moreover, $\mathcal{O}' \cap K = \mathcal{O}$ because $\mathcal{O}$ is integrally closed.[11]

Define $|x| = |\mathrm{Nm}_{L/K}(x)|^{1/d}$ for $x \in L$, where $d = [L : K]$. This restricts to the norm on K, is multiplicative, and $|x| \neq 0$ for $x \neq 0$. To show $|x + y| \leq \max\{|x|, |y|\}$, we use the fact that $\{|x| \leq 1\} = \mathcal{O}'$ is a valuation ring.

If $|\cdot|'$ is some other norm extending $|\cdot|$ to L, then since the corresponding valuation ring $\{|x|' \leq 1\}$ is integrally closed, it contains $\mathcal{O}'$. This implies that $|\cdot| \leq |\cdot|'$, and by Exercise 3 from Problem Set 1, we have $|\cdot|' = |\cdot|^c$ for some constant c. But $c = 1$ since the two agree on K. □

[11] Easy exercise: show that every valuation ring is integrally closed.

Theorem 2.5.4 (Krasner)**.** *Let K be a nonarchimedean field, and let $\overline{K}$ be an algebraic closure of K, which we endow with the unique extension of $|\cdot|$. Let $\widehat{\overline{K}}$ denote the completion of $\overline{K}$ with respect to this norm. Then $\widehat{\overline{K}}$ is algebraically closed.*

Proof. Let L be a finite extension of $\widehat{\overline{K}}$. By Theorem 2.5.1, there exists a unique norm on L extending the norm on $\widehat{\overline{K}}$ and L is complete with respect to that norm. To show $L = \widehat{\overline{K}}$, it therefore suffices to prove that $\widehat{\overline{K}}$ is dense in L.

Let $x \in L$ and let $1 > \rho > 0$. We shall find a $y \in \widehat{\overline{K}}$ with $|x - y| < \rho$. Without loss of generality, we may assume that $|x| \leq 1$. Let $f = \sum_{i=0}^{n} a_i X^i \in \widehat{\overline{K}}[X]$ be its minimal polynomial (with $a_n = 1$). Since $\overline{K}$ is dense in $\widehat{\overline{K}}$, we can find $b_i \in \overline{K}$ ($i = 0, \ldots, n$) with $|a_i - b_i| < \rho$ (and again $b_n = 1$). This implies that

$$|g(x)| = |g(x) - f(x)| = \left| \sum_{i=0}^{n} (a_i - b_i) x^i \right| < \rho.$$

Now, the polynomial $g = \sum_{i=0}^{n} b_i X^i$ splits completely in $\overline{K}$:

$$g = \prod_{i=1}^{n} (X - y_i), \quad y_1, \ldots, y_n \in \overline{K}.$$

Evaluating at x and taking absolute value, we obtain

$$\rho > |g(x)| = \prod_{i=1}^{n} |x - y_i|.$$

Therefore one of the factors is less than ρ. □

2.6 *Slopes of the Newton polygon*

We can now prove the promised strong form of Lemma 2.5.2. It will not be used later in the course.

Lemma 2.6.1. *If $f \in K[X]$ is irreducible, then all roots of f in $\overline{K}$ have the same norm.*

Proof. Let L/K be the splitting field of f and let $G = \mathrm{Gal}(L/K)$. Thus G acts transitively on the roots of f in L. Since the norm $|\cdot|$ on L extending the norm on K is unique, the group G acts on L by isometries. In particular, for any two roots α, β of f in L we can find $g \in G$ with $\beta = g(\alpha)$, and then

$$|\alpha| = |g(\alpha)| = |\beta|.$$

□

For a real number λ and $f \in K[X]$, we define the *slope multiplicity* $\mu(\lambda, f)$ of λ in $\mathrm{NP}(f)$ as the length of the projection on the x-axis of the segment in $\mathrm{NP}(f)$ with slope λ (zero if it does not exist), see Figure 2.6. Additivity of Newton polygons means precisely that

$$\mu(\lambda, fg) = \mu(\lambda, f) + \mu(\lambda, g) \quad \text{for every } \lambda \in \mathbf{R}.$$

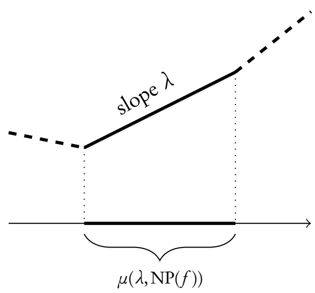


Figure 2.6: Slope multiplicity

Lemma 2.6.2. *For $f \in K[X]$ and $r > 0$, we have*

$$\#\left\{\alpha \in \overline{K} : f(\alpha) = 0 \text{ and } |\alpha| = r\right\} = \mu(\log r, f).$$

Proof. By additivity of both sides of the asserted equality, we may assume that f is irreducible, in which case all roots of f have the same absolute value ρ by Lemma 2.6.1. We may also assume that f is monic and $\rho \neq 0$, and write

$$f = \sum_{i=0}^{n} a_{n-i} X^i = \prod_{j=1}^{n} (X - \alpha_j), \quad |\alpha_j| = \rho.$$

Therefore for $0 < i \leq n$ we have

$$a_i = (-1)^i \sum_{0 \leq j_1 < \ldots < j_i \leq n} \alpha_{j_1} \cdot \ldots \cdot \alpha_{j_i},$$

and taking absolute values we obtain

$$|a_i| \leq \rho^i \quad \text{and} \quad |a_n| = |\alpha_1 \cdot \ldots \cdot \alpha_n| = \rho^n.$$

It follows that NP(f) is the segment connecting the points $(0, v(a_n)) = (0, -n\log\rho)$ and $(n, 0)$. This implies the asserted equality for $\rho = r$, with both sides equal to $n = \deg f$. Therefore for $r \neq \rho$ both sides are zero, and hence the assertion is true for every $r > 0$. □

Proof of the strong form of Lemma 2.5.2. Let $f \in K[X]$ be irreducible. By Lemma 2.6.1, all roots of f have the same absolute value. By Lemma 2.6.2, the Newton polygon NP(f) has a single slope, i.e. it is a segment. □

2.A Henselian rings

Hensel's lemma played an important in the proof of Theorem 2.5.1. The first goal of this section is to elucidate its role by introducing the notion of a *henselian local ring*. Roughly speaking, it is a local ring in which the assertion of Hensel's lemma holds. There are however many equivalent characterizations of this class of local rings, reviewed in Proposition 2.A.1 below, and the reader familiar with the étale topology will surely appreciate the topological flavor of some of them. The second goal is to prove Hensel's lemma in its general form: *a local ring complete with respect to a* $\mathfrak{m}$*-primary ideal is henselian*.

Our treatment follows the Stacks Project [14, Tag 04GE].

The ultimate reference is Raynaud's book *Anneaux locaux henseliens*.

Proposition 2.A.1. *Let A be a local ring with maximal ideal $\mathfrak{m}$. We set $k = A/\mathfrak{m}$, $x = \operatorname{Spec} k$, $X = \operatorname{Spec} A$, $i : x \to X$ the inclusion. The following conditions are equivalent:*

[14, Tag 04GG]

(a) If $f \in A[T]$ is monic and $t_0 \in k$ is a root of $\bar{f} = f \bmod \mathfrak{m} \in k[T]$ such that $\bar{f}'(t_0) \neq 0$, then there exists a unique root $t \in A$ of f such that $t \bmod \mathfrak{m} = t_0$.

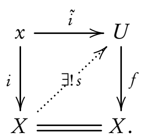
(b) If $f \in A[T]$ is monic and $\bar{f} = gh$ is a factorization of $\bar{f} = f \bmod \mathfrak{m} \in k[T]$ with $g, h \in k[T]$ coprime, then there exists a factorization $f = \tilde{g}\tilde{h}$ with $\tilde{g}, \tilde{h} \in A[T]$ such that $\tilde{g} \bmod \mathfrak{m} = g$, $\tilde{h} \bmod \mathfrak{m} = h$, and $\deg \tilde{g} = \deg g$.

(c) Every finite A-algebra is a product of local rings.

(d) For every étale A-algebra B and every prime $\mathfrak{p} \subseteq B$ lying over $\mathfrak{m}$ and such that $k(\mathfrak{p}) = k$, there exists a section $s : B \to A$ of $A \to B$ with $\mathfrak{p} = s^{-1}(\mathfrak{m})$.

(e) For every étale morphism $f : U \to X$ and every lifting $\tilde{i} : x \to U$ of i (i.e. $i = f \circ \tilde{i}$) there exists a unique section $s : X \to U$ such that $s \circ i = \tilde{i}$.[12]

[12] Useful to picture this condition as a lifting problem:



Proof. Maybe I'll write something here later. □

Definition 2.A.2. (a) A local ring A is *henselian* if the equivalent conditions of Proposition 2.A.1 hold.

(b) A local ring A is *strictly henselian* if it is henselian and its residue field k is separably closed.[13]

[13] Equivalently: every étale cover of $\operatorname{Spec} A$ admits a section.

(c) A valued field (K, v) is *henselian* if the valuation ring $\mathcal{O} = \{x \mid v(x) \geq 0\}$ is henselian.

Remark 2.A.3. Condition (d) of Proposition 2.A.1 allows one to construct the *henselization* of a local ring A as the direct limit

$$A^h = \varinjlim_{(B,s) \in \mathscr{C}_A} B$$

where $\mathscr{C}_A$ is the category of pairs (B, s) with B an étale A-algebra and $s : B \to k$ a homomorphism extending $A \to k$. (This category is filtering and essentially small.)

Universal property: $A \to A^h$ is a local homomorphism into a henselian local ring which is initial among such (in the category of local rings and local homomorphisms).

Similarly, given a separable closure k^{sep} of k, we can construct the *strict henselization* A^{sh} by considering the category of étale A-algebras endowed with a homomorphism to k^{sep} extending $A \to k^{\mathrm{sep}}$. (Using the algebraic closure $\bar{k}$ instead of k^{sep} gives the same result.)

Remark 2.A.4. The strict henselization of a local ring is the local ring for the étale topology. To make this precise, we reformulate everything in terms of geometry. Recall that a *geometric point* of a scheme X is a map $\bar{x} \to X$ with $\bar{x} = \operatorname{Spec} k(\bar{x})$ for some separably closed field $k(\bar{x})$. (Again, one can use algebraically closed fields instead.) An *étale neighborhood* of a geometric point $\bar{x}$ of X is an étale morphism $U \to X$ endowed with a lifting $\bar{x} \to U$ of $\bar{x} \to X$. Étale neighborhoods of $\bar{x}$ in X form a cofiltering category $N(X, \bar{x})$, and the colimit

$$\mathcal{O}_{X,\bar{x}} = \varinjlim_{U \in N(X,\bar{x})} \Gamma(U, \mathcal{O}_U)$$

is isomorphic to the strict henselization $\mathcal{O}_{X,x}^{\mathrm{sh}}$ of $\mathcal{O}_{X,x}$ where x is the image of $\bar{x}$ in X (and where we use the separable closure of $k(x)$ in $k(\bar{x})$ as $k(x)^{\mathrm{sep}}$). [14]

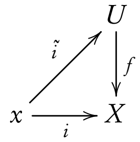
[14] Similarly, the henselization is related in the same way to local rings for the Nisnevich topology.

Proposition 2.A.5 (Hensel's lemma)**.** *Every local ring A which is J-adically complete and separated for an $\mathfrak{m}$-primary*[15] *ideal $J \subseteq A$ is henselian. In particular, every complete local ring is henselian.*

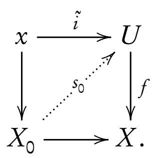
[15] This means that for $x \in \mathfrak{m}$ we have $x^N \in J$ for $N \gg 0$ depending on x.

For fans of the étale topology, we give a geometric proof:

Proof. We prove condition (e). Let $X = \operatorname{Spec} A$ and $x = \operatorname{Spec} k$ as before, and let



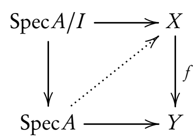
be an étale neighborhood of $x \to X$. Set $X_n = \operatorname{Spec} A/J^{n+1}$ for $n \geq 0$. First, consider the diagram



Since $x \to X_0$ is an immersion defined by the nil ideal[16] $\mathfrak{m}/J \subseteq A/J$, by the infinitesimal criterion for étaleness[17] there exists a unique diagonal arrow s_0 making the square commute.

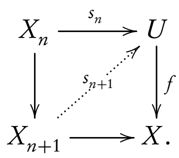
[16] An ideal in a commutative ring is *nil* (*locally nilpotent* in [14]) if it consists of nilpotent elements.

[17] **Infinitesimal criterion for étale maps:** A morphism $f: X \to Y$ locally of finite presentation is étale if and only if for every ring A and nil ideal $I \subseteq A$ (equivalently, every square zero ideal), and every commutative square of solid arrows



there exists a unique dotted arrow making the diagram commute.

Starting from s_0, we shall successively build maps $s_n: X_n \to U$ lifting $X_n \to X$ along f. It suffices to apply the infinitesimal criterion to the squares



Since A is J-adically complete, in the limit, the maps give the desired section $s: X \to U$.[18] □

[18] If you are confused with the last step, set $U = \operatorname{Spec} B$ and temporarily revert to commutative algebra.

Remark 2.A.6. The most common proof uses condition (a) of Proposition 2.A.1, and uses "Newton's method" to iteratively construct the desired root t using explicit induction steps. Proofs in [5, Appendix A] and [12] use condition (b), which gives a more direct approach to proving Theorem 2.5.1, but makes for a messier and less illuminating argument.

Corollary 2.A.7. *Every nonarchimedean field is henselian.*

Proof. Let K be a nonarchimedean field, let $\mathcal{O} \subseteq K$ be its valuation ring, and let $t \in \mathcal{O}$ be a pseudouniformizer. Apply Proposition 2.A.5 with $A = \mathcal{O}$ and $J = (t)$. □

Lemma 2.A.8. *The following are equivalent for a field K endowed with a height one valuation ν.*

(a) K is henselian.

(b) The assertion of Lemma 2.5.2 holds.

Proof. Left as exercise. □

The universal property of henselization induces a map $A^h \to \widehat{A}$.

Proposition 2.A.9. *For a valued field (K, v), the following are equivalent:*

(a) K is henselian,

(b) every finite extension L of K admits a unique extension of the valuation v.

Proof. Suppose that K is henselian. Given Lemma 2.A.8, we can repeat the proof of Proposition 2.5.3 word for word. The first paragraph of the proof of Theorem 2.5.1 shows that we can extend the valuation ring of K to L, which gives an extension of the valuation, easily seen to be unique. For the reverse direction, see [12, Theorem II 6.6]. □

Henselian rings will appear later in the course: the local ring $\mathcal{O}_{X,x}$ of a point x on a rigid analytic space X is not complete, but it is henselian.[19]

[19] The same holds for complex analytic spaces, e.g. the local ring $\mathbf{C}\{t\}$ of power series with positive radius of convergence is henselian.

3
The Tate algebra

In this chapter, we fix a nonarchimedean field K. We denote by $\mathcal{O}$ its valuation ring, by $k = \mathcal{O}/\mathfrak{m}$ its residue field, and by $t \in \mathfrak{m}$ a fixed pseudouniformizer.

We first introduce the Tate algebra, slightly emphasizing the "algebraic" point of view. We equip it with the Gauss norm, for which we give a geometric interpretation which facilitates the verification of some basic properties like multiplicativity or the Maximum Principle. The Gauss norm makes the Tate algebra into a Banach K-algebra; we prove that it satisfies a universal property in the category of Banach K-algebras. Next, we prove that the Tate algebra satisfies a number of favorable algebraic or topological properties, namely: [1]

- it satisfies a version of Noether normalization,
- it is Noetherian,
- all of its ideals are closed,
- the residue fields of its maximal ideals are finite extensions of K.

[1] I mostly managed to avoid the rather tedious arguments using the Weierstrass Preparation Theorem and the theory of bald and B-rings used in [5, Chapter 2]. Matter of taste, I guess.

In the appendix, written jointly with Alex Youcis, we figure out one can view Banach spaces over K algebraically through the lens of $\mathcal{O}/t^n$-modules.

3.1 Definition of the Tate algebra

Definition 3.1.1. The *algebra of restricted power series* in r variables is the t-adic completion of the polynomial algebra $\mathcal{O}[X_1,\ldots,X_r]$:

$$\mathcal{O}\langle X_1,\ldots,X_r\rangle = \varprojlim_n \mathcal{O}[X_1,\ldots,X_r]/(t^n) = \varprojlim_n \left((\mathcal{O}/t^n)[X_1,\ldots,X_r]\right).$$

The *Tate algebra* in r variables is the localization

$$K\langle X_1,\ldots,X_r\rangle = \mathcal{O}\langle X_1,\ldots,X_r\rangle \otimes_{\mathcal{O}} K = \mathcal{O}\langle X_1,\ldots,X_r\rangle\left[\frac{1}{t}\right].$$

Let $\mathfrak{n} = (t, X_1,\ldots,X_r) \subseteq \mathcal{O}[X_1,\ldots,X_r]$. The $\mathfrak{n}$-adic completion of $\mathcal{O}[X_1,\ldots,X_r]$ is the ring of formal power series

$$\mathcal{O}[[X_1,\ldots,X_r]] = \varprojlim_n \mathcal{O}[X_1,\ldots,X_r]/\mathfrak{n}^n.$$

Since $\mathfrak{n} \supseteq (t)$, we get the induced map on the respective completions:

$$\mathcal{O}\langle X_1,\ldots,X_r\rangle \to \mathcal{O}[[X_1,\ldots,X_r]]. \tag{3.1}$$

Lemma 3.1.2. *The map* (3.1) *is injective, and its image consists of the power series whose coefficients tend to zero:* [2]

$$\mathcal{O}\langle X_1,\dots,X_r\rangle \simeq \left\{ \sum_{n\in\mathbf{N}^r} a_n \mathbf{x}^n \in \mathcal{O}[[X_1,\dots,X_r]] : a_n \to 0 \text{ as } |n|\to\infty \right\}.$$

[2] Here we use the multi-index notation: if $n=(n_1,\dots,n_r)\in\mathbf{N}^r$, we set $\mathbf{X}^n = X_1^{n_1}\cdot\ldots\cdot X_r^{n_r}$ and $|n| = n_1+\ldots+n_r$.

Proof. We define the inverse homomorphism φ. Let $f=\sum a_n\mathbf{X}^n \in \mathcal{O}[[\mathbf{X}]]$ be an element of the right hand side. The condition that $a_n\to 0$ means precisely that for every $m\geq 0$, all but finitely many of the coefficients a_n are divisible by t^m. Thus, for every $m\geq 0$, the image f_m of f in $\mathcal{O}[[\mathbf{X}]]/t^m = (\mathcal{O}/t^m)[[\mathbf{X}]]$ is a polynomial. The elements $f_m\in(\mathcal{O}/t^m)[\mathbf{X}]$ form a compatible system, and give rise to an element $\varphi(f)$ of $\mathcal{O}\langle\mathbf{X}\rangle$. One easily checks that φ is the inverse to (3.1). □

By inverting t, we obtain an isomorphism

$$K\langle X_1,\dots,X_r\rangle \simeq \left\{ \sum_{n\in\mathbf{N}^r} a_n \mathbf{X}^n \in K[[X_1,\dots,X_r]] : a_n \to 0 \text{ as } |n|\to\infty \right\}.$$

As we have observed in §1.1, the right hand side is precisely the algebra of power series with coefficients in K which converge in the unit disc

$$\mathbf{D}^r(K) = \{(x_1,\dots,x_r)\in K : |x_i|\leq 1 \text{ for } i=1,\dots,r\}.$$

In particular, this implies that if for $f\in K[[X_1,\dots,X_r]]$ the series $f(\mathbf{x})$ converges for all $\mathbf{x}\in\mathbf{D}^r(K)$, then it also converges for all $\mathbf{x}\in\mathbf{D}^r(\overline{K})$.

3.2 The topology on $K\langle X_1,\dots,X_r\rangle$ and the Gauss norm

The ring $\mathcal{O}\langle X_1,\dots,X_r\rangle$, being defined as a completion, carries a natural inverse limit topology, called the *t-adic topology*. It extends uniquely to a topology of the Tate algebra $K\langle X_1,\dots,X_r\rangle$ for which $\mathcal{O}\langle X_1,\dots,X_r\rangle$ is an open subring; that topology can be described as the inductive limit topology, since

$$K\langle X_1,\dots,X_r\rangle = \bigcup_{n\geq 0} t^{-n}\mathcal{O}\langle X_1,\dots,X_r\rangle.$$

Compare with Exercise 2 on Problem Set 2.

Below, we describe the natural norm inducing these topologies.

Definition 3.2.1. The *Gauss norm* on $K\langle X_1,\dots,X_r\rangle$ is defined by

$$|f| = \max\{|a_n| : n\in\mathbf{N}^r\} \quad \text{if} \quad f=\sum_{n\in\mathbf{N}^r} a_n\mathbf{x}^n.$$

In other words, $|f|$ is the infimum of the values of $|c|$ for $c\in K^\times$ such that $c^{-1}f\in\mathcal{O}\langle X_1,\dots,X_r\rangle$. In particular, we have

$$\mathcal{O}\langle X_1,\dots,X_r\rangle = \{f\in K\langle X_1,\dots,X_r\rangle : |f|\leq 1\}.$$

The topology on $\mathcal{O}\langle X_1,\dots,X_r\rangle$ induced by the metric $d(x,y)=|x-y|$ is the t-adic topology.

The geometric interpretation: suppose that K is discretely valued, and that $t\in\mathcal{O}$ is a uniformizer. Then $X=\operatorname{Spec}\mathcal{O}[X_1,\dots,X_r]=\mathbf{A}^r_{\mathcal{O}}$ is a Noetherian regular scheme, and $Y=\{t=0\}=\mathbf{A}^r_k$ is a prime divisor on X. Therefore Y defines a valuation of height one v_Y on $k(X)$ ("order of zero or pole along Y"). It agrees with the Gauss norm in the weak sense that for $f\in K[X_1,\dots,X_r]\subseteq K\langle X_1,\dots,X_r\rangle$, we have

$$|f|_{\text{Gauss}} = |t|^{-v_Y(f)}.$$

In fact, $K[X_1,\dots,X_r]$ is dense in $K\langle X_1,\dots,X_r\rangle$ with respect to the t-adic topology, and the Gauss norm is the unique continuous extension of the norm $|t|^{-v_Y(f)}$ to $K\langle X_1,\dots,X_r\rangle$.

The proofs of the following two easy results employ the above intuition.

Lemma 3.2.2 (The Gauss norm is multiplicative)**.** *We have* $|fg| = |f| \cdot |g|$ *for* $f, g \in K\langle X_1, \ldots, X_r \rangle$.

Proof. Clearly this holds if $f \in K$ is a constant. We can therefore rescale f and g so that $|f| = 1 = |g|$. Equivalently $f, g \in \mathcal{O}\langle X_1, \ldots, X_r \rangle$ and their residues modulo the maximal ideal $\mathfrak{m} \subseteq \mathcal{O}$

$$\bar{f}, \bar{g} \in \mathcal{O}\langle X_1, \ldots, X_r \rangle / \mathfrak{m} = k[X_1, \ldots, X_r]$$

are nonzero. *Since* $k[X_1, \ldots, X_r]$ *is a domain*, $fg \in \mathcal{O}\langle X_1, \ldots, X_r \rangle$ has nonzero image $\bar{f}\bar{g}$ in $k[X_1, \ldots, X_r]$, and hence $|fg| = 1 = |f| \cdot |g|$. □

Proposition 3.2.3 (The Maximum Principle)**.** *For* $f \in K\langle X_1, \ldots, X_r \rangle$, *we have*

$$|f| = \sup \left\{ |f(x_1, \ldots, x_r)| : (x_1, \ldots, x_r) \in \overline{K}^r, |x_i| \leq 1 \right\}.$$

Proof. As in the previous proof, we can reduce to the case $|f| = 1$. Clearly, the right hand side is ≤ 1; we will show it equals 1. We have $f \in \mathcal{O}\langle X_1, \ldots, X_r \rangle$ and its image $\bar{f} \in k[X_1, \ldots, X_r]$ is nonzero. We can therefore find a point $(\bar{\xi}_1, \ldots, \bar{\xi}_r) \in \bar{k}^r$ such that $\bar{f}(\bar{\xi}_1, \ldots, \bar{\xi}_r) \neq 0$. Now $\bar{k}$ is the residue field of (the integral closure of $\mathcal{O}$ in) $\overline{K}$; let $(\xi_1, \ldots, \xi_r) \in \overline{K}^r$ be an element lifting $(\bar{\xi}_1, \ldots, \bar{\xi}_r)$. Then $|\xi_i| \leq 1$ and $|f(\bar{\xi}_1, \ldots, \bar{\xi}_r)| = 1$. □

Remark 3.2.4. The above proof shows three things in addition. First, the supremum is a maximum, and therefore attained in L^r for L a finite extension of K. Second, if the residue field k is infinite, the above maximum is attained at a point in K^r. Lastly, the maximum is attained at a point with $|x_1| = \cdots = |x_r| = 1$.

The Gauss norm makes the Tate algebra into a Banach K-algebra, as defined below.

Definition 3.2.5 (Banach spaces and Banach algebras)**.** Let V be a vector space over K. A *vector space norm* on V is a function

$$|\cdot| : V \to [0, \infty)$$

such that

i. $|xv| = |x| \cdot |v|$ for $x \in K$, $v \in V$,

ii. $|v| = 0$ if and only if $v = 0$,

iii. $|v + w| \leq \max\{|v|, |w|\}$ for $v, w \in V$.

It is called a *Banach norm* if V is complete with respect to the induced metric $d(x, y) = |x - y|$. A *Banach space* over K is a vector space over K equipped with a Banach norm.[3]

Let A be a K-algebra.[4] A *K-algebra norm* on A is a vector space norm $|\cdot|$ on A which satisfies

iv. $|ab| \leq |a| \cdot |b|$ for $a, b \in A$.

It is a *Banach algebra norm* if $|\cdot|$ is a Banach norm. A *Banach K-algebra* is a K-algebra equipped with a Banach algebra norm.

Let M be a module over a Banach K-algebra A. An *A-module norm* on M is a vector space norm $|\cdot|$ on M which satisfies

i'. $|am| \leq |a| \cdot |m|$ for $a \in A$, $m \in M$.

It is a Banach module norm if $|\cdot|$ is a Banach norm. A *Banach A-module* is an A-module equipped with a Banach A-module norm.

[3] Note that in axiom i. it is enough to require $|xv| \leq |x| \cdot |v|$. Indeed, for $x \neq 0$ we have

$$|x| \cdot |v| = |x| \cdot |x^{-1} \cdot xv| \leq |xv| \leq |x| \cdot |v|.$$

[4] In this course, all K-algebras are commutative.

A linear map $f: V \to W$ between Banach spaces over K is continuous if and only if it is *bounded* in the sense that $|f(v)| \leq C \cdot |v|$ ($v \in V$) for some constant C independent of v. This implies in particular that a continuous $f: V \to W$ is uniformly continuous.

Proposition 3.2.6. *The Tate algebra $K\langle X_1,\dots,X_r\rangle$ is a Banach algebra when equipped with the Gauss norm.*

Proof. Axioms i.–iii. are clear, and iv. follows from Lemma 3.2.2. It remains to show that $K\langle X_1,\dots,X_r\rangle$ is complete. It suffices to show that the closed unit ball $\{|f| \leq 1\} = \mathcal{O}\langle X_1,\dots,X_r\rangle$ is complete. This in turn follows from the fact that $\mathcal{O}\langle X_1,\dots,X_r\rangle$ is t-adically complete. □

Corollary 3.2.7. *The Tate algebra $K\langle X_1,\dots,X_r\rangle$ is the completion of $K[X_1,\dots,X_r]$ with respect to the Gauss norm.*

Proof. It suffices to observe that $\mathcal{O}[X_1,\dots,X_r]$ is dense in $\mathcal{O}\langle X_1,\dots,X_r\rangle$, which follows from the definition (and the fact that the metric topology induced by the Gauss norm agrees with the t-adic topology). □

3.3 The universal property

Definition 3.3.1. Let A be a Banach K-algebra. An element $a \in A$ is *powerbounded* if the set $\{a^n : n \geq 1\}$ is bounded, meaning that $\{|a^n| : n \geq 1\}$ is bounded from above. We denote the set of powerbounded elements by $A^\circ \subseteq A$.

The subset $A^\circ \subseteq A$ is a subring. If the norm on A is multiplicative, then $a \in A^\circ$ if and only if $|a| \leq 1$; therefore $A^\circ = \{|a| \leq 1\}$ is an open subring. Thus for $A = K\langle X_1,\dots,X_r\rangle$ we have $A^\circ = \mathcal{O}\langle X_1,\dots,X_r\rangle$.

Every continuous homomorphism $A \to B$ maps A° into B°. Since the element $X \in K\langle X\rangle$ is powerbounded, for every Banach K-algebra we obtain a map

$$\varphi \mapsto \varphi(X) \quad : \quad \mathrm{Hom}_K(K\langle X\rangle, A) \to A^\circ, \tag{3.2}$$

where for Banach K-algebras A and B, $\mathrm{Hom}_K(B,A)$ denotes the set of all *continuous* K-algebra homomorphisms $B \to A$.

Warning. If A is not reduced, then the subring A° is not very well-behaved.
For example, if $A = K\langle X\rangle/(X^2)$ then $A^\circ = \mathcal{O} \oplus K \cdot X$ is neither bounded nor t-adically separated.

Lemma 3.3.2. *The maps* (3.2) *are bijective and define an isomorphism between the functors $A \mapsto \mathrm{Hom}_K(K\langle X\rangle, A)$ and $A \mapsto A^\circ$ from Banach K-algebras to sets. In other words, $K\langle X\rangle$ represents the functor $A \mapsto A^\circ$.*

Similarly, $K\langle X_1,\dots,X_r\rangle$ represents the functor $A \mapsto (A^\circ)^r$.

Proof. Since $K[X]$ is dense in $K\langle X\rangle$ (Corollary 3.2.7), any two continuous K-algebra homomorphsims $\varphi, \psi : K\langle X\rangle \to A$ with $\varphi(X) = \psi(X)$ have to coincide. This shows injectivity. To show that $\varphi \mapsto \varphi(X)$ is surjective, let $a \in A^\circ$ and let $\varphi : K[X] \to A$ be the unique K-algebra homomorphism sending X to a. To extend φ to the completion $K\langle X\rangle$ of $K[X]$ with respect to the Gauss norm, it suffices to show that φ is (uniformly) continuous, i.e. that

$$|\varphi(f)| \leq C \cdot |f| \quad \text{for some } C > 0.$$

Since a is powerbounded, there exists a C such that $|a^n| \leq C$ for all $n \geq 0$. But then, for $f = \sum_{i=0}^m b_i X^i \in K[X]$, we have

$$|\varphi(f)| = \left|\sum_{i=0}^m b_i a^i\right| \leq \max\{|b_i|\} \cdot \max\{|a^n|\} \leq |f| \cdot C.$$ □

3.4 *The Tate algebra is Noetherian*

The goal of this section is to prove that $K\langle X_1,\dots,X_r\rangle$ is Noetherian.

Proposition 3.4.1 (Warm-up). *Suppose that K is discretely valued, i.e. $\mathcal{O}$ is a dvr. Then $\mathcal{O}\langle X_1,\dots,X_r\rangle$ and $K\langle X_1,\dots,X_r\rangle$ are Noetherian.*

Proof. Since $\mathcal{O}$ is Noetherian, so is the polynomial algebra $\mathcal{O}[X_1,\dots,X_r]$. The completion of a Noetherian ring with respect to an ideal is Noetherian [3, Theorem 10.26], thus $\mathcal{O}\langle X_1,\dots,X_r\rangle$ is Noetherian. Finally, the localization of a Noetherian ring is Noetherian, and therefore $K\langle X_1,\dots,X_r\rangle$ is Noetherian as well. □

However, if the valuation is nondiscrete, then $\mathcal{O}$ will not be Noetherian: the maximal ideal is not finitely generated, in fact it satisfies $\mathfrak{m}=\mathfrak{m}^2$. Thus $\mathcal{O}\langle X_1,\dots X_r\rangle$ is non-Noetherian as well, for the same reason. That reason disappears when we invert t.

The proof below loosely follows Tian's notes [15], with some simplifications.

Proposition 3.4.2 (Noether normalization). *Let $I\subseteq K\langle X_1,\dots,X_r\rangle$ be a closed ideal.*[5] *Then there exists a finite and injective K-algebra homomorphism*

$$K\langle Y_1,\dots,Y_s\rangle\hookrightarrow K\langle X_1,\dots,X_r\rangle/I\quad \textit{for some } s\le r.$$

[5] We shall soon prove that every ideal in $K\langle X_1,\dots,X_r\rangle$ is closed.

Proof. The idea of the proof is to deduce the statement from the usual Noether normalization lemma over k. We shall use the algebra $\mathcal{O}\langle X_1,\dots,X_r\rangle$ as an intermediary between the Tate algebra $K\langle X_1,\dots,X_r\rangle$ and the polynomial ring $k[X_1,\dots,X_r]$.

Let $J=I\cap\mathcal{O}\langle X_1,\dots,X_r\rangle$ and $B=\mathcal{O}\langle X_1,\dots,X_r\rangle/J$. Note that J is open in I, we have $I=J\cdot K\langle X_1,\dots,X_r\rangle$, and J is closed in $\mathcal{O}\langle X_1,\dots,X_r\rangle$. The last fact implies that

$$B\simeq\varprojlim_n B/t^n,\quad B/t^n=(\mathcal{O}/t^n)[X_1,\dots,X_r]/J.$$

Noether normalization applied to $B/\mathfrak{m}=k[X_1,\dots,X_r]/J$ produces a finite injective map

$$k[Y_1,\dots,Y_r]\to B/\mathfrak{m}$$

which we can lift to a map $\mathcal{O}\langle Y_1,\dots,Y_s\rangle\to B$. Indeed, we can certainly lift it to an $\mathcal{O}$-algebra map $\mathcal{O}[Y_1,\dots,Y_r]\to B$, and upon taking t-adic completion we obtain the desired $\mathcal{O}\langle Y_1,\dots,Y_s\rangle\to B$ (because B is t-adically complete). We want to show that the latter map is finite and injective as well.

Injectivity is easy: let $f\in\mathcal{O}\langle Y_1,\dots,Y_s\rangle$ and write $f=cg$ with $c\in\mathcal{O}$ and $|g|=1$. Then g has nonzero image in $k[Y_1,\dots,Y_s]$, and hence its image in $B/\mathfrak{m}$ is nonzero. Since B is $\mathcal{O}$-torsion free (being a submodule of the K-module $K\langle X_1,\dots,X_r\rangle/I$), we see that f maps to zero only for $c=0$.

For finiteness, as an intermediate step we will show that

$$\mathcal{O}/t[Y_1,\dots,Y_s]\to B/t$$

is finite. It suffices to show that the images of X_i in B/t are integral over $\mathcal{O}/t[Y_1,\dots,Y_s]$. Since their images in $B/\mathfrak{m}$ are integral over $k[Y_1,\dots,Y_s]$, there exist monic polynomials $P_i\in\mathcal{O}\langle Y_1,\dots,Y_s\rangle[X]$ with $P_i(X_i)\in\mathfrak{m}B$. But then for $N\gg 0$ we have $P_i^N(X_i)\in tB$, i.e. the X_i are integral over $\mathcal{O}/t[Y_1,\dots,Y_s]$.

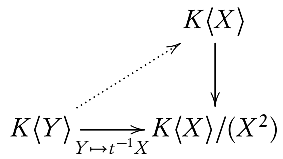
Now, let $\{Z_\alpha\}$ be a finite set of elements of B which generate B/t as a $\mathcal{O}/t[Y_1,\dots,Y_s]$-module. Fix $W_0 \in B$ and write

$$\begin{aligned} W_0 &= \sum_\alpha f_{0,\alpha} Z_\alpha + t\,W_1 \\ &= \sum_\alpha (f_{0,\alpha} + t f_{1,\alpha}) Z_\alpha + t^2 W_2 \\ &= \dots \stackrel{?}{=} \sum_\alpha f_\alpha Z_\alpha \end{aligned}$$

where $f_\alpha = \sum_n f_{n,\alpha} t^n$. Indeed, the difference of the two sides of $\stackrel{?}{=}$ belongs to $\bigcap_n t^n B = 0$. Therefore Z_α generate B over $\mathcal{O}\langle Y_1,\dots,Y_s\rangle$. [6] □

[6] The argument presented in the final paragraph shows more generally that if A is a t-adically complete $\mathcal{O}$-algebra and M is a t-adically separated A-module, then elements $e_1,\dots,e_n \in M$ which generate M/t also generate M ("t-adic Nakayama's lemma").

Remark 3.4.3. The above proof shows that we can choose the finite injective map so that it factors through $K\langle X_1,\dots,X_r\rangle$. This is not automatic, for example in the situation below



there does not exist a dotted arrow making the triangle commute. Indeed, the element $t^{-1}X \in A$ is nilpotent and hence power-bounded, but it cannot be lifted to a power-bounded element of $K\langle X\rangle$. We will need this observation in §4.2, where the above issue will be clarified.

Proposition 3.4.4. *The Tate algebra $K\langle X_1,\dots,X_r\rangle$ is Noetherian.*

Proof. We prove this by induction on r. Let $I \subseteq K\langle X_1,\dots,X_r\rangle$ be a nonzero ideal. Pick $f \in I$ with $|f| = 1$. It is enough to show that $K\langle X_1,\dots,X_r\rangle/(f)$ is Noetherian, for then the image $I/(f)$ is finitely generated and hence so is I.

The ideal (f) is closed, as multiplication by f

$$f : K\langle X_1,\dots,X_r\rangle \to K\langle X_1,\dots,X_r\rangle$$

is an isometry onto its image (f). We can therefore apply Noether normalization (Proposition 3.4.2) to obtain a finite and injective homomorphism

$$K\langle Y_1,\dots,Y_s\rangle \hookrightarrow K\langle X_1,\dots,X_r\rangle/(f).$$

Moreover, since $|f| = 1$, we must have $s < r$ by construction. By induction, $K\langle Y_1,\dots,Y_s\rangle$ is Noetherian and hence so is $K\langle X_1,\dots,X_r\rangle/(f)$. □

Proposition 3.4.5. *Every ideal in $K\langle X_1,\dots,X_r\rangle$ is closed.*

Proof. Let $I \subseteq K\langle X_1,\dots,X_r\rangle$ be an ideal and let $\overline{I}$ be its closure. Then $\overline{I}$, again an ideal, is finitely generated: $\overline{I} = (f_1,\dots,f_s)$. Using the density of I in $\overline{I}$, we will show that we can find another system of generators $(g_1,\dots,g_s)$ with $g_i \in I$, showing $I = \overline{I}$.

Consider the surjective and bounded map of Banach spaces

$$K\langle X_1,\dots,X_r\rangle^{\oplus s} \to \overline{I}, \quad (h_1,\dots,h_s) \mapsto \sum h_i f_i.$$

By the Open Mapping Theorem[7], there exists a $C > 0$ such that for every $f \in \overline{I}$ there exist $h_1,\dots,h_s \in K\langle X_1,\dots,X_r\rangle$ with $f = \sum h_i f_i$ and $|h_i| \le C \cdot |f|$.

[7] **Open Mapping Theorem.** *A surjective continuous map $\pi\colon V \to W$ of Banach spaces over K is open. That is, there exists a $C > 0$ such that $\{|w| \le 1\}$ is contained in $\pi(\{|v| \le C\})$.*
Proof. Open your Functional Analysis textbook and check that the proof works without change in the non-Archimedean setting. □

Since $I \subseteq \overline{I}$ is dense, we can find $g_1,\dots,g_s \in I$ with $|g_i - f_i| < C^{-1}$. By the previous paragraph, there exist $h_{ij} \in K\langle X_1,\dots,X_r\rangle$ $(1 \le i,j \le s)$ such that

$$g_i - f_i = \sum_j h_{ij} f_j \quad \text{and} \quad |h_{ij}| < 1.$$

Rewrite this as

$$g_i = \sum_j H_{ij} f_j, \quad H_{ij} = h_{ij} + \delta_{ij},$$

so that the matrix $H = [H_{ij}]$ satisfies $|H - \mathrm{Id}| < 1$ (for the supremum norm on matrix entries). It is easy to see (see Problem 2 on PS3) that this implies that H is invertible, showing $\overline{I} = (f_1,\dots,f_s) \subseteq (g_1,\dots,g_s) \subseteq I$. □

3.5 *Maximal ideals*

Recall that by Nullstellensatz, for an algebraically closed field k, the maximal ideals in $k[X_1,\dots,X_r]$ are in bijection with k^r. If k is not necessarily algebraically closed, and $\overline{k}$ is an algebraic closure, then maximal ideals in $k[X_1,\dots,X_r]$ correspond to orbits of the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$ on $\overline{k}^r$. The case of the Tate algebra is similar.

Proposition 3.5.1. *There is a bijection between the set* $\mathrm{Max}\,K\langle X_1,\dots,X_r\rangle$ *of maximal ideals in* $K\langle X_1,\dots,X_r\rangle$ *and the set of orbits of the action of the Galois group* $\mathrm{Gal}(\overline{K}/K)$ *on*

$$\mathbf{D}^r(\overline{K}) = \{(x_1,\dots,x_r) \in \overline{K}^r : |x_i| \le 1\},$$

where $|\cdot|$ *is the unique extension of the norm on* K *to* $\overline{K}$.

Proof. For $x = (x_1,\dots,x_r) \in \mathbf{D}^n(\overline{K})$, let

$$\mathfrak{m}_x = \{f \in K\langle X_1,\dots,X_r\rangle : f(x) = 0\}$$

(note that $f(x)$ makes sense because $|x_i| \le 1$). This is a maximal ideal, as the image of the evaluation map

$$f \mapsto f(x) \colon K\langle X_1,\dots,X_r\rangle \to \overline{K}$$

is a subring of $\overline{K}$ containing K and hence is a field. Moreover, Galois conjugate points give the same ideal, so we get a map $x \mapsto \mathfrak{m}_x$ from one side to the other.

Conversely, let $\mathfrak{n}$ (the notation $\mathfrak{m}$ already being reserved for the maximal ideal in $\mathcal{O}$) be a maximal ideal in $K\langle X_1,\dots,X_r\rangle$. Applying Noether normalization, we see that the residue field $L = K\langle X_1,\dots,X_r\rangle/\mathfrak{n}$ is finite over $K\langle X_1,\dots,X_s\rangle$ for some s. But this implies that the latter ring is a field, so $s = 0$ and L is a finite extension of K. Embedding it into $\overline{K}$, we obtain a homomorphism

$$\varphi \colon K\langle X_1,\dots,X_s\rangle \to L \to \overline{K}.$$

Let $x_i = \varphi(X_i) \in \overline{K}$. Thus x_i are powerbounded, and hence $|x_i| \le 1$. This gives a point $x = (x_1,\dots,x_r) \in \mathbf{D}^r(\overline{K})$, well-defined up to the choice of the embedding of L in $\overline{K}$. This gives a map $\mathfrak{n} \mapsto x$ in the other direction.

As such embeddings are permuted by the Galois group, it is clear that $\mathfrak{m}_x \mapsto x$. If $\mathfrak{n} \mapsto x$, then $\mathfrak{n} \subseteq \mathfrak{m}_x$, and hence they are equal since both are maximal. We have thus constructed mutually inverse bijections. □

Corollary 3.5.2. *Every K-algebra homomorphism*

$$K\langle Y_1,\dots,Y_s\rangle \to K\langle X_1,\dots,X_r\rangle$$

is continuous.

Proof. By the Maximum Principle (Proposition 3.2.3), the Gauss norm on $K\langle X_1,\ldots,X_r\rangle$ agrees with the *supremum norm*

$$|f|_{\sup} = \sup\{|f \bmod \mathfrak{n}| : \mathfrak{n} \in \operatorname{Max} K\langle X_1,\ldots,X_r\rangle\},$$

where $|f \bmod \mathfrak{n}|$ is the norm of the image of f in the residue field $L = K\langle X_1,\ldots,X_r\rangle/\mathfrak{n}$. This definition of the Gauss norm is *intrinsic* to the K-algebra structure on $K\langle X_1,\ldots,X_r\rangle$. It is also straightforward to check using $|\cdot|_{\mathrm{Gauss}} = |\cdot|_{\sup}$ that for every K-algebra homomorphism

$$\varphi : K\langle Y_1,\ldots,Y_s\rangle \to K\langle X_1,\ldots,X_r\rangle$$

we have $|\varphi(f)| \le |f|$, i.e. f is not only continuous but even *contracting*. □

3.6 More commutative algebra

We state the following additional results without giving a proof.

Theorem 3.6.1. *(a) The Tate algebra is Jacobson (every prime ideal is the intersection of maximal ideals).* See [5, Proposition 2.2/16].

(b) The Tate algebra is regular, of Krull dimension n, and excellent. See [6, §1.1] and references therein.

(c) Every ideal $I \subseteq K\langle X_1,\ldots,X_r\rangle$ admits a system of generators $(f_1,\ldots,f_s)$ with $|f_i| = 1$ and such that every $f \in I$ we can write $f = \sum f_i g_i$ with $|g_i| \le |f|$ See [5, Corollary 2.3/7].

3.A Banach spaces (with Alex Youcis)

Still slightly incomplete.

The goal of this slightly persnickety appendix, only tangentially related to the lecture, is to explicate the notion of a Banach space over K in terms of $\mathcal{O}/t^n$-modules. The main result (Proposition 3.A.9) describes the category Ban_K of Banach spaces over K as a localization of the category $\mathrm{Mod}_{\mathcal{O}}^{\wedge}$ of *complete* $\mathcal{O}$-modules (which itself is the inverse limit of the categories $\mathrm{Mod}_{\mathcal{O}/t^n}$) with respect to *topological isogenies*, i.e. morphisms whose kernel and cokernel have dense torsion submodules.

As before, we work over a non-Archimedean field K, denote by $\mathcal{O} \subseteq K$ be its valuation ring, and fix a pseudouniformizer $t \in \mathcal{O}$.

3.A.1 Torsion-free $\mathcal{O}$-modules

Mod_A for a ring A is the category of all A-modules, and Mod_A^f is the full subcategory of flat A-modules.

For $M \in \mathrm{Mod}_{\mathcal{O}}$, we define its *torsion submodule*

$$M_{\mathrm{tors}} = \bigcup_{n\geq 0} \ker(t^n : M \to M).$$

The module M is *torsion* (resp. *torsion-free*) if $M_{\mathrm{tors}} = M$ (resp. $M_{\mathrm{tors}} = 0$). We have the following basic result:

Lemma 3.A.1. *An $\mathcal{O}$-module M is flat if and only if it is torsion-free.*

Since the module M/M_{tors} is torsion-free, we have a functorial way of making any given $\mathcal{O}$-module flat. Since every map $M \to N$ where N is torsion-free has to map M_{tors} to zero, we obtain:

Lemma 3.A.2. *The functor*

$$M \mapsto M/M_{\mathrm{tors}} : \mathrm{Mod}_{\mathcal{O}} \to \mathrm{Mod}_{\mathcal{O}}^f$$

is a left adjoint to the inclusion $\mathrm{Mod}_{\mathcal{O}}^f \subseteq \mathrm{Mod}_{\mathcal{O}}$.

3.A.2 Complete $\mathcal{O}$-modules

The *completion* of an $\mathcal{O}$-module M is the inverse limit

$$\widehat{M} = \varprojlim_n M/t^n M.$$

A $\mathcal{O}$-module M is *complete* if the natural map $M \to \widehat{M}$ is an isomorphism. We denote by $\mathrm{Mod}_{\mathcal{O}}^{\wedge}$ the full subcategory of $\mathrm{Mod}_{\mathcal{O}}$ consisting of complete $\mathcal{O}$-modules. The completion functor

$$M \mapsto \widehat{M} : \mathrm{Mod}_{\mathcal{O}} \to \mathrm{Mod}_{\mathcal{O}}^{\wedge}$$

is a left adjoint to the inclusion $\mathrm{Mod}_{\mathcal{O}}^{\wedge} \subseteq \mathrm{Mod}_{\mathcal{O}}$.

We denote by $\mathrm{Mod}_{\mathcal{O}}^{\wedge,f}$ the full subcategory of flat and complete $\mathcal{O}$-modules. The completion of a flat $\mathcal{O}$-module is again flat, and again the completion functor $\mathrm{Mod}_{\mathcal{O}}^f \to \mathrm{Mod}_{\mathcal{O}}^{\wedge,f}$ is a left adjoint to the inclusion functor.

We have equivalences of categories

$$\mathrm{Mod}_{\mathcal{O}}^{\wedge} = 2\text{-}\varprojlim_n \mathrm{Mod}_{\mathcal{O}/t^n} \quad \text{and} \quad \mathrm{Mod}_{\mathcal{O}}^{\wedge,f} = 2\text{-}\varprojlim_n \mathrm{Mod}_{\mathcal{O}/t^n}^f,$$

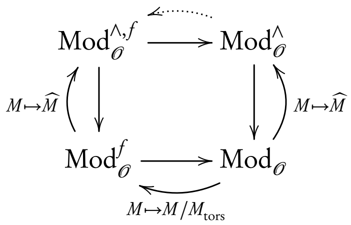
where for an inverse system of categories $(\mathscr{C}_n, \pi_n \colon \mathscr{C}_{n+1} \to \mathscr{C}_n)$, we define its 2-categorical inverse limit $2\text{-}\varprojlim_n \mathscr{C}_n$ as consisting of systems of objects and isomorphisms $(x_n \in \mathscr{C}_n, \iota_n \colon \pi_n(x_{n+1}) \simeq x_n)$, and where morphisms are systems of maps $(x_n' \to x_n)$ commuting with the maps ι_n', ι_n.

Warning: The category $\mathrm{Mod}_{\mathcal{O}}^{\wedge}$ has kernels and cokernels. The kernel is simply the kernel in $\mathrm{Mod}_{\mathcal{O}}$, and the cokernel is the completion of the usual cokernel. However, $\mathrm{Mod}_{\mathcal{O}}^{\wedge}$ is not abelian. The reason for that is that the image of a map need not be closed.

See [14, Tag 07JQ].

Lemma 3.A.3. *The functor $M \mapsto (M/M_{\mathrm{tors}})^{\wedge}$ is a left adjoint to the inclusion $\mathrm{Mod}_{\mathcal{O}}^{\wedge,f} \subseteq \mathrm{Mod}_{\mathcal{O}}^{\wedge}$.*

Proof. We have a commutative diagram of categories and functors



where the straight arrows are inclusion functors and the curvy (solid) arrows are their respective left adjoints. It follows formally that going down-left-up (i.e. $M \mapsto (M/M_{\mathrm{tors}})^{\wedge}$) in this diagram gives a dotted arrow which is a left adjoint to the top inclusion functor. □

Lemma 3.A.4. *Let M be an object of $\mathrm{Mod}_{\mathcal{O}}^{\wedge}$ and N a submodule of M. Then, there is a natural embedding*

$$M/\overline{N} \to (M/N)^{\wedge}$$

with dense image.

Proof. Let us note that

$$(M/N)^{\wedge} = \varprojlim (M/N)/t^n(M/N) = \varprojlim M/(t^n, N).$$

So, let us then observe that we have a natural map

$$M \to \varprojlim M/(t^n, N)$$

We claim that the kernel of this map is precisely $\overline{N}$. Indeed, to show that $\overline{N}$ is in the kernel we need to show that $\overline{N}$ projects to zero in (t^n, N) for every n. But, take x in $\overline{N}$ and write $x = \lim y_n$ with y_n in N for all n and $x - y_n \in t^n M$. Then, evidently x projects to 0 in $M/(t^n, N)$ since x is in $y_n + t^n M \subseteq (t^n, M)$. Conversely, suppose that x maps to zero in $\varprojlim M/(t^n, N)$. Then, by definition, for all $n \geqslant 0$ we have that we can write $x = y_n + t^n z_n$ for some y_n in N and z_n in M. In particular, from this we see that $x = \lim y_n$ and thus x is in $\overline{N}$.

From this we see that we get an injection

$$M/\overline{N} \to \varprojlim M/(t^n, N) = (M/N)^{\wedge}$$

To see that it has dense image it suffices to note that for all n we have the composition

$$M/\overline{N} \to \varprojlim M/(t^n, N) \to M/(t^n, N)$$

is surjective, from where the claim follows. □

From this we deduce the following:

Corollary 3.A.5. *Let M be an object of $\mathrm{Mod}_{\mathcal{O}}^{\wedge}$. Then, M_{tors} is dense in M if and only if $(M/M_{\mathrm{tors}})^{\wedge}$ is zero.*

3.A.3 Banach spaces

See Definition 3.2.5 for the definition of a Banach space. A linear map $f: V \to W$ between Banach spaces over K is called *bounded* if there exists a $c \in [0, \infty)$ such that

$$|f(v)| \leq c|v| \quad \text{for all } v \in V.$$

We denote by $\mathrm{Hom}(V, W)$ the linear space of such maps. It is stable under composition, and we denote the category of all Banach K-spaces and bounded maps by Ban_K.

We then have the following well-known result (e.g. see [4, §2.1.6] and [4, §2.1.8]):

Lemma 3.A.6. *Let V and W be Banach K-spaces. Then, a K-linear map $f : V \to W$ is bounded if and only if it's continuous. Moreover, the function*

$$|f| := \sup_{x \neq 0} \frac{|f(x)|}{|x|}$$

is a norm on $\mathrm{Hom}(V, W)$ *which endows* $\mathrm{Hom}(V, W)$ *with the structure of a Banach K-space. Moreover, the following properties hold:*

1. $|f| = \sup_{\substack{x \in V \\ |x|=1}} |f(x)|$
2. $|f(x)| \leqslant |f||x|$ *for all x in V.*
3. $|f \circ g| \leqslant |f||g|$ *for any continuous map of Banach K-spaces $g : W \to U$.*

3.A.4 Lattices

For $V \in \mathrm{Ban}_K$, we write $V_0 = \{|v| \leq 1\}$. We then have the following elemenary observation:

Lemma 3.A.7. *The subset V_0 is an $\mathcal{O}$-submodule which is $\mathcal{O}$-flat, complete, and such that the induced map $V_0 \otimes_{\mathcal{O}} K \to V$ is an isomorphism.*

Proof. Since $|xv| \leqslant |x||v|$ for all x in K and v in V we evidently see that V_0 is an $\mathcal{O}$-submodule of V. Since V is a K-module we know that it's $\mathcal{O}$-torsionfree and thus a fortori the same holds true for V_0 which implies that it's $\mathcal{O}$-flat. Finally, we note that the induced map $V_0 \otimes_{\mathcal{O}} K \to V$ is an isomorphism as follows. Since K is $\mathcal{O}$-flat we have that the induced map $V_0 \otimes_{\mathcal{O}} K \to V \otimes_{\mathcal{O}} K$ is injective. But, we note that $V \otimes_{\mathcal{O}} K \cong V$ via the map which maps $v \otimes x$ to xv. Thus, we see that the induced map $V_0 \otimes_{\mathcal{O}} K \to V$ is an isomorphism if and only if for all v in V one can write $v = xv_0$ with x in K and v_0 in V_0. But, this is clear since if $t^n v$ converges to 0 and so, since V_0 is open in V, must be in V_0 for some $n \geqslant 0$. We then can write $v = t^{-n}(t^n v)$. □

If $f: V \to W$ is a continuous map of Banach K-spaces, then for $c \in K$ we have $f(V_0) \subseteq cW_0$ if and only if $|c| \geq |f|$. In particular, we see that if we set

$$\mathrm{Hom}_0(V, W) := \{f \in \mathrm{Hom}(V, W) : |f| \leqslant 1\}$$

then we have the equality

$$\mathrm{Hom}_0(V, W) = \{f \in \mathrm{Hom}(V, W) : f(V_0) \subseteq W_0\}$$

We define the category $\mathrm{Ban}_{\mathcal{O}}$ to be the subcategory of Ban_K with the same underlying class of objects but where for V and W Banach K-spaces we set

$$\mathrm{Hom}_{\mathrm{Ban}_{\mathcal{O}}}(V, W) := \mathrm{Hom}_0(V, W)$$

and call it the category of *Banach lattices*.

Conversely, every torsion-free complete $\mathcal{O}$-module V_0 induces the structure of a Banach space on $V = V_0 \otimes_{\mathcal{O}} K$ by setting

$$|v| = \inf_{\substack{x \in K^\times \\ x^{-1}v \in V_0}} |x|$$

We have $V_0 = \{v \in V : |v| \leq 1\}$. In particular, if $f : V_0 \to W_0$ is an $\mathcal{O}$-module map, then the induced map $f : V \to W = W \otimes_{\mathcal{O}} K$ maps V_0 into W_0 and therefore it is continuous and $|f| \leq 1$.

Proposition 3.A.8. *The functors*

$$- \otimes K : \mathrm{Mod}_{\mathcal{O}}^{\wedge,f} \to \mathrm{Ban}_{\mathcal{O}}, \qquad V \mapsto V_0 : \mathrm{Ban}_{\mathcal{O}} \to \mathrm{Mod}_{\mathcal{O}}^{\wedge,f}$$

are mutually inverse equivalences of categories.

Proof. Follows from previous observations. □

3.A.5 Banach spaces in terms of complete modules

We would now like to put this altogether to obtain Ban_K is a localization of $\mathrm{Mod}_{\mathcal{O}}^{\wedge}$. Namely, let us define a morphism $f : V_0 \to W_0$ in $\mathrm{Mod}_{\mathcal{O}}^{\wedge}$ to be a *topological isogeny* if $\ker f$ and has dense torsion and if the cokernel of

$$(V_0/V_{0,\mathrm{tors}})^{\wedge} \to (W_0/W_{0,\mathrm{tors}})^{\wedge}$$

is annihilated by t^N for some N. We then have the following:

Proposition 3.A.9. *The functor*

$$F : \mathrm{Mod}_{\mathcal{O}}^{\wedge} \to \mathrm{Ban}_K : M \mapsto (M/M_{\mathrm{tors}})^{\wedge} \otimes_{\mathcal{O}} K$$

realizes Ban_K *as the localization of* $\mathrm{Mod}_{\mathcal{O}}^{\wedge}$ *with respect to topological isogenies.*

Proof. We first check that F turns topological isogenies into isomorphisms. Let $f : V_0 \to W_0$ be a topological isogeny. Since the functor F, being the composition of a left adjoint and of an exact functor, is right exact, we have an exact sequence

$$0 \to F(\ker f) \to F(V_0) \to F(W_0).$$

But $F(\ker f) = 0$ because $\ker f$ has dense torsion (Corollary 3.A.5). Thus $F(f)$ is injective.

By [**?** , Lemma 5.5] it suffices verify that F is essentially surjective, weakly full with fixed target (as in loc. cit.), and for all V in Ban_K we have that $F^{-1}(V)$ is a cofiltering category, and that $F(f)$ is an isomorphism if and only if f is topological isogeny.

To see that F is essentially surjective and weakly full with fixed target, we can apply Proposition 3.A.8.

To see that $F^{-1}(V)$ is cofiltering is clear

Finally, we verify that $F(f)$ is an isomorphism if and only if f is a topological isogeny. But, by the open mapping theorem we know that $F(f)$ is an isomorphism if and only if

$$\ker F(f) = \ker(f) \otimes_{\mathcal{O}} K, \qquad \mathrm{coker}(F(f)) = \mathrm{coker}(f) \otimes_{\mathcal{O}} K$$

(using the $\mathcal{O}$-flatness of K) are both trivial. Thus, it suffices to show that $F(M)$ is zero if and only if M_{tors} is dense in M. But, since $(M/M_{\mathrm{tors}})^{\wedge}$ is flat we know that $(M/M_{\mathrm{tors}})^{\wedge}$ embeds into $F(M)$ and thus $F(M)$ is zero if and only if $(M/M_{\mathrm{tors}})^{\wedge} = 0$. The claim then follows from Corollary 3.A.5. □

We summarize the preceding discussion with the following diagram

$$\begin{array}{ccccccc}
2\text{-}\varprojlim_n \mathrm{Mod}_{\mathcal{O}/t^n} & \simeq & \mathrm{Mod}_{\mathcal{O}}^{\wedge} & \longrightarrow & & & \mathrm{Mod}_{\mathcal{O}}^{\wedge}[(\text{top. isog.})^{-1}] \\
& & \Big\downarrow {\scriptstyle M \mapsto (M/M_{\mathrm{tors}})^{\wedge}} & & & & \Big\downarrow \simeq \\
2\text{-}\varprojlim_n \mathrm{Mod}^{f}_{\mathcal{O}/t^n} & \simeq & \mathrm{Mod}_{\mathcal{O}}^{\wedge,f} & \xrightarrow[\otimes K]{\sim} & \mathrm{Ban}_{\mathcal{O}} & \longrightarrow & \mathrm{Ban}_K.
\end{array}$$

4
Affinoid algebras and spaces

In this short section, we study quotients of Tate algebras, called *affinoid algebras*. The main result is that they carry natural equivalence classes of Banach K-algebra norms. Later, we will define their *affinoid spectra*, which will serve as building blocks for rigid-analytic spaces over K, just as spectra of finitely generated algebras over a field k are building blocks for schemes locally of finite type over k.

4.1 Affinoid algebras and the residue norm

Definition 4.1.1. Let K be a non-Archimedean field. An K-algebra A is an *affinoid algebra* if it is isomorphic to a quotient of the Tate algebra $K\langle X_1,\ldots,X_r\rangle$ for some $r\geq 0$.

The results from §3.4 and §3.6 imply the following.

Proposition 4.1.2. *Every affinoid K-algebra A is Noetherian, Jacobson, and there exists a finite and injective K-algebra homomorphism*

$$K\langle Y_1,\ldots,Y_s\rangle \hookrightarrow A$$

for some $s\geq 0$.

Let A be an affinoid K-algebra and let

$$\alpha\colon K\langle X_1,\ldots,X_r\rangle \to A$$

be a surjective homomorphism; set $I=\ker(\alpha)$. Since every ideal in the Banach K-algebra $K\langle X_1,\ldots,X_r\rangle$ is closed (Proposition 3.4.5), the quotient $A=K\langle X_1,\ldots,X_r\rangle/I$ is a Banach space for the *residue norm*

$$|f|_\alpha=\inf\{|g| : g\in\alpha^{-1}(f)\}.$$

Further, it is trivial to check that $|\cdot|_\alpha$ is sub-multiplicative, therefore making $(A,|\cdot|_\alpha)$ into a Banach K-algebra. We shall soon prove that different presentations α give rise to *equivalent* norms $|\cdot|_\alpha$.

4.2 The supremum norm

Our goal in this section is to show that the K-algebra structure on an affinoid K-algebra A determines its topology. This is similar to the fact that the t-adic topology on an $\mathcal{O}$-module is canonically determined.

Our main foothold will be the corresponding result for finite field extensions of K, Theorem 2.5.1. We already know that affinoid K-algebras are Jacobson, which means that their maximal ideals carry significant information, and that the residue fields at maximal ideals are finite extensions of K. Together, these observations allow us to define the *supremum semi-norm* on an affinoid K-algebra A by setting

$$|f|_{\mathrm{sup}} = \sup\{|f(x)| : x \in \operatorname{Max} A\},$$

where $|f(x)|$ is the absolute value of the image of f in the residue field L of x with respect to the unique extension of the norm on K to L. (We already saw a preview of this for $A = K\langle X_1, \dots, X_r\rangle$ in the proof of Corollary 3.5.2.)

Proposition 4.2.1 (Properties of the supremum semi-norm)**.** *Let A be an affinoid K-algebra.*

(a) The supremum semi-norm $|\cdot|_{\mathrm{sup}}$ on A satisfies the axioms (i), (iii), and (iv) of a Banach K-algebra norm (Definition 3.2.5). It is power-multiplicative, in the sense that $|a^n|_{\mathrm{sup}} = |a|_{\mathrm{sup}}^n$. For every K-algebra homomorphism $\varphi\colon A \to B$ between affinoid algebras, we have $|\varphi(a)|_{\mathrm{sup}} \le |a|_{\mathrm{sup}}$ for all $a \in A$.

(b) One has $|a|_{\mathrm{sup}} = 0$ if and only if a is nilpotent. If A is reduced, so that axiom (ii) of Definition 3.2.5 is also satisfied, then $|\cdot|_{\mathrm{sup}}$ is a Banach K-algebra norm.

(c) For $A = K\langle X_1, \dots, X_r\rangle$, the supremum norm coincides with the Gauss norm.

(d) (Maximum principle) For every $a \in A$ there exists an $x \in \operatorname{Max} A$ such that $|a|_{\mathrm{sup}} = |a(x)|$. In particular, there exists an $n \ge 1$ such that $|a|_{\mathrm{sup}}^n \in |K|$.

(e) For every residue norm $|\cdot|_\alpha$ on A, an element $a \in A$ is powerbounded (Definition 3.3.1) if and only if $|a|_{\mathrm{sup}} \le 1$.

Proof. Part (a) is clear. The first assertion of (b) follows from the fact that A is Jacobson, so that

$$\sqrt{(0)} = \bigcap_{\mathfrak{n} \in \operatorname{Max} A} \mathfrak{n}.$$

Completeness of a reduced A with respect to $|\cdot|_{\mathrm{sup}}$ is more involved and will not be needed; see [4, Theorem 6.2.4/1]. Part (c) was proved as part of the proof of Corollary 3.5.2.

For the remaining claims (d) and (e), we need some preparatory results. The following easy lemma says that one can estimate the absolute values of the roots of a polynomial by looking at its Newton polygon.

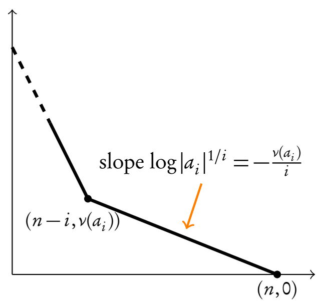


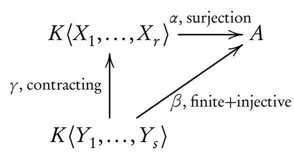
Figure 4.2.1: Proof of Lemma 4.2.2.

Lemma 4.2.2. *Let $f = X^n + a_1 X^{n-1} + \ldots + a_n \in K[X]$ be a polynomial, and let $\alpha_1, \dots, \alpha_n \in \overline{K}$ be its roots. Then*

$$\max_{i=1,\dots,n} |\alpha_i| = \max_{i=1,\dots,n} |a_i|^{1/i}.$$

Proof. The right-hand side is equal to $\exp(-\mu)$ where μ is the largest slope of $\mathrm{NP}(f)$ (see Figure 4.2). By Lemma 2.6.2, this equals $\max |\alpha_i|$. □

Let us fix a surjection $\alpha\colon K\langle X_1, \dots, X_r\rangle \to A$ and finite and injective homomorphism $\beta\colon K\langle Y_1, \dots, Y_s\rangle \to A$. By Remark 3.4.3, β can be lifted to a map $\gamma\colon K\langle Y_1, \dots, Y_s\rangle \to K\langle X_1, \dots, X_r\rangle$; part (c) implies that γ is contracting with respect to the Gauss norms.



We fix an $a \in A$; since a is integral over $K\langle Y_1, \dots, Y_s\rangle$, we fix a polynomial

$$f = X^n + f_1 X^{n-1} + \ldots + f_n \in K\langle Y_1, \dots, Y_s\rangle[X]$$

such that $f(a) = 0$. We make the following assumption:[1]

$$B = K\langle Y_1, \dots, Y_s\rangle[X]/(f) \to A \text{ is injective.}$$

[1] This assumption is satisfied for example if A is a domain, or just torsion-free as a $K\langle Y_1, \dots, Y_s\rangle$-module, and if f is of minimal degree, see [5, Lemma 3.1/13].

Note that $B = K\langle Y_1,\dots,Y_s,X\rangle/(f)$ is also an affinoid K-algebra. Under the above assumption, $\mathrm{Max}(A) \to \mathrm{Max}(B)$ is surjective. Therefore

$$\begin{aligned} |a|_{\sup} &= \sup_{x\in \mathrm{Max}(A)} |a(x)| = \sup_{x\in\mathrm{Max}(B)} |X(x)| \\ &= \sup_{y\in\mathrm{Max}(K\langle Y_1,\dots,Y_s\rangle} \max_{x\in\mathrm{Max}(B), x\mapsto y} |X(x)|. \end{aligned}$$

By Lemma 4.2.2, the maximum equals $\max |f_i(y)|^{1/i}$, and hence the above equals $\max |f_i|_{\sup}^{1/i} = \max |f_i|^{1/i}$.

We have thus, under our simplifying assumption, obtained the following assertion:

$$\text{One can find } f \text{ such that } |a|_{\sup} = \max_{i=1,\dots,n} |f_i|^{1/i}.$$

We omit the rather unenlightening reduction to this case, referring the reader to [5, §3.1].[2]

To prove (d), we apply the Maximum Principle (Proposition 3.2.3) to $g = f_1 \cdot \ldots \cdot f_n \in K\langle Y_1,\dots,Y_r\rangle$, obtaining a $y \in \mathrm{Max}\, K\langle Y_1,\dots,Y_r\rangle$ with $|g|_{\sup} = |g| = |g(y)|$. But this implies that $|f_i|_{\sup} = |f_i| = |f_i(y)|$ for every i, and hence

$$|a|_{\sup} = \max_{i=1,\dots,n} |f_i|^{1/i} = \max_{i=1,\dots,n} |f_i(y)|^{1/i} = \max_{x\mapsto y} |a(x)|.$$

To prove (e), the condition $|a|_{\sup} \le 1$ is equivalent to $|f_i| \le 1$ for all i. This implies that a is integral over $\mathcal{O}\langle Y_1,\dots,Y_r\rangle$. Since γ is contracting (Corollary 3.5.2), the images $a_i = \beta(f_i) = \alpha(\gamma(f_i)) \in A$ satisfy $|a_i|_\alpha \le 1$. This easily implies that a is power-bounded: if $C = \max\{|a^i|_\alpha : i < n\}$ then by induction we show that $|a^{n+m}|_\alpha \le C$ for all $m \ge 0$:

$$|a^{n+m}|_\alpha = \left| -\sum_{i=0}^{n-1} a_{n-i} a^{i+m} \right|_\alpha \le C.$$

Finally, if a is powerbounded, then $|a|_{\sup}^n = |a^n|_{\sup} \le |a^n|_\alpha$ is bounded, forcing $|a|_{\sup} \le 1$. □

Theorem 4.2.3. *Every K-algebra homomorphism $A \to B$ between affinoid K-algebras is continuous with respect to any choice of residue norms on the source and target. In particular, all residue norms on an affinoid K-algebra are equivalent.*

Proof. Fix a surjection $\alpha\colon K\langle X_1,\dots X_r\rangle \to A$ corresponding to a residue norm $|\cdot|_\alpha$ and let

$$\varphi\colon K\langle X_1,\dots X_r\rangle \to A \to B$$

be the composition. Since A has the quotient topology, it is enough to show that φ is continuous. In other words, we may replace A with $K\langle X_1,\dots X_r\rangle$ endowed with the Gauss norm.

The elements $b_i = \varphi(X_i) \in B$ are power-bounded by Proposition 4.2.1(e), since

$$|b_i|_{\sup} \le |X_i|_{\sup} = 1.$$

By the universal property of $K\langle X_1,\dots X_r\rangle$ among Banach K-algebras, there exists a unique *continuous* K-algebra homomorphism

$$\varphi'\colon K\langle X_1,\dots X_r\rangle \to B \quad \text{such that} \quad \varphi'(X_i) = b_i.$$

It suffices to show that $\varphi = \varphi'$. Fix $f \in K\langle X_1,\dots X_r\rangle$ and set $g = \varphi(f) - \varphi'(f) \in B$. For every maximal ideal $\mathfrak{n} \subseteq B$ and every $s \ge 1$, the quotient $B/\mathfrak{n}^s$ is a finite dimensional K-algebra, and therefore the composition

$$\pi\circ\varphi\colon K\langle X_1,\dots X_r\rangle \to B \to B/\mathfrak{n}^s$$

[2] Idea of the reduction [5, Lemma 3.1/14]: replace A with $\prod A/\mathfrak{p}_i$ where $\mathfrak{p}_1,\dots,\mathfrak{p}_m \subseteq A$ are the minimal prime ideals. For each i, the affinoid K-algebra $A/\mathfrak{p}_i$ is a domain, and we can apply [5, Lemma 3.1/13].

is continuous (since $B/\mathfrak{n}^s$ is finite-dimensional, all norms are equivalent). Indeed, we may assume that $\pi \circ \varphi$ is surjective, and then $\pi \circ \varphi$ is continuous with respect to the residue norm it induces on $B/\mathfrak{n}^s$). This forces $\pi\varphi = \pi\varphi'$ by the universal property of $K\langle X_1, \dots X_r \rangle$ applied this time to $B/\mathfrak{n}^s$.

Thus g maps to zero in $B/\mathfrak{n}^s$ for every $\mathfrak{n}$ and s. Therefore for every $\mathfrak{n}$, the image of g in $A_\mathfrak{n}$ lies in $\bigcap_s \mathfrak{n}^s A_\mathfrak{n}$, which is zero (by Krull's intersection theorem). This implies that $g = 0$. □

TODO: Easier proof using the Closed Graph Theorem.

4.3 Completed tensor product

TODO: complete this section. Meanwhile, consult [5, Appendix B].

Definition 4.3.1. Let V and W be Banach spaces over K. A *completed tensor product* $V\widehat{\otimes}W$ is a Banach space representing the functor

$$\mathrm{Bilinear}(V,W;-)\colon \{\text{Banach spaces over } K\} \to \mathbf{Sets},$$

$$\mathrm{Bilinear}(V,W;U) = \{\text{continuous bilinear maps } V \times W \to U\}.$$

Proposition 4.3.2. *The completed tensor product exists for any two Banach spaces over K.*

Proof. We use the ideas of Appendix REF. Let $V_0 = \{|v| \le 1\} \subseteq V$ and $W_0 = \{|w| \le 1\} \subseteq W$ be the corresponding lattices, and let M_0 be the t-adic completion of $V_0 \otimes_{\mathcal{O}} W_0$. Since V_0 and W_0 are flat, so is M_0. Thus $M = M_0 \otimes_{\mathcal{O}} K$ is a Banach space over K with $M_0 = \{|m| \le 1\} \subseteq M$. The bilinear map

$$(v_0, w_0) \mapsto v_0 \otimes w_0 \colon V_0 \otimes W_0 \to M_0$$

extends uniquely to a K-bilinear map $\mu\colon V \otimes W \to M$. The map μ is continuous (WHY?).

We claim that the map μ exhibits M as a completed tensor product of V and W. Thus, let U be a Banach space over K, and let $\alpha\colon V \times W \to U$ be a continuous bilinear map. Rescaling the norm on U, we may assume that $\alpha(V_0 \times W_0) \subseteq U_0$. The bilinear map of complete $\mathcal{O}$-modules $V_0 \times W_0 \to U_0$ extends uniquely to a linear map $V_0 \otimes_{\mathcal{O}} W_0 \to U_0$. Since U_0 is complete and completion is a left adjoint, this map factors uniquely through the completion M_0. Inverting t, we obtain the desired linear and continuous $M \to U$. □

Remark 4.3.3. Alternatively, one can construct $V\widehat{\otimes}W$ as the completion of $V \otimes W$ with respect to the norm ...

Lemma 4.3.4. *Let $A \to B$ be a homomorphism of affinoid K-algebras. Then there exists an $r \ge 0$, a (finitely generated and closed) ideal $I \subseteq A\langle X_1, \dots, X_s \rangle$, and an isomorphism of A-algebras*

$$B \simeq K\langle X_1, \dots, X_s \rangle / I.$$

Proposition 4.3.5 (Pushouts of Banach K-algebras)**.** *(a) The category of Banach K-algebras admits amalgamated coproducts (pushouts). The underlying Banach space of the pushout of*

$$B \leftarrow A \rightarrow C$$

is the completed tensor product $B\widehat{\otimes}_A C$.

(b) If A, B, and C are affinoid K-algebras, then so is $B\widehat{\otimes}_A C$.

(c) Completed tensor products of affinoid K-algebras can be computed in the usual way: if

$$B = A\langle X_1, \dots, X_r \rangle / (f_1, \dots, f_n), \quad C = A\langle Y_1, \dots, Y_s \rangle / (g_1, \dots, g_m),$$

then

$$B\widehat{\otimes}_A C \simeq A\langle X_1, \dots, X_r, Y_1, \dots, Y_s \rangle / (f_1, \dots, f_n, g_1, \dots, g_m).$$

5

Sheaves, sites, and topoi

In this chapter, we will learn how to deal with "spaces" without enough points, or with no points at all![1]

[1] Another joke [Gelfand] liked to tell involved the wireless telegraph: "At the beginning of the twentieth century, someone asks a physicist at a party: can you explain how it works? The physicist replies that it's very simple. First, you have to understand how the ordinary, wired, telegraph works: imagine a dog with its head in London and its tail in Paris. You pull the tail in Paris, and the dog barks in London. The wireless telegraph, says the physicist, is the same thing, but without a dog." After recounting the joke and waiting for the laughter to subside (even from those people in the audience who had heard it a thousand times), Gelfand would pivot to whatever math problem was being discussed. If he thought that the solution of the problem required a radically new approach, he would comment, "What I'm trying to say is we need to do it without a dog."
E. Frenkel *Love and Math*

5.1 Motivation: reinventing the real

Imagine being a geometer who does not believe in irrational numbers, perhaps for the fear of drowning. You study the geometry of the "line" $\mathbf{Q}$ and maybe the higher-dimensional spaces $\mathbf{Q}^r$. With the irrationals hiding in your blind spot, the "unit interval" $[0,1]_{\mathbf{Q}} = [0,1] \cap \mathbf{Q}$ appears to you as both connected and compact, in the naive sense that it is not the union of two disjoint intervals *with rational endpoints*, and that every family of such intervals in $\mathbf{Q}$ which covers $[0,1]_{\mathbf{Q}}$ admits a finite subcover. Further, the functor assigning to each interval with rational endpoints $(a,b)_{\mathbf{Q}} = (a,b) \cap \mathbf{Q}$ the set of all continuous piecewise linear functions $(a,b)_{\mathbf{Q}} \to \mathbf{Q}$ satisfies the sheaf condition for *finite* coverings by intervals with rational endpoints. [2]

[2] This toy example is largely taken from Brian Conrad's lecture notes.

Naturally, these properties fail to hold for the usual metric topology on $\mathbf{Q}$. Since we want to make do with what we have and avoid "filling in the holes," we need a different way of formalizing our naive thoughts above.

Definition 5.1.1. A closed (resp. open) *rational box* is a subset of $\mathbf{Q}^r$ of the form $\prod_{i=1}^r [a_i, b_i]_{\mathbf{Q}}$ (resp. $\prod_{i=1}^r (a_i, b_i)_{\mathbf{Q}}$) with $a_i, b_i \in \mathbf{Q}$.

Convention: $(a,b)_{\mathbf{Q}} = (a,b) \cap \mathbf{Q}$ etc.

(a) An open subset $U \subseteq \mathbf{Q}^r$ is an *admissible open* if for every closed rational box $K \subseteq U$ there exists a finite collection $V_1, \ldots, V_m$ of open rational boxes contained in U such that $K \subseteq \bigcup_{i=1}^m V_i$.

(b) A cover $U = \bigcup_{\alpha \in I} U_\alpha$ of an admissible open U by admissible opens U_α is an *admissible cover* if for every closed rational box $K \subseteq U$ there exists a finite collection $V_1, \ldots, V_m$ of open rational boxes contained in U such that $K \subseteq \bigcup_{i=1}^m V_i$ and each V_i is contained in some U_α.

Note that the intersection $U'' = U \cap U'$ of two admissible opens is again admissible. Indeed, if $K \subseteq U \cap U'$, we can find $V_1, \ldots, V_m$ and $V_1', \ldots, V_n'$ as in the definition. Then $V_{ij}'' = V_i \cap V_j$ $(1 \le i \le m, 1 \le j \le n)$ are rational boxes, are contained in U'', and cover K.

Definition 5.1.2. A *sheaf for the admissible topology* on $\mathbf{Q}^r$ is a functor[3]

$$\mathscr{F} : \{\text{admissible opens in } \mathbf{Q}^r\} \to \mathbf{Sets}$$

[3] Here we regard any poset C as a category with morphisms

$$\mathrm{Hom}_C(c,c') = \begin{cases} \{*\} & \text{if } c \le c' \\ \emptyset & \text{otherwise.} \end{cases}$$

such that for every admissible cover $U = \bigcup_{\alpha \in I} U_\alpha$ the sequence

$$\mathscr{F}(U) \to \prod_{\alpha \in I} \mathscr{F}(U_\alpha) \rightrightarrows \prod_{\alpha, \beta \in I} \mathscr{F}(U_\alpha \cap U_\beta) \qquad (5.1)$$

is exact.[4]

Recall some basic terminology: if $U = \bigcup_{\alpha \in I} U_\alpha$ is an open cover, we say that $\mathscr{F}$ *satisfies the sheaf condition* for $\{U_\alpha\}_{\alpha \in I}$ if (5.1) is exact. If $U = \bigcup_{\beta \in J} V_\beta$ is another cover, we say that $\{V_\beta\}_{\beta \in J}$ *refines* $\{U_\alpha\}_{\alpha \in I}$ if every V_β is contained in some U_α; more precisely, if there exists a map $f : J \to I$ such that $V_\beta \subseteq U_{f(\beta)}$ for every $\beta \in J$.

[4] Here **exact** is another name for an *equalizer*: the left map is injective and its image equals the set of elements whose images by the two parallel arrows are equal.

Proposition 5.1.3. *(a) Let $\mathscr{G}$ be a functor from the poset of* closed *rational boxes in $\mathbf{Q}^r$ to sets which satisfies the sheaf condition for every* finite *covering $K = \bigcup_{\alpha \in I} K_\alpha$. Then $\mathscr{G}$ extends uniquely to a sheaf for the admissible topology on $\mathbf{Q}^r$.*

The consideration of values on closed boxes is a bit artificial here. In algebraic geometry and rigid geometry, our basic opens (affine or affinoid opens) will be quasi-compact, and there will be no need to consider the values of a sheaf on closed sets.

(b) If $\overline{\mathscr{F}}$ is a sheaf on $\mathbf{R}^r$ (for the standard topology), then the functor associating to a closed rational box $K = \prod [a_i, b_i]_{\mathbf{Q}}$ the value

$$\overline{\mathscr{F}}\left(\prod [a_i, b_i]\right) := \varinjlim_{\prod [a_i, b_i] \subseteq U \subseteq \mathbf{R}^r} \overline{\mathscr{F}}(U)$$

satisfies the sheaf condition for every finite covering of a closed rational box by closed rational boxes, and therefore by (a) it extends uniquely to a sheaf for the admissible topology on $\mathbf{Q}^r$, denoted $\mathscr{F}$.

(c) The association $\overline{\mathscr{F}} \mapsto \mathscr{F}$ defines an equivalence of categories

$$\{\textit{sheaves on } \mathbf{R}^r\} \simeq \{\textit{sheaves for the admissible topology on } \mathbf{Q}^r\}.$$

Proof. Omitted, but see Problems 2 and 3 on Problem Set 4. □

In Appendix 5.A we will learn how to reconstruct certain topological spaces from their category of sheaves. In particular, we shall obtain:

Corollary 5.1.4 (See Appendix 5.A)**.** *The space $\mathbf{R}^r$ can be recovered from the category of sheaves for the admissible topology on $\mathbf{Q}^r$.*

Example 5.1.5. (a) Every open subset $U \subseteq \mathbf{Q}$ is admissible.

(b) However, the covering of $\mathbf{Q}$ by all open rational intervals $(a, b)_{\mathbf{Q}}$ such that $\sqrt{2} \notin (a, b)$ is not an admissible cover, since e.g. $K = [0, 1]_{\mathbf{Q}}$ cannot be covered by finitely many such intervals.

(c) The sheaf "skyscraper at $\sqrt{2}$," defined as

$$\overline{\mathscr{F}}(U) = \begin{cases} \mathbf{Z} & \text{if } \sqrt{2} \in U \\ 0 & \text{otherwise} \end{cases}$$

defines a nonzero sheaf $\mathscr{F}$ for the admissible topology on $\mathbf{Q}$ whose stalks at all points in $\mathbf{Q}$ (defined in the obvious way) are zero.

(d) The following is an example of an inadmissible open in $\mathbf{Q}^2$ (due to Zev Rosengarten):

$$U = \mathbf{Q}^2 \cap \left((0, \sqrt{2}) + \{x \geq -|y|\}\right) \quad \text{(see Figure 5.1).}$$

In this case, the closed box $K = [0, 1] \times [0, 2]$ does not admit a finite cover by open subsets contained in U.

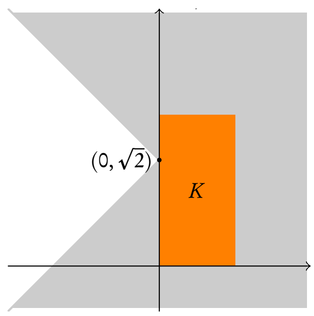


Figure 5.1.1: An inadmissible open subset of $\mathbf{Q}^2$.

5.2 Sites

Definition 5.2.1 (Site)**.** A *site* is a category $\mathscr{C}$ in which every object $c \in \operatorname{ob}\mathscr{C}$ is endowed with a collection $\operatorname{Cov} c$ of families of maps $\{c_\alpha \to c\}_{\alpha\in I}$, called *covering families*, satisfying the following axioms.

i. (ISOMORPHISM) If $c' \to c$ is an isomorphism then the singleton $\{c' \to c\}$ is a covering family of c,

ii. (PULLBACK) If $\{c_\alpha \to c\}_{\alpha\in I}$ is a covering family of c and if $c' \to c$ is a morphism, then the fiber products $c'_\alpha = c_\alpha \times_c c'$ exist and the family

$$\{c'_\alpha = c_\alpha \times_c c' \to c'\}_{\alpha\in I}$$

is a covering family of c'.

iii. (COMPOSITION) If $\{c_\alpha \to c\}_{\alpha\in I}$ is a covering family of c and for every $\alpha \in I$ we have a covering family $\{c_{\alpha\beta} \to c_\alpha\}_{\beta\in J_\alpha}$ of c_α, then

$$\{c_{\alpha\beta} \to c_\alpha \to c\}_{\alpha\in I, \beta\in J_\alpha}$$

is a covering family of c.

The basic example is of course the site $\operatorname{Op} X$ of opens in a topological space X, where morphisms are inclusions $U' \subseteq U$ of open subsets, and where $\{U_\alpha \subseteq U\}$ is a covering family precisely when $U = \bigcup U_\alpha$. Another one is provided by our toy example above: the category of admissible opens in $\mathbf{Q}^r$ where covering families are given by admissible covers. [5]

Note that axioms (i) and (ii) imply that an isomorphism $c' \to c$ induces a bijection between covering families of c and of c'. By abuse of terminology, we shall use the notation $\mathscr{C}$ to refer to both the site and the underlying category. A safer way would be to give a name such as τ to the choice of $\operatorname{Cov} c$ for every $c \in \mathrm{C}$ satisfying the above axioms (called a *Grothendieck (pre)topology* on the category $\mathscr{C}$) and define a site as a category $\mathscr{C}$ with a Grothendieck topology τ, denoted $(\mathscr{C}, \tau)$. This is sometimes useful, e.g. if one considers two sites with the same underlying category. [6]

Definition 5.2.2 (Sheaf)**.** Let $\mathscr{C}$ be a site. A *sheaf* on $\mathscr{C}$ is a contravariant functor

$$\mathscr{F} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Sets}$$

such that for every $c \in \operatorname{ob}\mathscr{C}$ and every covering family $\{c_\alpha \to c\}_{\alpha\in I}$ the sequence

$$\mathscr{F}(c) \to \prod_{\alpha\in I} \mathscr{F}(c_\alpha) \rightrightarrows \prod_{\alpha,\beta\in I} \mathscr{F}(c_\alpha \times_c c_\beta) \tag{5.2}$$

is exact (note that the fiber products $c_\alpha \times_c c_\beta$ exist thanks to axiom ii).

We denote by $\operatorname{Sh}\mathscr{C}$ the category of sheaves on $\mathscr{C}$, considered as a full subcategory of the category of *presheaves* $\operatorname{PSh}\mathscr{C} = \operatorname{Fun}(\mathscr{C}^{\mathrm{op}}, \mathbf{Sets})$. We call $\operatorname{Sh}\mathscr{C}$ the *topos* associated to $\mathscr{C}$.

In general, a *topos* (plural: topoi) is a category which is equivalent to $\operatorname{Sh}\mathscr{C}$ for some site $\mathscr{C}$ (with no extra structure!). Different sites can give rise to equivalent topoi, and so a topos is in a way a superior notion; we can regard a site as a particular presentation of the associated topos, just as a metric on a topological space is a useful but non-canonical "presentation" of its topology.

So far, to define sheaves and topoi, we only needed a part of axiom (ii), namely that suitable fiber products exist. To see the other axioms in action, let us show that familiar features of sheaf theory: refinement, (zeroth) Čech cohomology, and sheafification, work in a similar way in a site $\mathscr{C}$.

[5] **More examples of sites:**

- The *étale site* of a scheme X: the objects are étale morphisms $U \to X$, maps are morphisms over X, and covers $\{U_\alpha \to U\}$ are families of jointly surjective maps,
- Replacing *étale* with *flat and locally finitely presented* one obtains the *fppf site*.
- For a group G, the category of G-sets where covers are jointly surjective families of G-equivariant maps.

[6] The same happens in topology: one uses a letter such as X to denote both a topological space and the underlying set; if confusion is possible, one writes $(X, \mathscr{T})$ for the topological space.

Definition 5.2.3 (Refinement)**.** We say that a covering family $\{c'_\beta \to c\}_{\beta\in J}$ *refines* a covering family $\{c_\alpha \to c\}_{\alpha\in I}$ of the same object c if there exists a function $\varphi\colon J \to I$ and maps $\varphi_\beta\colon c'_\beta \to c_{\varphi(\beta)}$ fitting inside a commutative triangle

$$\begin{array}{ccc} c'_\beta & \xrightarrow{\varphi_\beta} & c_{\varphi(\beta)} \\ & \searrow \quad \swarrow & \\ & c. & \end{array}$$

This is an analog of the usual notion in topology: a cover $U = \bigcup U'_\beta$ refines $U = \bigcup U_\alpha$ if every U_β is contained in some U_α. The relation of refinement is clearly transitive. Further, axioms (ii) and (iii) imply that every two covering families $\{c_\alpha \to c\}_{\alpha\in I}$ and $\{c'_\beta \to c\}_{\beta\in J}$ admit a common refinement, namely [14, Tag 00W6]

$$\{c_\alpha \times_c c'_\beta \to c\}_{(\alpha,\beta)\in I\times J}.$$

Given a presheaf $\mathscr{F}\colon \mathscr{C}^{\mathrm{op}} \to \mathbf{Sets}$ and a covering family $\{c_\alpha \to c\}_{\alpha\in I}$ let us define $\mathscr{H}^0(\mathscr{F},\{c_\alpha\})$ as the equalizer of

$$\prod_{\alpha\in I}\mathscr{F}(c_\alpha) \rightrightarrows \prod_{\alpha,\beta\in I}\mathscr{F}(c_\alpha \times_c c_\beta).$$

Thus $\mathscr{F}$ satisfies the sheaf condition for $\{c_\alpha\}$ (meaning that (5.2) is exact) precisely when the canonical map

$$u\colon \mathscr{F}(c) \to \mathscr{H}^0(\mathscr{F},\{c_\alpha\})$$

is a bijection.

[14, Tag 00W7]

Lemma 5.2.4. *Let $\mathscr{F}$ be a presheaf on $\mathscr{C}$ and let $\{c_\alpha \to c\}_{\alpha\in I}$, $\{c'_\beta \to c\}_{\beta\in J}$ be two covering families of an object c such that $\{c'_\beta \to c\}$ refines $\{c_\alpha \to c\}$*

(a) Let $\varphi\colon J \to I$ and $\varphi_\beta\colon c'_\beta \to c_{\varphi(\beta)}$ be as in Definition 5.2.3. Then $(\varphi,\{\varphi_\beta\})$ induces a map

$$\mathscr{H}^0(\mathscr{F},\{c_\alpha\}) \to \mathscr{H}^0(\mathscr{F},\{c'_\beta\}).$$

(b) If $\varphi'\colon J \to I$, $\varphi'_\beta\colon c'_\beta \to c_{\varphi'(\beta)}$ is another such datum, then the two induced maps $\mathscr{H}^0(\mathscr{F},\{c_\alpha\}) \to \mathscr{H}^0(\mathscr{F},\{c'_\beta\})$ are equal.

See [5, Lemma 4.3/2]

(c) If $\mathscr{F}$ satisfies the sheaf condition for $\{c'_\beta \to c\}$, and if the canonical maps

$$u\colon \mathscr{F}(c_\alpha) \to \mathscr{H}^0(\mathscr{F},\{c'_\beta \times_c c_\alpha\}_{\beta\in J})$$

are injective for all $\alpha\in I$ (e.g. if $\mathscr{F}$ satisfies the sheaf condition also for $\{c'_\beta \times_c c_\alpha \to c_\alpha\}_{\alpha\in I}$), then it satisfies the sheaf condition for $\{c_\alpha \to c\}$.

Proof. Left as exercise. □

Parts (a) and (b) imply that we have a canonical map

$$\varphi\colon \mathscr{H}^0(\mathscr{F},\{c_\alpha\}) \to \mathscr{H}^0(\mathscr{F},\{c'_\beta\}).$$

Thus if we consider $\operatorname{Cov} c$ as a partially ordered set with respect to the relation of refinement (as we observed, this poset is cofiltering: every two elements have a common upper bound), we can define the *zeroth Čech cohomology* as the colimit

$$\check{\mathscr{H}}^0(\mathscr{F},c) = \varinjlim_{\{c_\alpha\}\in\operatorname{Cov} c} \mathscr{H}^0(\mathscr{F},\{c_\alpha\}).$$

Then $c \mapsto \check{\mathscr{H}}^0(\mathscr{F},c)$ is another presheaf on $\mathscr{C}$, denoted $\mathscr{F}^+$.

[14, Tag 00WB], [14, Tag 00WH]

Lemma 5.2.5 (Sheafification). *For every presheaf $\mathscr{F}$ on $\mathscr{C}$, the presheaf $(\mathscr{F}^+)^+$ is a sheaf. The functor $\mathscr{F} \mapsto \mathscr{F}^\# := (\mathscr{F}^+)^+$ is a left adjoint to the inclusion* $\mathrm{Sh}\,\mathscr{C} \subseteq \mathrm{PSh}\,\mathscr{C}$*, called the* sheafification functor.

Proof sketch. Let us say that $\mathscr{F}$ is *separated* if for every cover $\{c_\alpha \to c\}$, the canonical maps $u\colon \mathscr{F} \to \mathscr{H}^0(\mathscr{F}, \{c_\alpha\})$ are injective. One checks that:

1. For any presheaf $\mathscr{F}$, the presheaf $\mathscr{F}^+$ is separated.
2. If $\mathscr{F}$ is separated, then $\mathscr{F}^+$ is a sheaf.

Together, these imply that $\mathscr{F}^\#$ is always a sheaf. Moreover, since if $\mathscr{G}$ is a sheaf then the canonical map $\mathscr{G} \to \mathscr{G}^+$ is an isomorphism, we see using functoriality of $\mathscr{F} \to \mathscr{F}^+$ that every map $\mathscr{F} \to \mathscr{G}$ from a presheaf $\mathscr{F}$ to a sheaf $\mathscr{G}$ factors through $\mathscr{F}^\#$. Uniqueness of this factorization follows from the fact that every section of $\mathscr{F}^+$ locally comes from a section of $\mathscr{F}$. This implies that $\mathscr{F} \mapsto \mathscr{F}^\#$ is a left adjoint to the inclusion. □

Further, many other notions of sheaf theory: cohomology, continuous maps of sites $f\colon \mathscr{C} \to \mathscr{C}'$[7], push-forward and pull-back functors $f_*\colon \mathrm{Sh}\,\mathscr{C} \to \mathrm{Sh}\,\mathscr{C}'$ and $f^*\colon \mathrm{Sh}\,\mathscr{C}' \to \mathrm{Sh}\,\mathscr{C}$, and so on, exist and behave as one would expect.

Let us stop here the development of the general theory, referring the curious reader to [16], [13], [1], or [14, Tag 00UZ].

[7] By convention, this is a functor $f^{-1}\colon \mathscr{C}' \to \mathscr{C}$ in the opposite direction. It is assumed to map covering families to covering families and to preserve fiber products. These conditions ensure that the functor

$$(-) \circ f^{-1}\colon \mathrm{PSh}\,\mathscr{C} \to \mathrm{PSh}\,\mathscr{C}'$$

maps sheaves to sheaves, inducing a functor

$$f_*\colon \mathrm{Sh}\,\mathscr{C} \to \mathrm{Sh}\,\mathscr{C}'.$$

5.3 G-topologies

The admissible site of $\mathbf{Q}^r$ defined in §5.1 is fairly concrete: its objects are simply subsets of the set $\mathbf{Q}^r$. In other words, admissible opens and covers define a *G-topology* in the sense of the following definition.

Definition 5.3.1 (G-topology). A *G-topology* on a set X is a site whose underlying category is a full subcategory of the poset of subsets of X, which is stable under intersections and such that covering families are jointly surjective.

In other words, it is the data of a set $\mathscr{C}$ of subsets of X, called *admissible opens*, such that the intersection of two admissible subsets is again admissible, and for each admissible open $U \in \mathscr{C}$, a class of *admissible covers* $\{U_\alpha\}_{\alpha\in I}$ where $U = \bigcup_{\alpha\in I} U_\alpha$ and $U_\alpha \in \mathscr{C}$, such that the following axioms are satisfied

i. The cover $\{U\}$ is an admissible cover for every $U \in \mathscr{C}$.

ii. If $U' \subseteq U$ is an inclusion of admissible opens and if $\{U_\alpha\}$ is an admissible cover of U, then $\{U'_\alpha = U_\alpha \cap U'\}$ is an admissible cover of U'.

iii. If $\{U_\alpha\}_{\alpha\in I}$ is an admissible cover of an admissible open U and if for every $\alpha \in I$, $\{U_{\alpha\beta}\}_{\beta\in J_\alpha}$ is an admissible cover of U_α, then $\{U_{\alpha\beta}\}_{\alpha\in I,\beta\in J_\alpha}$ is an admissible cover of U.

A *G-topological space* is a set X endowed with a G-topology. A map $f\colon Y \to X$ between G-topological spaces is *continuous* if $f^{-1}(U)$ is an admissible open in Y for every admissible open $U \subseteq X$ and if $\{f^{-1}(U_\alpha)\}$ is an admissible cover of $f^{-1}(U)$ whenever $\{U_\alpha\}$ is an admissible cover of $U \subseteq X$.

Example 5.3.2. Let X be a separated scheme, and take as admissible opens the set of all affine open subsets $U \subseteq X$. Separatedness ensures that $\mathbf{C}$ is stable under pairwise intersection. There are two variants of admissible covers:

- (STRONG) A covering $\{U_\alpha\}_{\alpha\in I}$ of an affine open U by affine opens U_α is admissible if $U = \bigcup U_\alpha$.
- (WEAK) The same but with I finite.

Since every affine scheme is quasi-compact, both give rise to the same category of sheaves, which is moreover equivalent to the category of sheaves on the topological space X.

In a G-topological space, we say that some property holds *locally* if it does so on the members of an admissible covering.

Our goal in the next chapter will be to put a G-topology on the space of maximal ideals of an affinoid K-algebra, as well as a structure sheaf. We will then glue such spaces to obtain general rigid-analytic spaces. Gluing G-topologies is facilitated by the following properties:

Definition 5.3.3 (Completeness axioms)**.** Let X be a G-topological space.

(0) We say that X satisfies axiom (G_0) if $\emptyset$ and X are admissible opens.

(1) We say that X satisfies axiom (G_1) if "admissibility of a subset is a local condition": given a subset $V \subseteq U$ of an admissible open U, the set V is an admissible open if and only if there exists an admissible cover $\{U_\alpha\}$ of U such that $U_\alpha \cap V$ is an admissible open for all α.

(2) We say that X satisfies axiom (G_2) if a covering of an admissible open V by admissible opens $\{V_\alpha\}$ is admissible if it admits an admissible covering of V as a refinement.

Remark 5.3.4. Consider the following condition (G_2') "admissibility of a cover is a local condition": given an admissible open V contained in an admissible open U and a family $\{V_\beta\}$ of admissible open subsets of V, the family $\{V_\beta\}$ is an admissible cover of V if and only if there exists an admissible cover $\{U_\alpha\}$ of U such that $\{U_\alpha \cap V_\beta\}$ is an admissible cover of $U_\alpha \cap V$ for every α. Then $(G_2) \Rightarrow (G_2')$, and if the G-topology satisfies the additional property that every cover of the form $V = \bigcup V_\alpha$ of an admissible open by admissible opens such that $V = V_\alpha$ for some α is admissible (that is, "split" covers are admissible), then also $(G_2') \Rightarrow (G_2)$.

If U is an admissible open of a G-topological space X, then the set of all admissible opens $V \subseteq X$ and the datum of all admissible covers consisting of such subsets forms a G-topology on U, called the *induced G-topology*.

Proposition 5.3.5 (Gluing G-topologies)**.** *Let X be a set and let $U_\alpha \subseteq X$ ($\alpha \in I$) be subsets of X such that $X = \bigcup U_\alpha$. Suppose that* [5, Proposition 5.1/11]

- *each U_α is endowed with a G-topology satisfying axioms (G_0), (G_1), and (G_2), and*
- *$U_\alpha \cap U_\beta$ is an admissible open in both U_α and U_β for every $\alpha, \beta \in I$, and*
- *the G-topologies on U_α and U_β induce the same G-topology on $U_\alpha \cap U_\beta$.*

Then there exists a unique G-topology on X satisfying (G_0), (G_1), and (G_2) for which the U_α are admissible opens, for which $X = \bigcup U_\alpha$ is an admissible cover, and which induces the given topology on each U_α.

Proof. Condition (G_1) imposes that $V \subseteq X$ is admissible if and only if $V \cap U_\alpha$ is an admissible open of U_α for all α. Similarly, (G_2) forces declaring $\{V_\beta\}$ an admissible cover of $V = \bigcup V_\beta$ if $\{U_\alpha \cap V_\beta\}_\beta$ is an admissible cover of $U_\alpha \cap V$ (for the G-topology on U_α) for every α. This shows uniqueness, and we need to check that this defines a G-topology on X with the desired properties. This is rather straightforward and we omit the proof. □

5.A Sober topological spaces

For a topological space X, let $\mathrm{Op}(X)$ be the poset of open subsets of X, ordered by inclusion.[8] *Can we recover X from* $\mathrm{Op}(X)$*?* Clearly not always, for example if X has the indiscrete topology (the only opens are X and $\emptyset$) then $\mathrm{Op}(X)$ carries no information about the cardinality of X. More generally, if X is not T_0, i.e. there exist two points $x \neq x'$ which such that $x \in U \iff x' \in U$ for every open $U \subseteq X$, then $\mathrm{Op}(X)$ and $\mathrm{Op}(X/(x \sim x'))$ are isomorphic.

[8] The poset $\mathrm{Op}(X)$ is an example of a **locale**: a poset in which the supremum of every subset and the infimum of every finite subset exists, and which satisfies the *distributive law*

$$\inf\{x, \sup\{a_i\}_{i\in I}\} = \sup\{\inf\{x, a_i\}\}_{i\in I}.$$

Every locale forms a site where $\{a_i \leq a\}_{a\in A}$ forms a covering family if $a = \sup\{a_i\}$, and hence gives rise to a topos. For the locale $\mathrm{Op}(X)$, this is the usual sheaf theory on X.

Even axiom T_0 is not sufficient for the recovery of X for $\mathrm{Op}(X)$. For example, if $X = \mathbf{A}^1_k$ with the Zariski topology and $X' = X \setminus \{\eta\}$ is the set of all closed points of X (η is the generic point), then $\mathrm{Op}(X) \simeq \mathrm{Op}(X')$, since a non-empty $U \subseteq X'$ is open if and only if $U \cup \{\eta\}$ is open in X'.

Recall that a closed subset $Y \subseteq X$ of a topological space X is *irreducible* if it is not the sum of two proper closed subsets. If $Y = \overline{\{y\}}$ for some point $y \in Y$, we call y a *generic point* of Y.

Definition 5.A.1 (Sober space)**.** A topological space X is *sober* if every irreducible closed subset $Y \subseteq X$ has a unique generic point. We denote by $\mathbf{Top}^{\mathrm{sober}} \subseteq \mathbf{Top}$ the full subcategory of sober spaces.

See [9, 0 2.1.(b)]

Proposition 5.A.2. *The inclusion functor* $\mathbf{Top}^{\mathrm{sober}} \subseteq \mathbf{Top}$ *admits a left adjoint* $X \mapsto X^{\mathrm{sob}}$*, the* soberification*.*

Proof sketch. Let X be a topological space and let X^{sob} be the set of all irreducible closed subsets of X; we have a natural map $\tau_X : X \to X^{\mathrm{sob}}$ sending x to $\overline{\{x\}}$. We endow X^{sob} with the topology in which a subset $U \subseteq X^{\mathrm{sob}}$ open if there exists an open $U^\circ \subseteq X$ such that U equals the set of irreducible subsets which intersect U°. This topology makes $\tau_X : X \to X^{\mathrm{sob}}$ continuous. Moreover, the open subset U° is unique if it exists, so we have an order-preserving bijection $U \leftrightarrow U^\circ$ between opens in X and in X^{sob}.

The space X^{sob} is sober: if $Z \subseteq X^{\mathrm{sob}}$ is an irreducible closed subset, write its complement $U = X^{\mathrm{sob}} \setminus Z$ as the set of all closed irreducible $Y \subseteq X$ which intersect some open $U^\circ \subseteq X$. Set $W = X \setminus U^\circ$; it is easy to check that W is irreducible, and hence defines a point $[W] \in X^{\mathrm{sob}}$. One then checks that $Z = \overline{\{[W]\}}$, and that $[W]$ is the unique generic point of Z. Details omitted.

If $f : X \to X'$ is continuous, and $Y \subseteq X$ is closed and irreducible, then $f(Y) \subseteq X'$ is irreducible, and so is its closure $\overline{f(Y)}$. The map $f^{\mathrm{sob}} : X^{\mathrm{sob}} \to (X')^{\mathrm{sob}}$ defined by $Y \mapsto \overline{f(Y)}$ is continous. Moreover, the square

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle \tau_X}\downarrow & & \downarrow{\scriptstyle \tau_{X'}} \\ X^{\mathrm{sob}} & \xrightarrow[f^{\mathrm{sob}}]{} & (X')^{\mathrm{sob}} \end{array}$$

commutes. We have thus defined a functor $X \mapsto X^{\mathrm{sob}} : \mathbf{Top} \to \mathbf{Top}^{\mathrm{sober}}$ and a natural transformation τ which will serve as the unit of the adjunction.

Finally, we need to check that every map $X \to X'$ with X' sober factors uniquely through X^{sob}. This is equivalent to saying that $\tau_X : X \to X^{\mathrm{sob}}$ is a homeomorphism of X is sober. The inverse maps $[Y]$ to the unique generic point η_Y of Y; it is clearly an inverse bijection. It is also continuous, since the preimage of $\{[Y] : Y \cap U^\circ \neq \emptyset\}$ equals U°. □

Since $\tau_X : X \to X^{\mathrm{sob}}$ induces a bijection on open subsets, we have $\mathrm{Op}(X) \simeq \mathrm{Op}(X^{\mathrm{sob}})$ as posets. Conversely, the construction of the space X^{sob} only depends on the poset $\mathrm{Op}(X)$.

Indeed, the set of closed irreducible subsets Y of X is in bijection $Y \longleftrightarrow X \setminus Y = U$ with the set of open subsets $U \in \mathrm{Op}(X)$ which are not equal to the intersection $U_1 \cap U_2$ of two opens $U_1, U_2 \neq U$. Since $U_1 \cap U_2$ is the largest element of the poset $\mathrm{Op}(X)$ which is smaller than both U_1 and U_2, the latter depends only on the order on $\mathrm{Op}(X)$. Summarizing:

Corollary 5.A.3. *The soberification of a space X depends only on the poset $\mathrm{Op}(X)$, and the poset $\mathrm{Op}(X)$ depends only on the soberification of X. For two spaces X and Y, there exists an isomorphism of posets $\mathrm{Op}(X) \simeq \mathrm{Op}(Y)$ if and only if $X^{\mathrm{sob}} \simeq Y^{\mathrm{sob}}$.*

For a family $\{U_\alpha\}$ of open subsets of a space X, the union $U = \bigcup U_\alpha$ is the smallest element of the poset $\mathrm{Op}(X)$ which is larger than all U_α. It follows that the topos $\mathrm{Sh}(X)$ (the category of sheaves on X) depends only on the poset $\mathrm{Op}(X)$. In particular, X and X^{sob} have equivalent topoi.

It turns out that $\mathrm{Op}(X) \mapsto \mathrm{Sh}(X)$ does not lose any information, namely:

See [9, 0 2.7(a)].

Proposition 5.A.4. *Let X be a sober topological space. Then X can be reconstructed from the topos $\mathrm{Sh}(X)$.*

Proof. Note that every topos $T = \mathrm{Sh}\,\mathscr{C}$ admits a final object e, the sheaf whose value on every $c \in \mathrm{ob}\,\mathscr{C}$ is the singleton $\{*\}$. If $T = \mathrm{Sh}(X)$ for a topological space X, then $e = \mathrm{Hom}_{\mathrm{Op}(X)}(-, X)$ is simply the sheaf represented by X, the final object of the site $\mathrm{Op}(X)$.

Ignoring potential set-theoretic difficulties, let us consider the set $\mathrm{Op}(T)$ of sub-objects of e, i.e. isomorphism classes of objects $v \in \mathrm{ob}\,T$ such that the unique morphism $v \to e$ is a monomorphism. We endow $\mathrm{Op}(T)$ with the order where we declare $v \leq v'$ if there exists a morphism $v \to v'$.

Suppose now that $T = \mathrm{Sh}(X)$ for a topological space X. If $U \subseteq X$ is an open subset, then the sheaf $h_U = \mathrm{Hom}_{\mathrm{Op}(X)}(-, U)$ is a sub-object of the final object h_X; moreover, if $V \subseteq U$ then $h_V \leq h_U$, so we get a morphism of posets

$$\gamma\colon \mathrm{Op}(X) \to \mathrm{Op}(T).$$

We claim that γ is an isomorphism of posets. Indeed, if v is a sub-object of e, let U be the union of all opens $V \subseteq X$ such that $v(V) \neq \emptyset$. Since every $v(V)$ is a subset of $\{*\} = e(V)$, the sheaf condition implies that $v(U) = \{*\}$. By Yoneda's lemma, this gives a map of sheaves $h_U \to v$. This map is an isomorphism on stalks and hence is an isomorphism. This gives the inverse to γ, and we omit checking all the remaining details.

Finally, $X^{\mathrm{sob}} = X$ can be reconstructed from $\mathrm{Op}(X) \simeq \mathrm{Op}(\mathrm{Sh}(X))$ by Corollary 5.A.3. □

We come back to our toy example at the beginning of the chapter:

Corollary 5.A.5. *The space $\mathbf{R}^r$ can be reconstructed from the category $\mathrm{Sh}^{\mathrm{adm}}(\mathbf{Q}^r)$ of sheaves for the admissible topology on $\mathbf{Q}^r$.*

The same idea in rigid geometry recovers the *adic spectrum* $\mathrm{Spa}A$ of an affinoid K-algebra A in the sense of Huber from the affinoid space ($\mathrm{Sp}A = \mathrm{Max}A$, admissible topology), to be defined next. Thus a good understanding of the points of $\mathrm{Spa}A$ allows one to get rid of the G-topology in favor of usual topology.

6
The admissible topology

Let A be an affinoid K-algebra. Our goal is to:

1. Equip $\mathrm{Sp}A := \mathrm{Max}A$ with a G-topology called the *admissible topology*.
2. Construct a structure sheaf $\mathcal{O}$ on $\mathrm{Sp}A$ (the sheaf condition is the subject of the Tate Acyclicity Theorem).
3. This gives a locally ringed G-topological space $(\mathrm{Sp}A, \mathcal{O})$. We define a *rigid-analytic space* over K as a locally ringed space $(X, \mathcal{O}_X)$ which is locally (in the G-topology sense) isomorphic to $(\mathrm{Sp}A, \mathcal{O}_{\mathrm{Sp}A}$ for some affinoid K-algebra A.

Before we proceed, recall that the results of Chapter 4 endow A with a topology induced by an equivalence class of K-algebra norms. This allowed us to define the *Tate algebra over A*:

$$A\langle X_1, \dots, X_r \rangle = \left\{ f = \sum_{\mathbf{n} \in \mathbf{N}^r} a_{\mathbf{n}} \mathbf{X}^{\mathbf{n}} \in A[[\mathbf{X}]] : a_{\mathbf{n}} \to 0 \text{ as } |\mathbf{n}| \to \infty \right\}.$$

In particular, for $f, g \in A$, we can define the algebras

$$A\langle f \rangle = A\langle X \rangle / (X - f), \quad A\langle g^{-1} \rangle = A\langle Y \rangle / (gY - 1).$$

Note that the image of f in $A\langle f \rangle$ is powerbounded, and $A\langle f \rangle$ is universal with this property: every $\varphi \colon A \to B$ with $\varphi(f) \in B^\circ$ factors uniquely through $A\langle f \rangle$. The algebra $A\langle g^{-1} \rangle$ has a similar property with respect to maps sending g to a unit whose inverse is powerbounded.

6.1 The canonical topology

Let $A = K\langle X_1, \dots, X_r \rangle / (f_1, \dots, f_s)$ be an affinoid K-algebra. Then $\mathrm{Max}A$ is identified with

$$\{x \in \mathbf{D}^r(\overline{K}) : f_i(x) = 0\} / \mathrm{Gal}(\overline{K}/K).$$

Endowing $\overline{K}$ with the metric topology, $\overline{K}^r$ with the product topology, the set $\{f_i(x) = 0, |x_i| \leq 1\} \subseteq \overline{K}^r$ with the induced topology, and finally the above quotient by Galois action with the quotient topology, we obtain a topology on $\mathrm{Max}A$ called *canonical*. A more intrinsic (evidently independent of the presentation) definition is the following.

Definition 6.1.1. The *canonical topology* on $\mathrm{Sp}A$ is the topology generated by the subsets

$$X(f) = \{x \in \mathrm{Sp}A : |f(x)| \leq 1\}, \quad f \in A.$$

Lemma 6.1.2. *The following subsets of $X = \mathrm{Sp}A$ are open in the canonical topology:*

(a) $\{|f(x)| \square c\}$ *for $c > 0$ and $f \in A$, where* $\square \in \{<, \leq, =, \geq, >\}$,

(b) $\{|f(x)| \neq 0\}$ *for* $f \in A$ *(in particular, Zariski opens in* $\operatorname{Sp} A$ *are open),*

(c) $\{|f(x)| \leq |g(x)| \neq 0\}$ *for* $f, g \in A$,

Warning: $U = \{|f(x)| \leq |g(x)|\}$ is not always open, e.g. for $g = tf$ we have $U = \{f(x) = 0\}$.

(d) $f_1, \dots, f_n, g \in A$ *without a common zero, the subset (called a* rational domain*)*

$$X\left(\frac{f_1, \dots, f_n}{g}\right) = \{|f_i(x)| \leq |g(x)|, i = 1, \dots, n\}.$$

Proof. (a) First, if $c^m = |x|$ for some $x \in K^\times$ and $m \geq 1$, then $\{|f| \leq c\} = X(x^{-1} f^m)$ is open. Since the set S of such numbers is dense in $(0, \infty)$, this shows that $\{|f| < c\} = \bigcup_{c' < c, c' \in S} \{|f| \leq c'\}$ is open for $c > 0$.

For the rest, it suffices to treat $U = \{|f| = c\}$. Let $x \in U$; first, suppose for simplicity that $\alpha = f(x) \in K$. Setting $g = f - \alpha$, we have $x \in \{|g(y)| < c\} \subseteq U$, so U is open. In general, let $p(X) = \sum_{i=0}^{n} a_i X^{n-i} \in K[X]$ be the minimal (monic) polynomial of $\alpha = f(x)$. Since $\operatorname{NP}(p)$ is a segment (Lemma 2.5.2), we have $|a_i| \leq |\alpha|^i = c^i$ and $|a_n| = c^n$. Set $g = p(f) \in A$ and $V = \{|g| < c^n\}$ (which is open); we have $g(x) = 0$, so $x \in V$. We check that $V \subseteq U$. Indeed, we have

$$g(y) = f(y)^n + \sum_{i=1}^{n-1} a_i f(y)^{n-i} + a_n, \quad y \in X,$$

so if $|f(y)| > c$, then the first term above dominates and $|g(y)| = |f(y)|^n > c^n$, so $y \notin V$. Similarly, if $|f(y)| < c$, then the last term dominates and so $|g(y)| = |a_n| = c^n$, and $y \notin V$ again.

(b) Clear.

(c) Similar to the proof of (d).

(d) Let $x \in U = X(f_1, \dots, f_n / g)$; then $c := |g(x)| > 0$, otherwise $f_1(x) = \dots = f_n(x) = g(x) = 0$. Let $V = \{|g| = c, |f_1| \leq c, \dots, |f_n| \leq c\}$, then V is open by (a), $x \in V$, and we have $U \subseteq V$. Thus U is open. □

Definition 6.1.3. For $f_1, \dots, f_r \in A$, the set

$$X(f_1, \dots, f_r) = X(f_1) \cap \dots \cap X(f_r)$$

is called a *Weierstrass domain*. For $f_1, \dots, f_r, g_1, \dots, g_s \in A$, the set

$$X(f_1, \dots, f_r, g_1^{-1}, \dots, g_s^{-1}) = \{|f_i| \leq 1, |g_i| \geq 1\}$$

is called a *Laurent domain*.

Lemma 6.1.4. *For a homomorphism* $\varphi \colon A \to B$ *between affinoid* K*-algebras, the induced map*

$$\varphi \colon \operatorname{Sp} B \to \operatorname{Sp} A$$

is continuous with respect to the canonical topology.

Proof. The preimage of $X(f)$ is $X(\varphi(f))$. □

The following lemma says that $X(f)$ is a basic example of an *affinoid subdomain*, to be defined in the next section.

Lemma 6.1.5. *Let* $f \in A$, *and let* $A\langle f \rangle = A\langle X \rangle / (X - f)$. *The map*

$$\operatorname{Sp} A\langle f \rangle \to \operatorname{Sp} A$$

induced by $A \to A\langle f \rangle$ *is a homeomorphism onto the open set* $X(f)$. *Every map* $A \to B$ *such that* $\operatorname{im}(\operatorname{Sp} B \to \operatorname{Sp} A) \subseteq X(f)$ *factors uniquely through* $A\langle f \rangle$.

Proof. We start with the last assertion. By the universal property of $A\langle f\rangle$, we need to show that for a map $\varphi\colon A\to B$ we have $\operatorname{im}(\operatorname{Sp}B\to\operatorname{Sp}A)\subseteq X(f)$ if and only if $\varphi(f)\in B$ is powerbounded. But $\varphi(f)$ is powerbounded if and only if $|\varphi(f)|_{\sup}\leq 1$ (Proposition 4.2.1(e)). The latter is equivalent to $|\varphi(f)(x)|=|f(\varphi(x))|\leq 1$ for all $x\in\operatorname{Sp}B$, which means precisely that $\varphi(x)\in X(f)$ for all $x\in\operatorname{Sp}B$.

The universal property applied to B a finite extension of K makes it clear that $\operatorname{Sp}A\langle f\rangle\to\operatorname{Sp}A$ is a bijection onto $X(f)$.

It remains to show that $\varphi\colon\operatorname{Sp}A\langle f\rangle\to\operatorname{Sp}A$ is an open map. Since A is dense in $A\langle f\rangle$, for $g\in A\langle f\rangle$ we can find an $h\in A$ such that $|g-\varphi^*(h)|_{\sup}\leq 1$. But then $X(g)=X(h)$ and so

$$\varphi(X(g))=X(f)\cap X(h)$$

is open in $\operatorname{Sp}A$. □

The above lemma implies that the ring $A\langle f\rangle$ depends only on the open set $X(f)$. An iterated application of the lemma says even that for $f_1,\dots,f_r\in A$, the ring $A\langle f_1,\dots,f_r\rangle$ has the same property with respect to $X(f_1,\dots,f_r)$. Since the subsets $X(f_1,\dots,f_r)$ form a basis $\mathscr{B}$ for the canonical topology on X, we obtain a presheaf of rings

$$\mathscr{O}^w\colon\mathscr{B}^{\mathrm{op}}\to\mathbf{Rings},\qquad \mathscr{O}^w(X(f_1,\dots,f_r))=A\langle f_1,\dots,f_r\rangle.$$

As explained in §5, this presheaf gives rise to a sheaf of rings in the canonical topology

$$\mathscr{O}^{\mathrm{wobbly}}\colon\operatorname{Op}(X)^{\mathrm{op}}\to\mathbf{Rings},$$

called the sheaf of *wobbly analytic functions* (see §1.1). Its sections can locally be described as elements of $A\langle f\rangle$ on opens of the form $X(f)$. However, since X is typically disconnected, we will have $\mathscr{O}^{\mathrm{wobbly}}(X)\neq A$ except in trivial cases.

6.2 *Affinoid subdomains*

Recall that if $X=\operatorname{Spec}A$ is an affine scheme and $U=\operatorname{Spec}A_U\subseteq X$ is an affine open subset, then for every ring B we have

$$\operatorname{Hom}(A_U,B)=\{f\colon A\to B\,:\,\operatorname{im}(\operatorname{Spec}B\to\operatorname{Spec}A)\subseteq U\}.$$

In other words, by Yoneda, the algebra A_U is determined by the open set U.

See Alex's blog post: link.

Definition 6.2.1 (Affinoid subdomain)**.** Let A be an affinoid K-algebra and let $X=\operatorname{Sp}A$. A subset $U\subseteq X$ is an *affinoid subdomain* if the functor

$$h_{A,U}\colon\{\text{affinoid }K\text{-algebras}\}\to\mathbf{Sets},$$

$$B\mapsto\{f\colon A\to B\,:\,\operatorname{im}(f\colon\operatorname{Sp}B\to\operatorname{Sp}A)\subseteq U\}$$

is representable by an affinoid K-algebra A_U.

With the notation $h_A=\operatorname{Hom}(A,-)$ for the (contravariant) Yoneda embedding, the functor $h_{A,U}$ equals h_{A_U} and is a subfunctor of h_A. The following easy claim follows formally from the definition and the fact that the category of affinoid K-algebras admits amalgamated coproducts (pushouts).

Lemma 6.2.2. *The intersection of two affinoid subdomains is an affinoid subdomain.*

Proof. Let $U, V \subseteq X$ be affinoid subdomains. Let

$$A_{U\cap V} = A_U \widehat{\otimes}_A A_V \quad \text{(completed tensor product, see §4.3).}$$

We claim that $A_{U\cap V}$ represents the functor $h_{A,U\cap V}$. Indeed, this functor is equal to the fiber product of functors

$$h_{A,U} \times_{h_A} h_{A,V} = h_{A_U} \times_{h_A} h_{A_V}.$$

Since $A_{U\cap V}$ is the pushout of $A_U \leftarrow A \rightarrow A_V$ (Proposition 4.3.5), the latter functor equals $h_{A_{U\cap V}}$, as desired. □

Proposition 6.2.3. *Let $U \subseteq X = \mathrm{Sp}A$ be an affinoid subdomain. Then U is open and the map $\varphi\colon \mathrm{Sp}A_U \to \mathrm{Sp}A$ induced by $A \to A_U$ is a homeomorphism onto U. Moreover, A_U is flat over A.*

Proof. If $x \in U$ and $L = A/\mathfrak{m}_x$, then $A \to A/\mathfrak{m}_x$ factors through A_U, showing that $\mathrm{Sp}A_U \to U$ is surjective. Further, $A_U/\mathfrak{m}_x = A_U \otimes_A L$ is initial among affinoid L-algebras, i.e. $L \simeq A_U/\mathfrak{m}_x$. In other words, the fibers of $\mathrm{Sp}A_U \to \mathrm{Sp}A$ are single points, i.e. φ is injective and we have

$$\mathfrak{m}_x \cdot A_U = \mathfrak{m}_y \quad \text{and} \quad A/\mathfrak{m}_x \xrightarrow{\sim} A_U/\mathfrak{m}_y,$$

where $x = \varphi(y)$. Moreover, replacing $A/\mathfrak{m}_x$ with $A/\mathfrak{m}_x^n$ above, we can upgrade the latter isomorphism to

$$A/\mathfrak{m}_x^n \xrightarrow{\sim} A_U/\mathfrak{m}_y^n. \quad \text{for all } n \geq 1.$$

Since A and A_U are Noetherian and Jacobson, [14, Tag 0523] implies that A_U is flat over A.

We will show that U is open, that is, that for every $x \in U$ there exists an $f \in A$ such that $X(f) \subseteq U$. In proving so, we may replace A with $A\langle f\rangle$ and A_U with $A_U \widehat{\otimes}_A A\langle f\rangle$ for any f with $|f(x)| \leq 1$.

As an intermediate step, we show that after passing from X to $X(f)$ for an appropriate f as above, the map $A \to A_U$ becomes surjective. Suppose we know that $A \to A_U$ is surjective, with kernel I. Then A/I^2, a square zero extension of $A_U = A/I$, satisfies

$$\mathrm{im}(\mathrm{Sp}A/I^2 \to \mathrm{Sp}A) = \mathrm{im}(\mathrm{Sp}A_U \to \mathrm{Sp}A) = U,$$

and hence $A \to A/I^2$ factors through $A \to A/I$. This forces $I = I^2$; since A is Noetherian, this implies that $I = (e)$ for an idempotent e[1], and $e(x) = 1$ since $x \in U$. Passing to $A\langle e^{-1}\rangle$, we obtain $I = 0$, and hence $A \to A_U$ is an isomorphism.

[1] **Easy commutative algebra exercise:** Show that a finitely generated ideal I in a commutative ring A satisfying $I = I^2$ is generated by an element $e \in A$ such that $e = e^2$.

Let $\alpha_1, \ldots, \alpha_r \in A_U$ be affinoid generators, i.e. powerbounded elements such that the associated map

$$A\langle X_1, \ldots, X_r\rangle \to A_U, \quad X_i \mapsto \alpha_i$$

is surjective. We pick a residue norm $|\cdot|$ on A and consider the residue norm on A_U induced by the above presentation. If $\mathfrak{m}_x = (f_1, \ldots, f_s)$, then since $A/\mathfrak{m}_x \simeq A_U/\mathfrak{m}_x A_U$ there exist $a_1, \ldots, a_r \in A$ and $c_{ij} \in A_U$ such that

$$\alpha_i - \varphi(a_i) = \sum c_{ij} f_j.$$

Since $f_i(x) = 0$ for all i, shrinking X we may assume that $|\alpha_i - \varphi(a_i)| < 1$. Similarly, since $|\alpha_i(x)| \leq 1$, we have $|a_i(x)| \leq 1$, and we may assume that $|a_i| \leq 1$. In particular, the a_i are powerbounded. We conclude using the following lemma. □

Lemma 6.2.4. *Let $\varphi\colon A \to B$ be a homomorphism between affinoid K-algebras. Suppose that there exist affinoid generators $b_1, \ldots, b_n$ of B and powerbounded elements $a_1, \ldots, a_n \in A$ such that*

$$|b_i - \varphi(a_i)| < 1$$

with respect to the residue norm on B induced by $K\langle X_1,\dots,X_n\rangle \to B$, $X_i \mapsto b_i$. Then φ is surjective.

Proof. Left as exercise. □

Lemma 6.2.5. *An affinoid subdomain of an affinoid subdomain is an affinoid subdomain.*

Proof. Straightforward. □

Proposition 6.2.6. *Let $X = \mathrm{Sp}A$ for an affinoid algebra A.*

(a) Every Laurent domain $U = X(f_1,\dots,f_r,g_1^{-1},\dots,g_s^{-1}) \subseteq X$ is an affinoid subdomain with

$$A_U = A\langle X_1,\dots,X_r,Y_1,\dots,Y_s\rangle/(X_i - f_i, g_jY_j - 1).$$

(b) Every rational domain $U = X(f_1,\dots,f_n/g) \subseteq X$ is an affinoid subdomain with

$$A_U = A\langle X_1,\dots,X_n\rangle/(f_i - gX_i).$$

Proof. Let U and A_U be as described as in (a) or (b), and let $\varphi: A \to B$ be a homomorphism such that $\mathrm{im}(\mathrm{Sp}B \to \mathrm{Sp}A) \subseteq U$. We need to show (1) that $\mathrm{im}(\mathrm{Sp}A_U \to \mathrm{Sp}(A)) \subseteq U$ and that (2) φ factors uniquely through $A \to A_U$.

(1) Let $\pi: A_U \to L$ be a K-algebra homomorphism onto a finite extension L of K corresponding to a point $x \in X$. Then in (a), the images $x_i = \pi(X_i) = \pi(f_i)$ and $y_j = \pi(Y_j) = 1/\pi(g_j)$ are powerbounded in L, which forces $|f_i(X)| \le 1$ and $g_i(X) \ge 1$. Similarly, in (b) the images $\pi(X_i)$ are powerbounded, which implies $|f_i(x)| \le |g(x)|$ unless $g(x) = 0$. But if $g(x) = 0$, then $0 = \pi(f_i - gX_i) = \pi(f_i)$, so $f_i = 0$ for all i, contradicting the assumption that $f_1,\dots,f_n,g$ have no common zero in X.

(2, a) Let $b_i = \varphi(f_i)$. Since $\varphi(\mathrm{Sp}(B)) \subseteq U \subseteq X(f_i)$, we have $|b_i|_{\sup} \le 1$, so b_i is powerbounded and hence there exists a unique homomorphism $\alpha_i: A\langle X_i\rangle \to B$ sending X_i to b_i. Similarly, if $c_j = \varphi(g_j)$, then $c_j \in B^\times$ and $1/c_j \in B$ is powerbounded; indeed, c_j does not belong to any maximal ideal, and $|c_j|_{\sup} = (\inf_{x\in \mathrm{Sp}B} |g_j(\varphi(x)|)^{-1} \le 1$. Therefore we obtain a unique $\beta_i: A\langle Y_i\rangle \to B$ with $\beta_i(Y_i) = c_j^{-1}$. The tensor product

$$\bigotimes_i \alpha_i \otimes \bigotimes_j \beta_j : A\langle X_1,\dots,X_r,Y_1,\dots,Y_s\rangle \to B$$

factors uniquely through A_U, giving the desired factorization $A_U \to B$.

(2, b) Left as exercise. □

The following rather difficult theorem (see [5, Corollary 4.2/12]) is often used as one of the key steps in the proofs of facts about affinoid subdomains.

Theorem 6.2.7 (Gerritzen–Grauert)**.** *Every affinoid subdomain $U \subseteq X$ is a finite union of rational domains.*

Of course not every finite union of rational domains is an affinoid subdomain, just as the union of two distinguished affine opens of a scheme need not be affine.

6.3 The admissible topology

Recall that every affine scheme is quasi-compact, meaning that every open cover in the Zariski topology has a finite subcover. The basic feature of the admissible topology on affinoid spaces defined below is that it forces them (as well as their affinoid subdomains), to be quasi-compact in the G-topology sense that every admissible cover admits a finite admissible subcover.

Definition 6.3.1. Let $X = \mathrm{Sp}A$ for an affinoid K-algebra A.

Compare with Definition 5.1.1 in our toy example: open boxes are replaced with affinoid subdomains, and closed boxes with sets of the form $\mathrm{im}(\mathrm{Sp}B \to X)$.

(a) An open subset $U \subseteq X$ is an *admissible open* if for every map of affinoid algebras $A \to B$ such that $W = \mathrm{im}(\mathrm{Sp}\, B \to X) \subseteq U$ there exists a finite collection $V_1, \dots, V_m$ of affinoid subdomains contained in U such that $W \subseteq \bigcup_{i=1}^m V_i$.

(b) A cover $U = \bigcup_{\alpha \in I} U_\alpha$ of an admissible open by admissible opens U_α is an *admissible cover* if for every map of affinoid algebras $A \to B$ such that $W = \mathrm{im}(\mathrm{Sp}\, B \to X) \subseteq U$ there exists a finite collection $V_1, \dots, V_m$ of affinoid subdomains contained in U such that $W \subseteq \bigcup_{i=1}^m V_i$ and each V_i is contained in some U_α.

Theorem 6.3.2. *Let $X = \mathrm{Sp}\, A$ for an affinoid K-algebra A.*

(a) Admissible opens and admissible covers define a G-topology on $\mathrm{Sp}\, A$, called the (strong) admissible topology.

(b) The admissible topology satisfies the completeness axioms (G_0), (G_1), and (G_2) of Definition 5.3.3.

(c) Every functor

$$\mathscr{F} : \{\textit{affinoid subdomains of}\ \mathrm{Sp}\, A\}^{\mathrm{op}} \to \mathbf{Sets}$$

which satisfies the sheaf condition for finite *covers of affinoid subdomains by affinoid subdomains*[2] *extends uniquely to a sheaf for the admissible topology on* $\mathrm{Sp}\, A$.

[2] Such an $\mathscr{F}$ is often called a sheaf for the *weak admissible topology*, see e.g. [5, ??].

Proof. This is completely formal, except for the fact that if $W = \mathrm{im}(\mathrm{Sp}\, B \to X)$ for some $A \to B$ and if $U = \mathrm{Sp}\, A_U \subseteq X$ is an affinoid subdomain, then $W \cap U$ is of the same type as W, namely

$$W \cap U = \mathrm{im}(\mathrm{Sp}\, B_U \to X) \quad \text{where} \quad B_U = B \widehat{\otimes}_A A_U. \qquad \square$$

Proposition 6.3.3. *Finite Boolean combinations of sets of the type*

$$\{|f(x)| \Box c\} \quad (c > 0, \Box \in \{\le, <, =, >, \ge\}), \quad \{f(x) \ne 0\}$$

are admissible.

Proof. See [4, Lemma 9.1.4/6]. $\square$

Compare with Example 5.1.5!

Example 6.3.4. (a) Every open subset $U \subseteq \mathbf{D}^1_K = \mathrm{Sp}\, K\langle X \rangle$ for the canonical topology is admissible.

(b) Not every finite cover by admissible opens is admissible, e.g.

$$\mathbf{D}^1_K = \{|X| = 1\} \cup \{|X| < 1\}$$

is an inadmissible cover of $\mathbf{D}^1_K = \mathrm{Sp}\, K\langle X \rangle$. Indeed, otherwise $\{|X| < 1\}$ would be a finite union of affinoid subdomains, contradicting the maximum principle.

(c) Not every covering of an affinoid subdomains by affinoid subdomains is admissible, e.g.

$$\mathbf{D}^1_K = \{|X| = 1\} \cup \bigcup_{n \ge 1} \{|X|^n \le |t|\}, \quad t \text{ pseudouniformizer.}$$

(d) ("Skyscraper sheaf at the Gauss point") Suppose for simplicity that $K = \overline{K}$, so that $\mathbf{D}^1_K = \{x \in K \,|x| \le 1\}$. Let us call an open subset $U \subseteq \mathbf{D}^1_K$ *huge* if it contains a non-empty affinoid subdomain of the form

$$\{x \in K : |x| \le 1, |x - a_i| \ge \rho_i\}.$$

The presheaf on $\mathbf{D}^1_K$ defined by

$$\mathscr{F}(U)=\begin{cases}\mathbf{Z} & \text{if } U \text{ is huge,}\\ 0 & \text{otherwise}\end{cases}$$

is a nonzero sheaf for the admissible topology whose stalks at all points in $\mathbf{D}^1_K$ are zero.

(e) For an example of an inadmissible open subset in $\mathbf{D}^2_K$, see Problem X on Problem Set 5.

6.4 The structure sheaf

The following key theorem will be proved in the next chapter.

Theorem 6.4.1 (Tate acyclicity). *Let $X=\operatorname{Sp}A$ for an affinoid K-algebra A. Then the functor*

$$U\mapsto A_U\colon \{\textit{affinoid subdomains of } \operatorname{Sp}A\}^{\mathrm{op}}\to \mathbf{Rings}$$

satisfies the sheaf condition for finite covers of affinoid subdomains by affinoid subdomains.

Combining this with Theorem 6.3.2(c), we obtain:

Corollary 6.4.2 (Structure sheaf on $\operatorname{Sp}A$). *There is a unique sheaf for the admissible topology $\mathscr{O}_X$ on $X=\operatorname{Sp}A$ such that for every affinoid subdomain $U\subseteq X$ we have*

$$\mathscr{O}_X(U)=A_U.$$

We shall now briefly discuss the stalks of $\mathscr{O}_X$. For a motivating example, consider the point $0\in\mathbf{D}^1_K=\operatorname{Sp}K\langle X\rangle$; the sets $U_n=\{|X|\le|t|^n\}$ form a fundamental system of neighborhoods of 0, and hence

$$\mathscr{O}_{\mathbf{D}^1_K,x}=\varinjlim_n \mathscr{O}_X(U_n)=\varinjlim_n K\langle X\rangle\left\langle\frac{X}{t^n}\right\rangle.$$

This can be identified with the subring of $K[[X]]$ (which is the completion $\widehat{K\langle X\rangle}_{(X)}$ of $K\langle X\rangle$ with respect to X) consisting of power series $\sum a_nX^n$ with positive radius of convergence, i.e. $|a_n|=O(\rho^n)$ for some $\rho<\infty$. Therefore we have strict inclusions

$$K\langle X\rangle_{(X)}\subsetneq\mathscr{O}_{\mathbf{D}^1_K,x}\subsetneq\widehat{K\langle X\rangle}_{(X)}=K[[X]].$$

The completion $\widehat{\mathscr{O}}_{\mathbf{D}^1_K,x}$ of $\mathscr{O}_{\mathbf{D}^1_K,x}$ coincides with $K[[X]]$.

Proposition 6.4.3. *Let $x\in X=\operatorname{Sp}A$, corresponding to a maximal ideal $\mathfrak{m}_x\subseteq A$. The stalk $\mathscr{O}_{X,x}$ is a Noetherian local ring with maximal ideal $\mathfrak{m}_x\cdot\mathscr{O}_{X,x}$. The completion of $\mathscr{O}_{X,x}$ coincides with the $\mathfrak{m}_x$-adic completion of A.*

Proof. We can write $\mathscr{O}_{X,x}=\varinjlim_U A_U$, the colimit taken over all affinoid subdomains $U\subseteq X$ containing x. The proof of Proposition 6.2.3 shows that for each such U, the ideal $\mathfrak{m}_x\cdot A_U$ is the maximal ideal of A_U corresponding to $x\in U=\operatorname{Sp}A_U$, and the map $A\to A_U$ induces isomorphism of $\mathfrak{m}_x$-adic completions. We thus have exact sequences

$$0\to\mathfrak{m}_x^s\cdot A_U\to A_U\to A/\mathfrak{m}_x^s\to 0$$

and passing to the inductive limit over U yields $\mathscr{O}_{X,x}/\mathfrak{m}_x^s\cdot\mathscr{O}_{X,x}\simeq A/\mathfrak{m}_x^s$ for all $s\ge1$. This shows that the $\mathfrak{m}_x$-adic completion of $\mathscr{O}_{X,x}$ is also equal to the $\mathfrak{m}_x$-adic completion of A.

We will now show that $\mathcal{O}_{X,x}$ is local. Let $f \in \mathcal{O}_{X,x} \setminus \mathfrak{m}_x \cdot \mathcal{O}_{X,x}$, we have to show f is invertible. Now f is the image of some $f \in A_U \setminus \mathfrak{m}_x \cdot A_U$. Find $\alpha \in K^\times$ and $m \geq 1$ such that $|f(x)|^m = |\alpha|$. Then $V = \mathrm{Sp} A_U\langle(\alpha^{-1} f^m)^{-1}\rangle$ is an affinoid subdomain neighborhood of x contained in V and f is invertible in A_V. Thus f is invertible in $\mathcal{O}_{X,x}$.

Finally we show that $\mathcal{O}_{X,x}$ is Noetherian by a standard argument (used also e.g. in the proof that the henselization of a Noetherian local ring is Noetherian [14, Tag 06LJ]). Set $B = A_{\mathfrak{m}_x}$ and let us look at the inclusions

$$B \subseteq \mathcal{O}_{X,x} \subseteq \widehat{B}.$$

We will show that $\widehat{B}$ is faithfully flat over $\mathcal{O}_{X,x}$. Indeed, since $\widehat{B}$ coincides with the $\mathfrak{m}_x$-adic completion of every A_U and each A_U is Noetherian, $\widehat{B}$ is flat over every A_U and hence over $\mathcal{O}_{X,x}$ [14, Tag 05UU]. Since $\mathcal{O}_{X,x} \to \widehat{B}$ is a local homomorphism, it is faithfully flat [14, Tag 00HR]. If now $\{I_n\}$ is an increasing sequence of ideals in $\mathcal{O}_{X,x}$, then $I_n \otimes \widehat{B} = I_n \cdot \widehat{B}$ is stationary because $\widehat{B}$ is Noetherian. But this implies that $(I_{n+1}/I_n) \otimes \widehat{B} = 0$ for $n \gg 0$, and since $\widehat{B}$ is faithfully flat over $\mathcal{O}_{X,x}$, we see that $I_{n+1}/I_n = 0$, so $\{I_n\}$ is stationary (see [14, Tag 033E]). Thus $\mathcal{O}_{X,x}$ is Noetherian. □

Definition 6.4.4. A G-topological space X satisfying axiom (G_0) of Definition 5.3.3 is *connected* if it does not admit an admissible cover of the form

$$X = \left(\bigcup_{\alpha \in I} U_\alpha\right) \cup \left(\bigcup_{\beta \in J} V_\beta\right)$$

with $\bigcup_{\alpha\in I} U_\alpha \neq \emptyset \neq \bigcup_{\beta \in J} V_\beta$ and $(\bigcup_{\alpha \in I} U_\alpha) \cap (\bigcup_{\beta\in J} V_\beta) = \emptyset$.

Remark 6.4.5. If X satisfies axioms (G_0), (G_1), and (G_2) of Definition 5.3.3, then X is connected if and only if it does not admit an admissible cover of the form $X = U \cup V$ with $U \neq \emptyset \neq V$ and $U \cap V = \emptyset$.

Remark 6.4.6. Connectedness is a topos-theoretic notion. Namely, a topos $\mathcal{T} = \mathrm{Sh}\,\mathcal{C}$ is *connected* if $H^0(X, \mathbf{Z}) = \mathbf{Z}$ where $\mathbf{Z}$ is the constant sheaf on T with value $\mathbf{Z}$. Then a G-topological space X is connected if and only if its associated topos $\mathrm{Sh}\,X$ is connected.

Proposition 6.4.7. *The affinoid space $X = \mathrm{Sp} A$ (endowed with the admissible topology) is connected if and only if A does not have nontrivial idempotents.*[3]

[3] I.e. if $\mathrm{Spec}\,A$ is connected.

Proof. If $e \in A$ is a nontrivial idempotent, then $X = V(e^{-1}) \cup V((1-e)^{-1})$ is an admissible cover, showing that X is not connected. Conversely, suppose that $X = U \cup V$ is an admissible cover with $U \neq \emptyset \neq V$ and $U \cap V = \emptyset$. The sheaf condition for this covering and the structure sheaf $\mathcal{O}_X$ yields

$$A = \mathcal{O}_X(X) \simeq \mathcal{O}_X(U) \times \mathcal{O}_X(V),$$

with both factors being nonzero rings. Then the element $e \in A$ corresponding to $(1,0) \in \mathcal{O}_X(U) \times \mathcal{O}_X(V)$ is a nontrivial idempotent. □

Definition 6.4.8 (Rigid-analytic space)**.** A *rigid-analytic space* over K is a locally G-ringed space $(X, \mathcal{O}_X)$ whose G-topology satisfies the completeness axioms (G_0), (G_1), and (G_2) of Definition 5.3.3, and which admits an admissible cover $X = \bigcup_{\alpha \in I} U_\alpha$ where each $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ is isomorphic as a locally G-ringed space with $(\mathrm{Sp} A_\alpha, \mathcal{O}_{\mathrm{Sp} A_\alpha})$ for some K-affinoid algebra A_α.

6.A Affinoid neighborhoods of Zariski closed subsets

The following result is easier to prove using other approaches to rigid geometry, such as Berkovich theory or formal schemes (see [7, §5.2]). We will need it in one of the homework problems.

A subset $Y \subseteq X$ of $X = \mathrm{Sp} A$ is *Zariski closed* if it is closed in the topology induced by the inclusion $X \subseteq \mathrm{Spec} A$. Equivalently, there exists an ideal $I \subseteq A$ such that Y is the image of $\mathrm{Sp} A/I \to X$. As in algebraic geometry, there is an inclusion-reversing bijection between Zariski closed subsets of X and radical ideals of A.

Theorem 6.A.1. *Let $X = \mathrm{Sp} A$ for an affinoid algebra A and let $Y \subseteq X$ be a Zariski closed subset cut out by an ideal $I = (f_1, \dots, f_r) \subseteq A$. Let $U \subseteq X$ be an affinoid subdomain containing Y. Then there exists a $\varepsilon > 0$ such that U contains the open subset*

$$\{|f_i(x)| \leq \varepsilon,\, i = 1, \dots, r\}.$$

Note that if $Y = \{f_i(x) = 0,\, i = 1, \dots, r\}$ is a closed subset of a compact Hausdorff space X cut out by continuous functions $f_1, \dots, f_r : X \to \mathbf{R}$, then the sets $\{|f_i(x)| < \varepsilon,\, i = 1, \dots, r\}$ form a basis of open neighborhoods of Y in X.

See [8, Exercise 4.1.8] for a complicated proof without using adic spaces, and [7, §5.2] for a proof using Berkovich spaces. The proof presented here secretly uses Berkovich spaces as well: the Berkovich space associated to X is the universal separated quotient of the valuative space X^{ad}.

Proof. Write $U = \mathrm{Sp} A_U$. We will work with the associated affinoid adic spaces X^{ad}, U^{ad}, and Y^{ad}. We may assume that the f_i are powerbounded. If

$$f : (f_1, \dots, f_r) : X^{\mathrm{ad}} \to (\mathbf{D}_K^r)^{\mathrm{ad}},$$

then $Y^{\mathrm{ad}} = f^{-1}(0)$. The map $U^{\mathrm{ad}} \to X^{\mathrm{ad}}$ is an open immersion, and hence we treat U^{ad} as an open subset of X^{ad}. Let $W = X^{\mathrm{ad}} \setminus U^{\mathrm{ad}}$, which is a closed subset of X^{ad} and hence it is quasi-compact because X^{ad} is. Consider $Z = f(W) \subseteq (\mathbf{D}_K^r)^{\mathrm{ad}}$, which is again quasi-compact and does not contain the classical point 0. Since $(\mathbf{D}_K^r)^{\mathrm{ad}}$ is a coherent valuative space, every two points x, y without a common generalization admit disjoint open neighborhoods [9, 0 2.3.18(2)]. Moreover, the rational opens

$$U_n = \{|X_i| \leq |t^n| \, i = 1, \dots, r\}$$

form a basis of open neighborhoods of 0. For every $z \in Z$ we find an open neighborhood V_z of z in $(\mathbf{D}_K^r)^{\mathrm{ad}}$ and an integer n_z such that $U_{n_z} \cap V_z = \emptyset$. Since Z is quasi-compact, finitely many of the V_z cover Z, and then the intersection of the corresponding U_{n_z} produces an n such that $U_n \cap Z = \emptyset$. Then $f^{-1}(U_n) \subseteq U^{\mathrm{ad}}$ and hence $\{x \in X : |f_i(x)| \leq |t^n|\} \subseteq U$. □

7
Tate's Acyclicity Theorem

Theorem 7.0.1 (Tate acyclicity). *Let $X = \mathrm{Sp}\,A$ for an affinoid K-algebra A and let $X = U_1 \cup \ldots \cup U_m$ be a finite covering of X by affinoid subdomains $U_i = \mathrm{Sp}\,A_i$. Each finite intersection $U_I = \cap_{i\in I} U_i$ ($I \subseteq \{1,\ldots,m\}$) is an affinoid subdomain,*

$$U_I = \mathrm{Sp}\,A_I \quad \textit{where} \quad A_{i_1\cdots i_k} = A_{i_1}\widehat{\otimes}_A \cdots \widehat{\otimes}_A A_{i_k}.$$

Then the sequence

$$0 \to A \to \prod_i A_i \to \prod_{i<j} A_{ij} \to \cdots \to \prod_{|I|=k+1} A_I \to \cdots \tag{7.1}$$

is exact.

Here, (7.1) is the alternating augmented Čech complex of the presheaf $U \mapsto A_U$ defined on affinoid subdomains $U \subseteq X$ with respect to the covering $X = U_1 \cup \ldots \cup U_m$, with differentials defined as alternating sums of the obvious restriction maps $\delta_i : A_I \mapsto A_J$ for $J = I \cup \{i\}$. Thus Theorem 7.0.1 implies that $U \mapsto A_U$ extends uniquely to a sheaf for the admissible topology $\mathcal{O}_X$ on X, and that this sheaf has vanishing higher Čech cohomology. This is a variant of the familiar assertion for affine schemes.

Outline of the proof. 1. An easy formal result (Lemma 7.2.2) shows that it is enough to check the sheaf property after refining the covering.

2. By the Gerritzen–Grauert Theorem (Theorem 6.2.7), we therefore may assume that each U_i is a rational domain.

3. Refining the covering further, we may put it in the form of a "rational covering"

$$U_i = X(\{f_j : j \neq i\}/f_i), \quad i = 1,\ldots,n$$

   for some $f_1,\ldots,f_n$ without common zero.[1] (Lemma 7.2.2)

4. A further trick allows one to reduce to "Laurent coverings," i.e. coverings of the form

$$U_\varepsilon = X(f_1^{\varepsilon_1},\ldots,f_n^{\varepsilon_n}), \quad \varepsilon \in \{-1,1\}^n.$$

   By induction, it suffices to treat the case $n = 1$, i.e. coverings of the form $X = X(f) \cup X(f^{-1})$.

5. This is handled by a direct computation, see Proposition 7.1.1. □

[1] In this and the next step, the U_i's and f_i's have nothing to do with their values in the previous step.

7.1 Proof of Tate Acyclicity (I): Basic case

The following special case of a covering $X = \{|f(x)| \leq 1\} \cup \{|f(x)| \geq 1\}$ will be the key step in the proof of Tate acyclicity.

Proposition 7.1.1. *Let $X = \mathrm{Sp}\, A$ for an affinoid K-algebra A and let $f \in A$. Then the sequence*

$$0 \to A \to A\langle f \rangle \times A\langle f^{-1} \rangle \to A\langle f, f^{-1} \rangle \to 0$$

Proof. By definition, we have

$$\begin{aligned} A\langle f \rangle &= A\langle X \rangle/(X - f), \\ A\langle f^{-1} \rangle &= A\langle Y \rangle/(fY - 1) = A\langle X^{-1} \rangle/(fX^{-1} - 1), \\ A\langle f, f^{-1} \rangle &= A\langle X, Y \rangle/(X - f, fY - 1) = A\langle X, X^{-1} \rangle/(X - f). \end{aligned}$$

We first note that the sequence

$$0 \to A \to A\langle X \rangle \times A\langle X^{-1} \rangle \to A\langle X, X^{-1} \rangle \to 0.$$

(where $A\langle X, X^{-1} \rangle = A\langle X, Y \rangle/(XY - 1)$) is exact. Indeed, if

$$(g = \sum_{n \geq 0} a_n X^n, h = \sum_{n \leq 0} b_n X^n) \in A\langle X \rangle \times A\langle X^{-1} \rangle$$

then its image in $A\langle X, X^{-1} \rangle$ equals

$$\sum_{n \in \mathbf{Z}} c_n X^n, \quad c_n = \begin{cases} a_n & n > 0 \\ -b_n & n < 0 \\ a_0 - b_0 & n = 0. \end{cases}$$

This vanishes precisely if $a_n = 0 = b_n$ for $n > 0$ and $a_0 = b_0 = a \in A$, and then (g, h) is the image of a. Exactness on the right is also clear.

The sequence in the assertion consists of the cokernels of the vertical arrows in the commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & (X - f)A\langle X \rangle \times (fX^{-1} - 1)A\langle X^{-1} \rangle & \xrightarrow{\beta} & (X - f)A\langle X, X^{-1} \rangle & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & A\langle X \rangle \times A\langle X^{-1} \rangle & \longrightarrow & A\langle X, X^{-1} \rangle & \longrightarrow & 0 \end{array}$$

Once we check that the top arrow β is an isomorphism, the diagram has exact rows, and the snake lemma implies the required assertion. Injectivity of β is clear: if (g, h) is in the kernel, it has to be of the form (a, a) for $a \in A$, and cannot lie in the product of the two ideals unless $a = 0$.

For surjectivity, take

$$g = (X - f)h, \quad h = \sum_{n \in \mathbf{Z}} c_n X^n \in A\langle X, X^{-1} \rangle$$

and set $g^+ = (X - f)\sum_{n \geq 0} c_n X^n$,

$$g^- = g^+ - g = (f - X)\sum_{n < 0} c_n X^n = (fX^{-1} - 1)\sum_{n \leq 0} c_{n-1} X^n.$$

Clearly $g^+ \in (X - f)A\langle X \rangle$, $g^{-1} \in (fX^{-1} - 1)A\langle X^{-1} \rangle$, and g is the image of (g^+, g^-). $\square$

Corollary 7.1.2. *Let $f_1,\dots,f_n \in A$, and for $\varepsilon \in \{-1,1\}^n$, let*

$$A\langle f^\varepsilon\rangle = A\langle f_1^{\varepsilon_1},\dots,f_n^{\varepsilon_n}\rangle.$$

Then the sequence

$$0 \to A \to \prod_\varepsilon A\langle f^\varepsilon\rangle \to \prod_{\varepsilon,\tau} A\langle f^\varepsilon, f^\tau\rangle \to \cdots$$

is exact.

Above, the complex

$$A\langle f^\varepsilon\rangle \to \prod_{\varepsilon,\tau} A\langle f^\varepsilon, f^\tau\rangle \to \cdots \tag{7.2}$$

is the total complex of the completed tensor product (over A) of two-term complexes

$$\bigotimes_{i=1}^n \left[A\langle f_i\rangle \times A\langle f_i^{-1}\rangle \to A\langle f_i, f_i^{-1}\rangle\right].$$

Since the terms of these complexes are flat over A, Proposition 7.1.1 implies Corollary 7.1.2. Note that (7.2) is the Čech complex for the covering $X = \bigcup_\varepsilon X(f^\varepsilon)$, and so Corollary 7.1.2 implies that the assertion of Theorem 7.0.1 holds for such coverings (called Laurent coverings in the literature).

7.2 *Proof of Tate Acyclicity (II): Rational coverings*

Definition 7.2.1. Let $f_0,\dots,f_n \in A$ be elements generating the unit ideal. The associated *rational covering* is the covering of $X = \operatorname{Sp} A$ by the rational domains

$$U_i = X\left(\frac{f_0,\dots,f_n}{f_i}\right) = \{|f_j(x)| \le |f_i(x)|,\ j \neq i\}, \quad i = 0,\dots,n.$$

We have $U_i = \operatorname{Sp} A_i$ where[2]

$$A_i = A\langle X_{ij},\ j\neq i\rangle/(f_j - X_{ij}f_i,\ j\neq i).$$

Lemma 7.2.2. *Every finite covering of $X = \operatorname{Sp} A$ by affinoid subdomains admits a rational covering as a refinement.*

Proof. By the Gerritzen–Grauert Theorem (Theorem 6.2.7), every affinoid subdomain is a finite union of rational domains. It therefore suffices to show that every finite covering $X = U_1 \cup \ldots \cup U_n$ by rational domains is refined by a rational covering. Write

$$U_i = X\left(\frac{f_{i1},\dots,f_{iN}}{f_{i0}}\right), \quad f_{ij} \in A$$

where $f_{i0},\dots,f_{iN}$ do not have a common zero for every i. Define

$$I = \{\varphi\colon \{1,\dots,n\} \to \{0,\dots,N\} : \exists_i \varphi(i) = 0\},$$

and for $\varphi \in I$ set

$$g_\varphi = \prod_{i=1}^n f_{i\varphi(i)} = f_{1\varphi(1)}\cdot\ldots\cdot f_{n\varphi(n)}.$$

We claim that $\{g_\varphi\}_{\varphi\in I}$ do not have a common zero and that the rational covering $\{V_\varphi\}_{\varphi\in I}$ defined by the f_φ refines $\{U_i\}_{i=1,\dots,n}$.

For the first claim, let $x \in X$. Since the U_i cover X, we have $x \in U_{i_0}$ for some i_0, and in particular $f_{i_0 0}(x) \neq 0$; we set $\varphi(i_0) = 0$. For every $i \neq i_0$, since the f_{ij} ($j = 0,\dots,N$) generate

[2] Recall that if $I = (f_0,\dots,f_n) \subseteq A$ is a finitely generated ideal of a ring A cutting out a closed subscheme

$$Y = V(I) \subseteq X = \operatorname{Spec} A,$$

then the blow-up $X' = \operatorname{Bl}_Y X$ is covered by the open affines

$$U_i = \operatorname{Spec} A[\{X_j : j\neq i\}]/(f_j - X_j f_i).$$

Thus a rational covering can be thought of as the covering induced by the blow-up in the unit ideal $A = (f_0, f_1,\dots,f_n)$ of $X = \operatorname{Sp} A$. This perspective will become very important when we study formal models in §REF.

the unit ideal, there exists a j such that $f_{ij}(x) \neq 0$; we set $\varphi(i) =$ any such j. This defines a $\varphi \in I$ such that $g_\varphi(x) \neq 0$.

For the second claim, let $\varphi \in I$. By definition, we have $\varphi(i_0) = 0$ for some i_0; we claim that $V_\varphi \subseteq U_{i_0}$. Let $x \in V_\varphi$, we need to prove that for every $j = 1, \ldots, N$ we have $|f_{i_0 j}(x)| \leq |f_{i_0 0}(x)|$. Since the U_i cover X, we have $x \in U_{i_1}$ for some i_1, and in particular

$$|f_{i_1 j}(x)| \leq |f_{i_1 0}(x)| \neq 0. \tag{7.3}$$

If $i_1 = i_0$ then we are done, so suppose $i_1 \neq i_0$. We set

$$\psi(i) = \begin{cases} j & i = i_0 \\ 0 & i = i_1 \\ \varphi(i) & i \neq i_0, i_1 \end{cases}.$$

Since $x \in U_\varphi$, we have $|g_\psi(x)| \leq |g_\varphi(x)| \neq 0$. Diving out the terms with $i \neq i_0, i_1$ yields

$$|f_{i_0 j}(x)| \cdot |f_{i_1 0}(x)| \leq |f_{i_0,0}(x)| \cdot |f_{i_1 \varphi(i_1)}(x)|.$$

Dividing out (7.3) yields $|f_{i_0 j}(x)| \leq |f_{i_0 0}(x)|$, as desired. □

7.3 Proof of Tate Acyclicity (III): Conclusion

Let us call a covering $X = \bigcup_{i=1}^m U_i$ by affinoid subdomains *acyclic* if for every affinoid subdomain $W \subseteq X$, the augmented alternating Čech complex for the presheaf $U \mapsto A_U$ with respect to the induced covering $W = \bigcup_{i=1}^m W \cap U_i$, i.e.

$$0 \to A_W \to \prod_i A_{W \cap U_i} \to \prod_{i<j} A_{W \cap U_{ij}} \to \cdots$$

is exact.

Lemma 7.3.1. *Suppose that a covering $X = V_1 \cup \ldots \cup V_m$ by affinoid subdomains refines a covering $X = U_1 \cup \ldots \cup U_n$ by affinoid subdomains and that $X = \bigcup V_i$ is acyclic. Then $X = \bigcup U_i$ is acyclic as well.*

Proof. This is formal, see [4, Corollary 8.1.4/3]. □

Combining this with Lemma 7.2.2, we obtain:

Corollary 7.3.2. *It is enough to prove Theorem 7.0.1 for rational coverings.*

Lemma 7.3.3. *Given a covering $X = U_1 \cup \ldots \cup U_n$ by affinoid subdomains and $f \in A$, suppose that the assertion of the theorem holds for the induced coverings of $X(f)$, $X(f^{-1})$, and $X(f, f^{-1})$. Then the assertion of the theorem holds for $X = U_1 \cup \ldots \cup U_n$.*

Proof. Write $A_i = A_{U_i}$ and $A_{ij} = A_{U_i \cap U_j} = A_i \widehat{\otimes}_A A_j$. We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & \prod_i A_i & \longrightarrow & \prod_{i,j} A_{ij} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A\langle f \rangle \times A\langle f^{-1} \rangle & \longrightarrow & \prod_i A_i\langle f \rangle \times A_i\langle f^{-1} \rangle & \longrightarrow & \prod_{i,j} A_{ij}\langle f \rangle \times A_{ij}\langle f^{-1} \rangle & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A\langle f, f^{-1} \rangle & \longrightarrow & \prod_i A_i\langle f, f^{-1} \rangle & \longrightarrow & \prod_{i,j} A_{ij}\langle f, f^{-1} \rangle & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

whose columns are exact by Proposition 7.1.1, and rows 2. and 3. are exact by assumption. Treating this as a short exact sequence of complexes, the long exact sequence of cohomology implies that the top row is exact. □

Proof of Theorem 7.0.1. By Corollary 7.3.2, it is enough to consider rational coverings $X = \bigcup_{i=0}^{n} U_i$ defined by $f_0, \ldots, f_n \in A$. We proceed by induction on n, and for a fixed n by induction on the number of indices i such that f_i is a non-unit.

If $f_0, \ldots, f_n$ are all units, then the covering is refined by the Laurent covering defined by the functions $g_{ij} = f_i/f_j$ ($i, j = 0, \ldots, n$), and we conclude by Corollary 7.1.2 and Lemma 7.3.1.

Suppose that f_n is a non-unit. Since the f_i do not have a common zero, the number $c = \inf_{x \in X} \max_i |f_i(x)|$ is positive, and we fix an N such that $|t^N| < c$. Set $f = f_n/t^N$ and consider the covering $X = X(f) \cup X(f^{-1})$. The induced covering of $X(f)$ is the rational covering defined by the restrictions of $f_1, \ldots, f_{n-1}$ (which do not have a common zero on $X(f)$ as $X(f) \cap U_n = \emptyset$). By induction assumption, the assertion ot the theorem is satisfied for the induced covering of $X(f)$. On $X(f^{-1})$ and $X(f, f^{-1})$, the function f_n becomes a unit, and hence the induced coverings satisfy the assertion ot the theorem by induction assumption. We conclude by Lemma 7.3.3. □

8
Rigid-analytic spaces

In this chapter, we define and study the basic properties of rigid-analytic spaces. This will be largely parallel to the basics of scheme theory: fiber products, coherent sheaves, line bundles and divisors, flat and smooth morphisms etc. work basically in the same way. There are however some important differences, notably the lack of a theory of quasi-coherent sheaves, and a rather complicated definition of a proper morphism. The analogy with scheme theory is amplified by the existence of the analytification functor from schemes locally of finite type over K to rigid-analytic spaces over K.

8.1 The category of rigid-analytic spaces

By definition, a scheme is locally ringed space which is locally of the form $\operatorname{Spec} A$ for a ring A. In rigid geometry, the situation is similar, but one uses the G-topology in place of ordinary topology. The completeness axioms (G_0), (G_1), and (G_2) of Definition 5.3.3 are imposed to facilitate the gluing.

Definition 8.1.1. A *locally ringed G-topological space* is a pair $(X, \mathcal{O}_X)$ of a G-topological space X and a sheaf of rings $\mathcal{O}_X$ on X whose stalks $\mathcal{O}_{X,x}$ $(x \in X)$ are local rings. A morphism of locally ringed G-topological spaces $f:(Y,\mathcal{O}_Y) \to (X,\mathcal{O}_X)$ consists of a continuous (see Definition 5.3.1) map of G-topological spaces $f: Y \to X$ and a map of sheaves of rings $f^*: \mathcal{O}_X \to f_*\mathcal{O}_Y$ such that for every $y \in Y$, the induced homomorphism $f^*: \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is local.

As in the case of schemes, we will often write X in place of $(X, \mathcal{O}_X)$.

Proposition 8.1.2. *(a) For every affinoid K-algebra A, the pair* $(X = \operatorname{Sp} A, \mathcal{O}_X)$ *is a locally ringed G-topological space, called an* affinoid space *over K. It satisfies the completeness axioms* (G_0), (G_1), *and* (G_2) *of Definition 5.3.3.*

(b) For every homomorphism $\varphi: A \to B$ *of affinoid K-algebras, the induced map* $\operatorname{Sp} A \to \operatorname{Sp} B$ *is a morphism of locally ringed G-topological spaces.*

(c) The association $A \mapsto \operatorname{Sp} A$ *defines an equivalence of categories*

$$\operatorname{Sp}\colon \{\textit{affinoid } K\textit{-algebras}\} \xrightarrow{\sim} \{\textit{affinoid spaces over } K\}$$

where the target category is considered as a full subcategory of the slice category of locally ringed G-spaces over the object $\operatorname{Sp} K$.

Proof. Parts (a) and (b) follow from the results of §6.4. For (c), we show that $X \mapsto \Gamma(X, \mathcal{O}_X)$ is a quasi-inverse functor. Indeed, we have $\Gamma(\operatorname{Sp} A, \mathcal{O}_{\operatorname{Sp} A}) = A$ by definition of the structure sheaf, and for $f: A \to B$ inducing a map $\mathrm{Sp}\,B \to \mathrm{Sp}\,A$, the induced map $\Gamma(\mathrm{Sp}\,A, \mathcal{O}_{\mathrm{Sp}\,A}) \to \Gamma(\mathrm{Sp}\,B, \mathcal{O}_{\mathrm{Sp}\,B})$ equals f. It remains to show that if two morphisms $\varphi, \psi\colon \mathrm{Sp}\,B \to \mathrm{Sp}\,A$ (of locally ringed G-spaces over K) induce the same morphism $\varphi^* = \psi^*: A \to B$, then $\varphi = \psi$. We first show that φ and ψ induce the same map of underlying sets. Let $y \in \mathrm{Sp}\,B$ be a point and let $\mathfrak{m}_y \subseteq B$ be the corresponding maximal ideal.

We claim that $\varphi(y) \in \mathrm{Sp}\,A$ corresponds to the maximal ideal $(\varphi^*)^{-1}(\mathfrak{m}_y)$. Indeed, we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\varphi^*} & B \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathrm{Sp}\,A,\varphi(y)} & \xrightarrow[\varphi^*]{} & \mathcal{O}_{\mathrm{Sp}\,B,y} \end{array}$$

in which the bottom map is a local homomorphism. Moreover, the maximal ideal of $\mathcal{O}_{\mathrm{Sp}\,A,\varphi(y)}$ (resp. $\mathcal{O}_{\mathrm{Sp}\,B,y}$) is $\mathfrak{m}_{\varphi(y)} \cdot \mathcal{O}_{\mathrm{Sp}\,A,\varphi(y)}$ (resp. $\mathfrak{m}_y \cdot \mathcal{O}_{\mathrm{Sp}\,B,y}$) and the induced maps

$$A/\mathfrak{m}_{\varphi(y)} \to \mathcal{O}_{\mathrm{Sp}\,A,\varphi(y)}/\mathfrak{m}_{\varphi(y)} \cdot \mathcal{O}_{\mathrm{Sp}\,A,\varphi(y)} \quad \text{and} \quad B/\mathfrak{m}_y \to \mathcal{O}_{\mathrm{Sp}\,B,y}/\mathfrak{m}_y \cdot \mathcal{O}_{\mathrm{Sp}\,B,y}$$

are isomorphisms. It follows that φ^* maps $\mathfrak{m}_{\varphi(y)}$ into $\mathfrak{m}_y$. Consequently, $(\varphi^*)^{-1}(\mathfrak{m}_y)$ contains $\mathfrak{m}_{\varphi(y)}$, and hence they are equal because $\mathfrak{m}_{\varphi(y)}$ is maximal. This proves the claim, and since the same is true for ψ we deduce that $\varphi(y) = \psi(y)$.

Finally, we compare the maps induced by φ and ψ on the structure sheaves. Let $U = \mathrm{Sp}\,A_U \subseteq \mathrm{Sp}\,A$ be an affinoid subdomain, and let $V = \varphi^{-1}(U) = \psi^{-1}(U) \subseteq \mathrm{Sp}\,B$ be its preimage, which is an affinoid subdomain of $\mathrm{Sp}\,B$. Let $\delta: \mathrm{Sp}\,B_V \to \mathrm{Sp}\,A$ be the map induced by $A \to B \to B_V$ where the first map is $\varphi^* = \psi^*$. Since the image of $\delta: V = \mathrm{Sp}\,B_V \to \mathrm{Sp}\,A$ is contained in U, by definition of an affinoid subdomain there exists a unique A-algebra homomorphism $f: A_U \to B_V$ factoring δ^*. Since $\varphi^*: A_U \to B_V$ and $\psi^*: A_U \to B_V$ are two such homomorphism, they must be equal. □

Definition 8.1.3. A *rigid-analytic space* over K is a locally ringed G-topological space over $\mathrm{Sp}\,K$ whose topology satisfies completeness axioms (G_0), (G_1), and (G_2) of Definition 5.3.3 which admits an admissible cover $X = \bigcup U_i$ where each $(U_i, \mathcal{O}_X|_{U_i})$ is an affinoid space over K.

Remark 8.1.4 (Gluing). By definition, a rigid-analytic space is constructed from affinoid spaces by gluing. To make this precise, suppose that we are given rigid-analytic spaces X_α ($\alpha \in I$), and for every $\alpha, \beta \in I$ admissible open subspaces $X_{\alpha\beta} \subseteq X_\alpha$ such that $X_{\alpha\alpha} = X_\alpha$, together with isomorphisms

$$X_\alpha \quad \supseteq \quad X_{\alpha\beta} \xrightarrow[\varphi_{\alpha\beta}]{\sim} X_{\beta\alpha} \quad \subseteq \quad X_\beta$$

such that $\varphi_{\alpha\beta} \circ \varphi_{\beta\alpha} = \mathrm{id}$, $\varphi_{\alpha\alpha} = \mathrm{id}$, and that $\varphi_{\alpha\beta}$ identifies $X_{\alpha\beta} \cap X_{\alpha\gamma}$ with $X_{\beta\alpha} \cap X_{\beta\gamma}$ for every $\alpha, \beta, \gamma \in I$. In addition, we assume that the triangles

$$\begin{array}{ccccc} X_{\alpha\beta} \cap X_{\alpha\gamma} & & \xrightarrow{\varphi_{\alpha\beta}} & & X_{\beta\alpha} \cap X_{\beta\gamma} \\ & \searrow^{\varphi_{\alpha\gamma}} & & \nearrow_{\varphi_{\gamma\beta}} & \\ & & X_{\gamma\alpha} \cap X_{\gamma\beta} & & \end{array}$$

commute. Then there exists a unique rigid-analytic space X admitting an admissible cover $X = \bigcup X_\alpha$ with $X_{\alpha\beta} = X_\alpha \cap X_\beta$.

Similarly, if $X = \bigcup X_\alpha$ is an admissible cover of a rigid-analytic space X and if Y is a rigid-analytic space, then given morphisms $f_\alpha: X_\alpha \to Y$ such that $f_\alpha = f_\beta$ on $X_\alpha \cap X_\beta$, there exists a unique morphism $f: X \to Y$ such that $f|_{X_\alpha} = f_\alpha$. In other words, the functor $U \mapsto \mathrm{Hom}(U, Y)$ is a sheaf for the admissible topology on X.

The last remark implies that for $Y = \mathrm{Sp}\,B$ affinoid, the equivalence of categories from Proposition 8.1.2(c) implies that we have a functorial isomorphism

$$\mathrm{Hom}(X, \mathrm{Sp}\,B) \simeq \mathrm{Hom}_K(B, \mathcal{O}_X(X)).$$

If $\mathcal{O}_X(X)$ was always an affinoid K-algebra, this would mean that $X \mapsto \mathcal{O}_X(X)$ is a left adjoint to $B \mapsto \mathrm{Sp}\,B$ (as in the case of schemes). However, this is not always the case, see Example 8.3.1.

Proposition 8.1.5. *The category of rigid-analytic spaces over K admits fiber products.*

Proof. This is entirely similar to the proof for schemes [10, II 3.3]: if $X = \mathrm{Sp}\,A$ and $Y = \mathrm{Sp}\,B$ are affinoid spaces over an affinoid space $S = \mathrm{Sp}\,R$, then $X \times_S Y = \mathrm{Sp}\,A \widehat{\otimes}_R B$. In general, given maps $f: X \to S$ and $g: Y \to S$ in Rig_K, find an admissible cover $S = \bigcup S_i$ by affinoids and for each i, admissible affinoid covers $f^{-1}(S_i) = \bigcup_{j \in J_i} X_{ij}$ and $g^{-1}(S_i) = \bigcup_{k \in K_i} Y_{ik}$. Set $Z_{ijk} = X_{ij} \times_{S_i} Y_{ik}$, which exists by previous considerations. Then glue these together to form the desired fiber product $Z = X \times_S Y$ (details omitted). □

8.2 Coherent sheaves

Every A-module M defines a presheaf on affinoid subdomains $U \subseteq X = \mathrm{Sp}\,A$ as follows:

$$U \quad \mapsto \quad M_U := A_U \otimes_A M.$$

Corollary 8.2.1. *In the situation of Theorem 7.0.1, let M be an A-module and set $M_I = M_{U_I} = A_{U_I} \otimes_A M$. Then the sequence*

$$0 \to M \to \prod_i M_i \to \prod_{i<j} M_{ij} \to \cdots \to \prod_{|I|=k+1} M_I \to \cdots$$

is exact.

Proof. Follows from Theorem 7.0.1 since the A_U are flat over A (Proposition 6.2.3). □

Again, this means that M defines a sheaf of $\mathcal{O}_X$-modules $\tilde{M} = M \otimes \mathcal{O}_X$ on X, and that this sheaf has vanishing higher Čech cohomology.

Proposition 8.2.2. *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on a rigid-analytic space X. The following conditions are equivalent:*

1. *There exists an admissible affinoid cover $X = \bigcup U_\alpha$, $U_\alpha = \mathrm{Sp}\,A_\alpha$ and finitely generated modules M_α over A_α such that $\mathcal{F}_{U_\alpha} \simeq \widetilde{M}_\alpha$ for every α.*
2. *For every admissible affinoid cover $X = \bigcup U_\alpha$, $U_\alpha = \mathrm{Sp}\,A_\alpha$ and finitely generated modules M_α over A_α such that $\mathcal{F}_{U_\alpha} \simeq \widetilde{M}_\alpha$ for every α.*
3. *The sheaf $\mathcal{F}$ is finitely presented: there exists an admissible cover $X = \bigcup U_\alpha$ and short exact sequences*

$$\mathcal{O}_{U_\alpha}^n \to \mathcal{O}_{U_\alpha}^m \to \mathcal{F}|_{U_\alpha} \to 0.$$

Proof. See [5, Corollary 6.1/5]. □

Definition 8.2.3. A sheaf of $\mathcal{O}_X$-modules on a rigid-analytic space X is *coherent* if the equivalent conditions of Proposition 8.2.2 hold.

Proposition 8.2.4. *TODO: Standard properties of the category of coherent sheaves, computing cohomology on an affinoid covering.*

Example 8.2.5. TODO: Line bundles and Pic

Example 8.2.6. TODO: Differentials

Remark 8.2.7. TODO: Explain why quasi-coherent sheaves do not work.

8.3 Examples

Example 8.3.1. Let X be a rigid-analytic space and let $U \subseteq X$ be an admissible open subset. Then $(U, \mathcal{O}_X|_U)$ is a rigid analytic space. Indeed, the collection of all affinoid subdomains of X contained in U forms an admissible cover of U.

For example, the *open unit disc* $D^\circ = \{|X| < 1\} \subseteq D = \mathrm{Sp}\, K\langle X\rangle$ is a rigid-analytic space with

$$D^\circ = \bigcup_{n\geq 1} \{|X| \leq |t|^{1/n}\}$$

as one possible affinoid cover. We have

$$\Gamma(D^\circ, \mathcal{O}_{D^\circ}) = \varprojlim_n K\left\langle X, \frac{X^n}{t}\right\rangle = \left\{f = \sum a_n X^n \in K[[X]] \,:\, |a_n|\rho^n \to 0 \text{ for every } \rho < 1\right\}.$$

This is not an affinoid K-algebra, for example the element X does not satisfy the maximum principle.

Example 8.3.2. The functors represented by the closed disc D and the open disc D° have simple descriptions. Note that if X is a rigid-analytic space and $x \in X$, then for $f \in \mathcal{O}_{X,x}$ the expression $|f(x)|$ makes sense. We define $\mathcal{O}_X^\circ \subseteq \mathcal{O}_X$ to be the subsheaf of elements such that $|f(x)| \leq 1$ at all points, and $\mathcal{O}_X^{\circ\circ} \subseteq \mathcal{O}_X^\circ$ to be defined by $|f(x)| < 1$. Then we have functorial isomorphisms

$$\mathrm{Hom}(X, D) \simeq \Gamma(X, \mathcal{O}_X^\circ) \quad \text{and} \quad \mathrm{Hom}(X, D^\circ) \simeq \Gamma 9X, \mathcal{O}_X^{\circ\circ}).$$

Example 8.3.3 (Affine line)**.** To go beyond admissible opens inside affinoids, let us define the rigid affine line $\mathbf{A}_K^{1,\mathrm{an}}$ as the union of discs of larger and larger radii. Formally, fix a pseudouniformizer $t \in K$ and define $D_n = \mathrm{Sp}\, K\langle X_n\rangle$ $(n \geq 1)$ and maps

$$j_n : D_n \to D_{n+1}, \quad j_n^* X_{n+1} = t X_n.$$

The map j_n identifies D_n with the affinoid subdomain

$$D_{n+1}(X_{n+1}/t) = \mathrm{Sp}\, K\left\langle X_{n+1}, \frac{X_{n+1}}{t}\right\rangle \subseteq D_{n+1},$$

and hence if we regard D_1 as the unit disc with coordinate X, we can regard D_n as the disc with radius $|t|^{-n}$. We define $\mathbf{A}_K^{1,\mathrm{an}}$ as the increasing union $\bigcup_{n\geq 1} D_n$ along the open immersions j_n (or taking the inductive limit in locally ringed G-topological spaces): an subset $U \subseteq \mathbf{A}_K^{1,\mathrm{an}}$ is an admissible open if and only if $U \cap D_n$ is an admissible open for every n, and similarly for admissible covers.

The functions $x_n = t^n X_n$ satisfy $j_n^* x_{n+1} = x_n$, and hence define a global section $X \in \Gamma(\mathbf{A}^{1,\mathrm{an}}, \mathcal{O}_{\mathbf{A}^{1,\mathrm{an}}})$ whose restriction to the unit disc $D_1 = D$ is the coordinate $X_1 = X$. Lemma 8.3.4 below states that $\mathbf{A}^{1,\mathrm{an}}$ together with the function X represent the functor $Y \mapsto \Gamma(Y, \mathcal{O}_Y)$. In particular, the result of the ad hoc construction using a particular sequence of discs does not depend on the choice of the pseudouniformizer t.

Lemma 8.3.4. *The association $\varphi \mapsto \varphi^* X$ defines an isomorphism of functors* $\mathrm{Rig}_K \to \mathrm{Sets}$*:*

$$\mathrm{Hom}(Y, \mathbf{A}^{1,\mathrm{an}}) \xrightarrow{\sim} \Gamma(Y, \mathscr{O}_Y).$$

Proof. For a fixed $Y \in \mathrm{Rig}_K$, both $U \mapsto \mathrm{Hom}(U, \mathbf{A}^{1,\mathrm{an}})$ and $U \mapsto \Gamma(U, \mathscr{O}_U)$ are sheaves for the admissible topology on Y. It therefore suffices to prove the assertion for $Y = \mathrm{Sp} A$ an affinoid space.

To show surjectivity, let $f \in A = \Gamma(Y, \mathscr{O}_Y)$, and let $n \geq 1$ be such that $|f|_{\sup} \leq |t|^{-n}$. Then $t^n f \in A^\circ$ defines a map

$$\varphi : Y \to D_n \hookrightarrow \mathbf{A}^{1,\mathrm{an}}$$

with $\varphi^* X_n = t^n f$, and hence $\varphi^* X = f$.

To show injectivity, suppose that $\varphi, \psi : Y \to \mathbf{A}^{1,\mathrm{an}}$ satisfy $\varphi^* X = \psi^* X$. □

Example 8.3.5 (Affine space)**.** Similarly, the r-fold product $\mathbf{A}^{r,\mathrm{an}} = (\mathbf{A}^{1,\mathrm{an}}_K)^r = \bigcup_n D_n^r$, called the *rigid affine r-space over K*, represents the functor

$$Y \mapsto \Gamma(Y, \mathscr{O}_Y)^r.$$

8.4 The analytification functor

Let us start by recalling the complex analytification functor. To a scheme locally of finite type X over $\mathbf{C}$ one can functorially attach a complex analytic space X^{an}. Its underlying set is the set $X(\mathbf{C})$ of $\mathbf{C}$-points (equivalently, closed points) of X, but it is equipped with the "analytic" topology deduced from the Archimedean metric on $\mathbf{C}$. Being a complex analytic space, it is endowed with a sheaf $\mathscr{O}_{X^{\mathrm{an}}}$ of holomorphic functions. In case $X \subseteq \mathbf{A}^n$ is affine, $X^{\mathrm{an}} = X(\mathbf{C}) \subseteq \mathbf{A}^n(\mathbf{C}) = \mathbf{C}^n$ with the induced topology from the metric topology on $\mathbf{C}^n$; if $\mathscr{O}_{\mathbf{C}^n}$ is the sheaf of holomorphic functions on $\mathbf{C}^n$, then $\mathscr{O}_{X^{\mathrm{an}}}$ is the restriction to X^{an} of the sheaf of rings $I_X \cdot \mathscr{O}_{\mathbf{C}^n}$, where $I_X \subseteq \mathbf{C}[x_1, \ldots, x_n]$ is the ideal of X. In general, to construct X^{an} one takes an affine open cover $X = \bigcup U_i$ and glues together the complex analytic spaces U_i^{an} constructed previously.

More intrinsically, X^{an} comes with a morphism of locally ringed spaces $\varepsilon : X^{\mathrm{an}} \to X$ which is final among all maps from complex analytic spaces to X: if Y is a complex analytic space and $f : Y \to X$ is a $\mathbf{C}$-linear map of locally ringed spaces, then $f = \varepsilon \circ \overline{f}$ for a unique map of complex analytic spaces $Y \to X^{\mathrm{an}}$. Then $X \mapsto X^{\mathrm{an}}$ is obviously a functor, and familiar notions are transfered through it: X is connected/reduced/smooth/separated/proper etc. if and only if X^{an} is connected/reduced/smooth/Hausdorff/compact. If $\mathscr{F}$ is a coherent sheaf (resp. a locally free sheaf) on X, then $\mathscr{F}^{\mathrm{an}} := \varepsilon^* \mathscr{F}$ is a coherent analytic sheaf (resp. a vector bundle) on X^{an}. Further, one has $(\Omega^1_X)^{\mathrm{an}} \simeq \Omega^1_{X^{\mathrm{an}}}$, i.e. the analytification of Kähler differentials is identified with holomorphic differentials.

If X is proper, then one can say a bit more. Serre's GAGA theorem states that the analytification functor on coherent sheaves

$$\mathscr{F} \mapsto \mathscr{F}^{\mathrm{an}} : \mathrm{Coh}\, X \to \mathrm{Coh}\, X^{\mathrm{an}}$$

is an equivalence. Further, the induced maps on sheaf cohomology

$$\varepsilon^* : H^q(X, \mathscr{F}) \to H^q(X^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}})$$

are isomorphisms for all $\mathscr{F} \in \mathrm{Coh}\, X$ and all $q \geq 0$. If Y is another proper scheme over $\mathbf{C}$, then the analytification map

$$f \mapsto f^{\mathrm{an}} : \mathrm{Hom}_{\mathbf{C}}(Y, X) \to \mathrm{Hom}(Y^{\mathrm{an}}, X^{\mathrm{an}})$$

is a bijection. (One can deduce this from the equivalence for coherent sheaves by looking at the structure sheaf of the graph of a map $Y^{\mathrm{an}} \to X^{\mathrm{an}}$, treated as a coherent sheaf on $(X \times Y)^{\mathrm{an}}$.)

The goal of this section is to describe an entirely analogous program over a non-Archimedean field K. We will come back to the GAGA theorems after we define proper morphisms of rigid-analytic spaces in §8.7 below. To simplify the notation a little bit, let us denote by $\mathscr{S}$ (S for spaces) the category of locally ringed G-topological spaces over the one-point space $\operatorname{Sp} K = \operatorname{Spec} K$ (in other words, the structure sheaves of rings are sheaves of K-algebras, and homomorphisms are K-algebra homomorphisms). By definition, the category Rig_K of rigid-analytic spaces over K is a full subcategory of $\mathscr{S}$. Since every topological space can be regarded as a G-topological space, the category Sch_K of schemes locally of finite type over K is a full subcategory of $\mathscr{S}$ as well.

Definition 8.4.1. The *analytification* of a scheme $X \in \mathrm{Sch}_K$ is a rigid-analytic space X^{an} which represents the functor

$$Y \mapsto \operatorname{Hom}_{\mathscr{S}}(Y, X)\colon \mathrm{Rig}_K^{\mathrm{op}} \to \mathrm{Sets}.$$

In other words, $X^{\mathrm{an}} \in \mathrm{Rig}_K$ is endowed with a morphism $\varepsilon\colon X^{\mathrm{an}} \to X$ in the category $\mathscr{S}$ such that every morphism $Y \to X$ in the category $\mathscr{S}$ with $Y \in \mathrm{Rig}_K$ factors uniquely through ε.

Lemma 8.4.2. *If $X \in \mathrm{Sch}_K$ is affine, then the analytification X^{an} exists.*

Proof. Write $X = \operatorname{Spec} R$, $R = K[X_1, \ldots, X_r]/(f_1, \ldots, f_s)$. Let $\mathbf{A} = \mathbf{A}_K^{r,\mathrm{an}} = (\mathbf{A}_K^{1,\mathrm{an}})^r$ be the r-dimensional rigid-analytic affine space over K, as defined in Example 8.3.5. If $X_1, \ldots, X_r \in \Gamma(\mathbf{A}, \mathscr{O}_{\mathbf{A}})$ are its coordinates, then by Lemma 8.3.4, giving a morphism $\varphi\colon Y \to \mathbf{A}$ is equivalent to specifying $\varphi^* X_1, \ldots, \varphi^* X_r \in \Gamma(Y, \mathscr{O}_Y)$. The ideal generators $f_1, \ldots, f_s$, being polynomials in the X_i, can be treated as elements of $\Gamma(\mathbf{A}, \mathscr{O}_{\mathbf{A}})$ as well. We set $X^{\mathrm{an}} \subseteq \mathbf{A}$ to be the closed rigid-analytic subspace cut out by $f_1, \ldots, f_r$. More precisely, if $\mathbf{A} = \bigcup D_n^r$ where D_n is the disc of radius $|t|^{-n}$ as in Example 8.3.3, then $X^{\mathrm{an}} = \bigcup X_n^{\mathrm{an}}$ where $X_n^{\mathrm{an}} = \operatorname{Sp} \mathscr{O}(D_n)/(f_1, \ldots, f_s)$. Then a morphism $Y \to \mathbf{A}$ in Rig_K factors (uniquely) through the closed immersion $X^{\mathrm{an}} \to \mathbf{A}$ if and only if $\varphi^*(f_i) = 0$ for all i. In other words, X^{an} represents the functor

$$Y \mapsto \operatorname{Hom}_K(R, \Gamma(Y, \mathscr{O}_Y))\colon \mathrm{Rig}_K^{\mathrm{op}} \to \mathrm{Sets}.$$

It remains to show that $\operatorname{Hom}_K(R, \Gamma(Y, \mathscr{O}_Y)) \simeq \operatorname{Hom}_{\mathscr{S}}(Y, X)$. This is shown analogously to Proposition 8.1.2(c). (TODO) □

Lemma 8.4.3. *Let $X \in \mathrm{Sch}_K$ be an affine scheme and let $U \subseteq X$ be an open subscheme. Let $\varepsilon_X\colon X^{\mathrm{an}} \to X$ be the analytification of X, which exists by the previous lemma. Then $\varepsilon_X^{-1}(U) \subseteq X$ is an admissible open subset and $\varepsilon\colon \varepsilon_X^{-1}(U) \to U$ is an analytification of U.*

Proposition 8.4.4. *Every $X \in \mathrm{Sch}_K$ admits an analytification.*

Proof. Let $X = \bigcup X_i$ be a Zariski open cover by affines. Then the X_i^{an} exist by Lemma 8.4.2. By Lemma 8.4.3, so do the $(X_i \cap X_j)^{\mathrm{an}}$, and $X_i \subseteq X_j \hookrightarrow X_i$ induces an open immersion $(X_i \cap X_j)^{\mathrm{an}} \hookrightarrow X_i$. By Remark 8.1.4, we can construct a rigid-analytic space X^{an} by gluing the X_i along the identifications

$$X_i^{\mathrm{an}} \supseteq (X_i \cap X_j)^{\mathrm{an}} \subseteq X_j^{\mathrm{an}}.$$

Thus, giving a morphism $Y \to X^{\mathrm{an}}$ is equivalent to giving an admissible open cover $Y = \bigcup Y_i$ and maps $f_i\colon Y_i \to X_i^{\mathrm{an}}$ such that ... . It follows that $X^{\mathrm{an}} \to X$ is an analytification of X. □

Example 8.4.5 (Rigid projective space)**.** Let $\mathbf{P}_K^{n,\mathrm{an}}$ be the analytification of $\mathbf{P}_K^n$. By construction, $\mathbf{P}^{n,\mathrm{an}}$ admits an admissible cover by $n+1$ copies $U_0, \cdots, U_n$ of the rigid-analytic affine n-space $\mathbf{A}_K^{n,\mathrm{an}}$:

$$U_i = \{x_i \neq 0\} = \left(\operatorname{Spec} K[X_{ij} \mid j \neq i]\right)^{\mathrm{an}}$$

where $X_{ij} = x_j/x_i$ in homogeneous coordinates. A more economic choice is the finite affinoid cover by the polydiscs

$$V_i = \{|x_j| \leq |x_i|, j \neq i\} = \operatorname{Sp} K\langle X_{ij} \mid j \neq i\rangle \subseteq U_i.$$

It is straightforward to check that the functor of points of $\mathbf{P}_K^{n,\mathrm{an}}$ is as in the scheme case: maps $Y \to \mathbf{P}_K^{n,\mathrm{an}}$ in Rig_K correspond to isomorphism classes of $\mathscr{O}_Y$-module surjections $\mathscr{O}_Y^{n+1} \to L$ where L is an invertible sheaf on Y, up to isomorphism of objects under $\mathscr{O}_Y^{n+1}$.

8.5 Properties of morphisms of rigid-analytic spaces

Definition 8.5.1. Let $f: Y \to X$ be a morphism in Rig_K.

(a) The morphism f is *finite* if for every map from an affinoid space $\operatorname{Sp} A \to X$, the base change $Y \times_X \operatorname{Sp} A$ is isomorphic over $\operatorname{Sp} A$ to some $\operatorname{Sp} B$ where $A \to B$ is finite.

(b) The morphism f is a *closed immersion* if for every map from an affinoid space $\operatorname{Sp} A \to X$, the base change $Y \times_X \operatorname{Sp} A$ is isomorphic over $\operatorname{Sp} A$ to some $\operatorname{Sp} B$ where $A \to B$ is surjective.

(c) The morphism f is an *open immersion* if f induces an isomorphism between Y an an admissible open subspace of X.

(d) (Locally closed immersion?)

Remark 8.5.2. 1. A morphism $f: Y \to X$ is finite (resp. a closed immersion) if and only if there exists an admissible affinoid cover $X = \bigcup \operatorname{Sp} A_i$ such that for every i, the preimage $Y_i = f^{-1}(\operatorname{Sp} A_i)$ is affinoid and the map $A_i \to B_i = \mathscr{O}_{Y_i}(Y_i)$ is finite (resp. surjective).

2. Closed immersions are finite morphisms.

3. The category of finite morphisms $Y \to X$ with X fixed is equivalent to the opposite category of the category of coherent $\mathscr{O}_X$-algebras. Similarly, Closed immersions $Y \to X$ correspond bijectively to coherent sheaves of ideals $I \subseteq \mathscr{O}_X$.

4. Warning: there exist morphisms which are bijective on points and isomorphisms locally on the source but which are not open immersions (see Problem 2 on Problem Set 6).

Definition 8.5.3. Let $f: Y \to X$ be a morphism in Rig_K.

(a) The morphism f is *quasi-compact* if the preimage of every quasi-compact (see Definition REF) open subset of X is quasi-compact.

(b) The morphism f is *quasi-separated* (resp. *separated*) if the diagonal morphism $\Delta: Y \to Y \times_X Y$ is quasi-compact (resp. *a closed immersion*).

(c) In case $X = \operatorname{Sp} X$, we simply say that Y (rather than f) is quasi-separated or separated.

Remark 8.5.4. 1. Every map between affinoids is quasi-compact and separated. A morphism $f: Y \to X$ is quasi-compact if and only if the preimage of an affinoid open is a finite union of affinoid opens. A rigid space X is quasi-separated if and only if the intersection of two affinoid opens is a finite union of affinoid opens.

2. Finite morphisms are quasi-compact and separated. Consequently, separated maps are quasi-separated.

3. If X is separated then the intersection of two affinoid opens $U, V \subseteq X$ is an affinoid open. Indeed, the diagonal $U \cap V \to U \times V$ is a closed immersion and $U \times V$ is affinoid, forcing $U \cap V$ to be affinoid as well.

4. For an example of non-separated or non-quasi-separated rigid spaces, see Exercise 4 on Problem Set 6.

Definition 8.5.5. Let $f: Y \to X$ be a morphism in Rig_K.

(a) The morphism f is *flat* if the induced homomorphisms of local rings $f^*: \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ are flat ring homomorphisms.

(b) The morphism f is *unramified* if the induced homomorphisms $f^*: \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ are unramified, i.e. $\mathcal{O}_{Y,y}/\mathfrak{m}_{f(y)} \cdot \mathcal{O}_{Y,y}$ is a finite separable field extension of $\mathcal{O}_{X,f(y)}$.

(c) The morphism f is *étale* if it is flat and unramified.

(d) The morphism f is *smooth* if it locally factors as $Y \to X \times D^r \to X$ for some $r \geq 0$, where $D = \mathrm{Sp}\, K\langle X \rangle$ is the unit disc and where $Y \to X \times D^r$ is étale.

Remark 8.5.6. TODO: Standard facts, e.g. smooth = flat + $\Omega^1_{Y/X}$ locally free.

8.6 *Change of base field*

Let $K \subseteq K'$ be an extension of non-Archimedean fields: that is, the norm on K' restricts to the norm on K. Of particular interest are finite extensions, or K' being a completed algebraic closure of K (see Theorem 2.5.4), or a spherical completion of such (see []). We shall define a base change functor

$$(-)_{K'}: \mathrm{Rig}_K \to \mathrm{Rig}_{K'}.$$

(TODO, to be completed after HW 7 submission deadline.)

8.7 *Proper morphisms*

As in scheme theory or complex geometry, one has a notion of proper morphisms between rigid-analytic spaces. However, their definition looks quite different from its counterparts, and some proofs of "standard facts"are significantly more difficult. For example, the fact that the composition of two proper morphisms is proper is only proved by leaving the category of rigid-analytic spaces and working with either Berkovich spaces, adic spaces, or formal schemes. In fact, in all of the three named categories one has corresponding notions of proper morphisms which resemble more closely the standard scheme-theoretic notion.

Definition 8.7.1. Let V be an affinoid subdomain of an affinoid space U. We say that V is *relatively compact* in U (or *compactly contained in* U), and write $V \Subset U$, if there exist affinoid generators $x_1, \dots, x_r \in \mathcal{O}(U)$ whose restrictions to V are topologically nilpotent (equivalently, $|j^*(x_i)|_{\sup} < 1$ where $j: V \hookrightarrow U$).

More generally, suppose that $U \to Y$ is a map of affinoid spaces and that $V \subseteq U$ is an affinoid subdomain. We say that V is *relatively compact in U over Y*, and write $V \Subset_Y U$, if there exist affinoid generators $x_1, \dots, x_r \in \mathcal{O}(U)$ over $\mathcal{O}(Y)$, in the sense that we have a surjection

$$\mathcal{O}(Y)\langle X_1, \dots, X_r \rangle \to \mathcal{O}(U), \quad X_i \mapsto x_i,$$

whose restrictions to V are topologically nilpotent.

Proposition 8.7.2. *Suppose that* $\operatorname{Sp} B \subseteq_{\operatorname{Sp} R} \operatorname{Sp} A$ *and let* $f \in A^\circ$. *Then there exists a monic polynomial* $p = X^n + a_1 X^{n-1} + \cdots + a_0 \in R^\circ[X]$ *such that* $j^*(p(f)) \in B^{\circ\circ}$ *where* $j\colon \operatorname{Sp} B \hookrightarrow \operatorname{Sp} A$.

Proof. Let $x_1, \ldots, x_r \in A^\circ$ be affinoid generators of A over R such that $j^*(x_i) \in B^{\circ\circ}$. Since f is powerbounded, it is integral over $R^\circ\langle X_1, \ldots, X_r\rangle$ (proof of Proposition 4.2.1(e)), so there exists a monic polynomial

$$q = X^n + b_1 X^{n-1} + \cdots + b_n \in R^\circ\langle X_1, \ldots, X_r\rangle[X]$$

with $q(f) = 0$. Let $a_i \in R^\circ$ be the constant term of b_i, so that every $c_i = b_i - a_i$ is a power series in the variables X_i with zero constant term and coefficients in B°. It follows that $j^*(c_i) \in B^{\circ\circ}$. Set $p = X^n + a_1 X^{n-1} + \cdots + a_n \in R^\circ[X]$. Then

$$j^*(p(f)) = j^*(q(f) - \sum c_i f^{n-i}) = j^*(-\sum c_i f^{n-i}) = -\sum j^*(c_i) j^*(f)^{n-i} \in B^{\circ\circ}. \qquad \square$$

Definition 8.7.3. A morphism $f\colon X \to S$ of rigid-analytic spaces is *proper* if it is separated and there exists an affinoid admissible cover $S = \bigcup_{i\in I} S_i$ and for every i, two affinoid admissible covers $\{U_{ij}\}_{j\in J_i}$, $\{V_{ij}\}_{j\in J_i}$ of $X_i = f^{-1}(S_i)$ indexed by the same finite set J_i such that

$$V_{ij} \Subset_{S_i} U_{ij} \quad \text{for all } i \in I,\ j \in J_i.$$

Proposition 8.7.4. *For* $f\colon Y \to X$ *and* $g\colon Z \to Y$ *in* Rig_K, *if* $fg\colon Z \to X$ *is proper and* f *is separated, then* g *is proper. If* f *and* g *is proper, so is* fg.

Proof. First assertion: TODO. The second assertion is much more difficult, see Temkin []. $\square$

Example 8.7.5. The rigid projective space $\mathbf{P}^{n,\mathrm{an}}$ is proper (over $\operatorname{Sp} K$). Indeed, we have the standard affinoid covering (Example 8.4.5) by

$$V_i = \{|x_j| \le |x_i|\} = \operatorname{Sp} K\langle X_{ij}, j \neq i\rangle, \quad X_{ij} = x_j/x_i.$$

At the same time, we have the slightly bigger cover by

$$U_i = \{|t| \cdot |x_j| \le |x_i|\} = \operatorname{Sp} K\langle t X_{ij}, j \neq i\rangle,$$

and clearly $V_i \Subset U_i$.

Example 8.7.6. Finite morphisms are proper (Exercise 1 on Problem Set 7).

Example 8.7.7. If $X \to S$ is a proper morphism of schemes locally of finite type over K, then $X^{\mathrm{an}} \to S^{\mathrm{an}}$ is a proper morphism of rigid-analytic spaces. (See Exercise 2 on Problem Set 7).

Theorem 8.7.8 (Kiehl's finiteness theorem). *Let* $f\colon X \to S$ *be a proper morphism of rigid-analytic spaces. Then for every coherent sheaf* $\mathscr{F}$ *on* X, *the higher direct images*

$$R^i f_* \mathscr{F}$$

are coherent sheaves on S *for all* $i \ge 0$.

To get better acquainted with the notion of properness in rigid geometry, let us show the following easy case of Kiehl's finiteness theorem.

Proposition 8.7.9 (Very special case of Kiehl's theorem)**.** *Suppose that K is algebraically closed and let X be a non-empty connected, reduced and proper rigid-analytic space over K. Then $\mathcal{O}_X(X) = K$.*

Proof. Suppose that there exists a non-constant $f \in \mathcal{O}_X(X)$. Take $x \in X$ and replace f with $f - f(x)$, so that $f(x) = 0$. Next, since X is quasi-compact, by the Maximum Principle there exists a $y \in X$ so that $|f|_{\sup} = |f(y)| \neq 0$; replacing f with $f/f(y)$ we get an f with $f(x) = 0$, $f(y) = 1$ and $|f|_{\sup} = 1$.

Let $\{U_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ be finite affinoid coverings of X such that $V_j \Subset_S U_j$. By Proposition 8.7.2, for every i there exists a monic polynomial $p_i \in \mathcal{O}[T]$ such that $|p_i(f|_{V_i})|_{\sup} < 1$. Take $p = \prod_{i\in I} p_i$; since the V_i cover X, we get $|p(f)|_{\sup} < 1$.

Since K is algebraically closed, we may write $p = \prod(T - \alpha_j)$ for some $\alpha_j \in \mathcal{O}$. Then X is covered by the sets $W_j = \{|f - \alpha_j| < 1\}$, and in fact this is an admissible cover. Since $W_j \cap W_{j'} = \emptyset$ if $|\alpha_j - \alpha_{j'}| = 1$ and $W_j = W_{j'}$ otherwise, the fact that X is connected implies that there is an α such that $|f - \alpha|_{\sup} < 1$, contradicting the fact that f takes values 0 and 1. □

Corollary 8.7.10. *An affinoid* $\mathrm{Sp}A$ *is proper over K if and only if A is finite-dimensional over K.*

Proof. If A is finite-dimensional over K, then $\mathrm{Sp}A \to \mathrm{Sp}K$ is finite and hence proper (Example 8.7.6). It remains to show the converse.

Assume first that K is algebraically closed. Suppose that $\mathrm{Sp}A$ is proper, and let $A_{\mathrm{red}} = A/\sqrt{(0)}$ be its reduction. Then $\mathrm{Sp}A_{\mathrm{red}}$ is proper as well. Write $A_{\mathrm{red}} = \prod A_i$ where A_i have no nontrivial idempotents. Then $\mathrm{Sp}A_i$ are proper, connected, and reduced, and hence by Proposition 8.7.9 we have $A_i = \mathcal{O}(\mathrm{Sp}A_i) = K$. It follows that A is Artinian and hence finite-dimensional over K.

For general K, one needs to work harder (TODO). □

We record one important corollary of Kiehl's theorem.

Corollary 8.7.11 (Stein factorization)**.** *Let $f : Y \to X \in \mathrm{Rig}_K$ be a proper morphism. Then f factors uniquely as $f = gh$ where $g : X' \to X$ is finite and $h : Y \to X'$ has connected fibers. In particular, every quasi-finite (i.e. with finite fibers) proper map is finite.*

Proof. TODO □

Theorem 8.7.12 (Rigid-analytic GAGA)**.** *Let X be a proper scheme over K. Then X^{an} is a proper rigid-analytic space, and the analytification functor*

$$\mathscr{F} \mapsto \mathscr{F}^{\mathrm{an}} \colon \mathrm{Coh}\,X \to \mathrm{Coh}\,X^{\mathrm{an}}$$

is an equivalence of categories. For $\mathscr{F} \in \mathrm{Coh}\,X$, the maps induced on cohomology

$$H^i(X, \mathscr{F}) \to H^i(X^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}})$$

are isomorphisms for all $i \geq 0$. If Y is another proper scheme over K, then the map

$$f \mapsto f^{\mathrm{an}} \colon \mathrm{Hom}(Y, X) \to \mathrm{Hom}(Y^{\mathrm{an}}, X^{\mathrm{an}})$$

is bijective.

Finally, let us discuss more examples of proper rigid-analytic spaces, to show that the category is much bigger than proper schemes.

Example 8.7.13 (Non-Archimedean Hopf surface)**.**

Example 8.7.14. Generic fibers of formal deformations of abelian varieties and K3 surfaces.

9

Tate uniformization of elliptic curves

In this chapter we discuss the first serious application of the theory of rigid-analytic spaces, in fact the very motivation for their introduction by Tate, namely the uniformization of elliptic curves with split multiplicative reduction over a non-Archimedean field K.

In this context, the term *uniformization* refers to the possibility of expressing a family of geometric objects Y in a uniform way as quotient spaces X/Γ where neither the space X nor the group Γ depend on Y, only the action of Γ on X does.

For example, the classical *uniformization theorem* of complex geometry states that every simply connected Riemann surface X is isomorphic to $\mathbf{P}^1$, $\mathbf{C}$, or the unit disc Δ. Therefore a compact Riemann surface Y of genus g can be expressed as X/Γ_g where X is the universal covering of Y, which must belong to one of the three types, and $\Gamma_g = \pi_1(Y)$ is the surface group of genus g; we have $X = \mathbf{C}$ for $g = 1$ and $X = \Delta$ for $g \geq 2$. Both $\mathbf{C}$ and Δ admit natural metrics with constant curvature (Euclidean on $\mathbf{C}$ and hyperbolic on Δ), and the group Γ_g acts via isometries.

In the genus one case, one has $\Gamma_1 \simeq \mathbf{Z} \times \mathbf{Z}$, acting on $X = \mathbf{C}$ by translations by a lattice $\Lambda \subseteq \mathbf{C}$ depending on Y, and we can parametrize all genus one Riemann surfaces by lattices in $\mathbf{C}$ up to homothety. A somewhat more economic expression is obtained by writing $\Lambda \simeq \mathbf{Z} \oplus \mathbf{Z}\tau$ for some $\tau \in \mathbf{C} \setminus \mathbf{R}$ and noting that the quotient of $\mathbf{C}$ by the subgroup $\mathbf{Z} \subseteq \Lambda$ is simply $\mathbf{C}^\times$ (via the map $z \mapsto e^{2\pi i z}$). Thus, setting $q = e^{2\pi i \tau}$, we may write $Y \simeq \mathbf{C}^\times/q^{\mathbf{Z}}$. We shall review both of these uniformizations in §9.2–9.6 below.

Tate's idea was that while in the non-Archimedean case quotients of the type K/Λ (or rather $\mathbf{A}_K^{1,\mathrm{an}}/\Lambda$) are rather useless, the quotient $Y = \mathbf{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}$ (with parameter $0 < |q| < 1$) is indeed an analog of a genus one Riemann surface over K. It is the analytification of an elliptic curve E over K with split multiplicative reduction, and conversely every such curve arises in this way for a unique value of q.

Tate's uniformization has important applications to number theory. Its advantage over its Archimedean counterpart is that it keeps track of the arithmetic through the action of the Galois group. More precisely, if E is an elliptic curve over a number field K, and $\mathfrak{p}$ is a prime of K where E has multiplicative reduction, then the base change of E to a finite extension L of the local field at $\mathfrak{p}$ admits a Tate uniformization. Consequently, we have a $\mathrm{Gal}(\overline{L}/L)$-equivariant isomorphism

$$E(\overline{L}) \simeq \overline{L}^\times/q^{\mathbf{Z}}$$

for some $q \in L$ with $0 < |q| < 1/$ We shall discuss some simple applications of this type in §9.8.

9.1 *Elliptic curves over k as cubic curves*

Definition 9.1.1. An *elliptic curve* over a field k is a smooth and proper geometrically connected group scheme E over k of dimension one.

Replacing k by a ring R or a scheme S, we obtain the definition of an *elliptic scheme*.

Here "geometrically connected" means that $E_{\bar{k}}$ is connected. The group structure is automatically commutative, and therefore we write it additively and denote its neutral element by $0 \in E(k)$. Properness of E implies that every global differential form is invariant under translation; in particular, $\dim H^0(E, \omega_E) = 1$, so E has genus one. Every k-scheme map of $E \to E'$ between elliptic curves which sends 0 to 0 is automatically a group homomorphism, and every curve (smooth and proper geometrically connected scheme of dimension one over k) E of genus one with a chosen point $0 \in E(k)$ admits a unique group structure of an elliptic curve over k.

Let us recall how elliptic curves are classically understood by embedding them in $\mathbf{P}^2_k$. Let E be an elliptic curve over k and let L be the line bundle associated to the divisor 0. Thus L^n is associated to $n \cdot 0$ and $\Gamma(E, L^n)$ is the space of meromorphic functions on E with pole of order $\leq n$ at 0 and no other poles. The Riemann-Roch theorem [10, IV 1.3] asserts that we have

$$\dim \Gamma(E, L^n) = n \quad \text{for } n > 0,$$

and a corollary to that theorem [10, IV 3.1] says that L is ample, in fact, L^3 is very ample. Thus

$$E \simeq \operatorname{Proj} \bigoplus_{n \geq 0} \Gamma(E, L^n).$$

To find a presentation of the above graded ring, let $x \in \Gamma(E, L^2)$ be a function such that $\{1, x\}$ is a basis (that is, x has a double pole at zero and no other poles). Further, let $y \in \Gamma(E, L^3)$ be such that $\{1, x, y\}$ is a basis. Then the seven elements

$$1, x, x^2, x^3, y, xy, y^2 \in \Gamma(E, L^6)$$

lie in a six-dimensional space and hence satisfy an equation of the form

$$a + bx + cx^2 + dx^3 + ey + fxy + gy^2$$

and we must have $d, g \neq 0$ since y^2 and x^3 are the only functions with a pole of order six. We obtain a homomorphism of graded rings

$$\theta \colon k[t, x, y]/(at^6 + bxt^4 + cx^2t^2 + dx^3 + eyt^3 + fxyt + gy^2) \to \bigoplus_{n \geq 0} \Gamma(E, L^n)$$

where $\deg t = 1$, $\deg x = 2$, $\deg y = 3$, so that the equation is homogeneous of degree six. It is then easily seen that θ is an isomorphism. Further, L^3 is very ample and by passing to the subrings with degree divisible by three we obtain an isomorphism

$$\operatorname{Proj} k[X, Y, Z]/(aZ^3 + bXZ^2 + cX^2Z + dX^3 + eYZ + fXYZ + gY^2) \xrightarrow{\sim} \bigoplus_{n \geq 0} \Gamma(E, L^{3n})$$

where $\deg X = \deg Y = \deg Z = 1$ and the map sends $X \mapsto x$, $Y \mapsto y$, $z \mapsto 1$. In other words, we have

$$E \quad \simeq \quad \{aZ^3 + bXZ^2 + cX^2Z + dX^3 + eYZ + fXYZ + gY^2Z = 0\} \quad \subseteq \quad \mathbf{P}^2_k.$$

If $\operatorname{char} k \neq 2$, then doing some simple change of variables (see [10, IV 4.6]) one can put the equation in the Weierstrass form

$$Y^2Z = X^3 + AXZ^2 + BZ^3.$$

Conversely, every cubic equation as above defines an elliptic curve (i.e., the resulting plane curve is smooth) if and only if its *discriminant*

$$\Delta = -16(4A^3 + 27B^2)$$

is nonzero. One can wonder when two pairs (A,B), (A',B') with discriminants Δ, Δ' define isomorphic elliptic curves; if k is algebraically closed, this happens precisely when

$$\frac{A^3}{\Delta} = \frac{(A')^3}{\Delta'}.$$

The quantity $j(E) = -1728(4A)^3/\Delta$, called the *j-invariant*, does not depend on the choice of the equation, and so we have $E_{\bar{k}} \simeq E'_{\bar{k}}$ if and only if $j(E) = j(E')$.

9.2 *Elliptic curves over* $\mathbf{C}$ *as quotients* $\mathbf{C}/\Lambda$

Let $Y = E^{\mathrm{an}}$ be the complex analytification of an elliptic curve over $\mathbf{C}$. Let $\pi\colon X \to Y$ be its universal covering space (which is again a Riemann surface, though no longer compact). We know that Y is an orientable surface of genus 1, and hence by classification of orientable surfaces it is homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^1$. Let $\Lambda = \pi_1(Y,0) = H_1(Y,\mathbf{Z})$, which is a group isomorphic to $\mathbf{Z} \times \mathbf{Z}$. If $0 \in X$ is a basepoint lying over $0 \in Y$, then the group structure on Y lifts uniquely to a topological group structure on X with 0 as the neutral element for which $\pi\colon X \to Y$ is a group homomorphism. This makes X into a commutative complex Lie group of dimension one, and the kernel of π is canonically identified with the fundamental group Λ. The tangent space T_0X is identified with the tangent space $V = T_0Y$ (which can be further identified with $H^1(Y,\mathcal{O}_Y)$, or the dual of $H^0(Y,\Omega^1_Y)$). The exponential map

$$\exp\colon V \to X$$

(where V is regarded as a complex manifold non-canonically isomorphic to $\mathbf{C}$) is an isomorphism. We obtain a presentation

$$Y \simeq V/\Lambda \simeq H^0(Y,\Omega^1_Y)^\vee / H_1(Y,\mathbf{Z}).$$

It can be shown that the embedding $H_1(Y,\mathbf{Z}) \hookrightarrow H^0(Y,\Omega^1_Y)^\vee$ constructed above is induced by the integral:

$$(\gamma,\omega) \quad \mapsto \quad \frac{1}{2\pi i}\int_\gamma \omega.$$

Less canonically, we conclude that the analytification of every elliptic curve is isomorphic to the quotient $\mathbf{C}/\Lambda$ for some lattice $\Lambda \subseteq \mathbf{C}$, where by a lattice we mean a subgroup generated by a basis of $\mathbf{C}$ over $\mathbf{R}$.

Let now $\Lambda \subseteq \mathbf{C}$ be a lattice. We want to show that the quotient $Y = \mathbf{C}/\Lambda$ indeed is the analytification of an elliptic curve, find a Weierstrass equation and compute its j-invariant. To this end, we would like to produce some non-constant meromorphic functions on Y with poles only at 0 (equivalently, sections of powers of the line bundle $L = \mathcal{O}_Y(0)$ on Y). Such functions correspond to Λ-invariant meromorphic functions on Y with poles only in Λ. To find such functions, we may try our luck by taking a function of the form $1/z^k$ and summing its translates by Λ:

$$f_k(z) = \sum_{\lambda\in\Lambda} \frac{1}{(z-\lambda)^k}.$$

This series will converge uniformly in compact domains of $\mathbf{C} \setminus \Lambda$ for $k \geq 3$. For $k = 2$, the following "renormalization" trick allows one to get a convergent series:

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus 0}\left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right).$$

This is the famous *Weierstrass elliptic function*. It has a double pole at all points of Λ at no other poles, and hence the induced function on Y has a double pole at 0 and no other poles. It is therefore a candidate for the element $x \in \Gamma(Y, L^2)$ in the previous discussion. As a replacement for $y \in \Gamma(Y, L^3)$, we could pick $f_3(z)$, but it is more traditional to consider the derivative $\wp'(z)$ of the Weierstrass function. This is not much of a difference, since we have $\wp'(z) = -2f_3(z)$. One then has the following equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 \tag{9.1}$$

with

$$g_2 = 60 \sum_{\lambda \in \Lambda \setminus 0} \frac{1}{\lambda^4}, \quad g_3 = 140 \sum_{\lambda \in \Lambda \setminus 0} \frac{1}{\lambda^6}.$$

Veryfying (9.1) is easy: one only needs to check that the difference of the two sides of the equation has no pole at 0 and vanishes at zero, which simply entails calculating the coefficients of z^k for $-6 \le k \le 0$. Then the difference is a Λ-invariant holomorphic function, which must be bounded and hence constant.

The map $\sigma\colon Y \to \mathbf{P}^{2,\mathrm{an}}$ induced by $(\wp : \wp' : 1)$ has image in the elliptic curve E defined by the Weierstrass equation

$$Y^2Z = X^3 + AX^2Z + BZ^3, \quad A = -g_2/4, \quad B = -g_3/4.$$

This has discriminant

$$\Delta = -16(4A^3 + 27B^2) = g_2^3 - 27g_3^2$$

and j-invariant

$$j(E) = 1728\frac{g_2^3}{\Delta}.$$

One checks that $\sigma\colon Y \to E^{\mathrm{an}}$ is an isomorphism.

9.3 *Elliptic curves over* $\mathbf{C}$ *as quotients* $\mathbf{C}^\times/q^{\mathbf{Z}}$

If $c \in \mathbf{C}^\times$, then the lattices Λ and $c \cdot \Lambda$ define biholomorphic quotients; we may therefore assume that $1 \in \Lambda$ is part of a basis $\{1, \tau\}$ of Λ, i.e. $\Lambda = \mathbf{Z} \oplus \mathbf{Z}\tau$. We may also assume that $\tau \in \mathbf{H} := \{\operatorname{Im} z > 0\}$. Then the quantities g_2, g_3, Δ and $j(E)$ above can be treated as functions of the variable τ. Moreover, since τ and $\tau + 1$ define the same lattice Λ, we have

$$g_2(\tau + 1) = g_2(\tau), \quad g_3(\tau + 1) = g_3(\tau), \quad \text{etc.}$$

It follows that it will be more economical to treat those as functions of the variable

$$q = e^{2\pi i\tau}$$

(note that $\tau\colon e^{2\pi i\tau}$ identifies $\mathbf{H}/2\pi i\mathbf{Z}$ with the punctured disc $\{0 < |q| < 1\}$). Similarly, the Weierstrass function $\wp$ satisfies $\wp(z+1) = \wp(z)$ and can be treated as a function of two variables $x = e^{2\pi iz}$ and $q = e^{2\pi i\tau}$. In the following, we aim to write it as a series in those.

Lemma 9.3.1 (Lipshitz formula)**.** *For every* $k \ge 2$ *and* $z \in \mathbf{H}$*, one has*

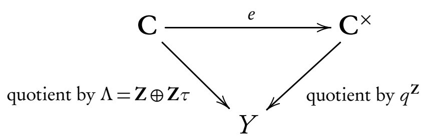
$$\sum_{n \in \mathbf{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1}x^m, \quad x = e^{2\pi iz}.$$

Corollary 9.3.2. *One has the following formulas*

$$
\begin{aligned}
g_2(q) &= \frac{4\pi^4}{3}\left(1+240\sum_{n\geq 1}\frac{n^3q^n}{1-q^n}\right)\\
g_3(q) &= \frac{8\pi^6}{27}\left(1-504\sum_{n\geq 1}\frac{n^5q^n}{1-q^n}\right)\\
\Delta(q) &= (2\pi)^{12}q\prod_{n\geq 1}(1-q^n)\\
j(q) &= q^{-1}+744+196884q+21493760q^2+864299970q^3+\cdots\\
\wp(w,q) &= C\left(\frac{1}{12}+\sum_{m\in\mathbf{Z}}\frac{wq^m}{(1-wq^m)^2}-2\sum_{m\geq 1}\frac{q^m}{(1-q^m)^2}\right)\\
&= C\left(\frac{1}{12}+\frac{w}{(1-w)^2}+\sum_{m\geq 1}\left(\frac{wq^m}{(1-wq^m)^2}+\frac{w^{-1}q^m}{(1-w^{-1}q^m)^2}-2\frac{q^m}{(1-q^m)^2}\right)\right)\\
\wp'(w,q) &= D\sum_{m\in\mathbf{Z}}\left(\frac{wq^m}{(1-wq^m)^2}+2\frac{(wq^m)^2}{(1-wq^m)^3}\right)\\
&= D\left(\frac{1}{C}\wp(w,q)-\frac{1}{12}-2\sum_{m\geq 1}\frac{q^m}{(1-q^m)^2}\right)-4D\sum_{m\in\mathbf{Z}}\frac{(wq^m)^2}{(1-wq^m)^3},
\end{aligned}
$$

where $C=-4\pi^2$ *and* $D=-8i\pi^3$.

In more geometric terms, singling out the subgroup $\mathbf{Z}\subseteq\Lambda=\pi_1(Y)$ corresponds to factoring the universal cover $\mathbf{C}\to Y$ as follows:



where $e(z)=\exp(2\pi iz)$ sends $\Lambda=\mathbf{Z}\oplus\mathbf{Z}\tau$ to $\Lambda/\mathbf{Z}\simeq q^{\mathbf{Z}}$ where $q=e(\tau)$. We therefore obtain

$$Y\simeq\mathbf{C}^\times/q^{\mathbf{Z}}.$$

The formulas in Corollary 9.3.2 imply that $Y\simeq E^{\mathrm{an}}$ where E is the elliptic curve described by explicit formulas in terms of q.

9.4 Formulas with integral coefficients

Setting $\overline{\wp}=\wp/C$ and $\overline{\wp}'=\wp'/D=w\frac{d}{dw}\overline{\wp}$, we obtain series in q and w with rational coefficients:

$$
\begin{aligned}
\overline{\wp}(w,q) &= \frac{1}{12}+\sum_{m\in\mathbf{Z}}\frac{wq^m}{(1-wq^m)^2}-2\sum_{m\geq 1}\frac{q^m}{(1-q^m)^2}\\
&= \frac{1}{12}+\frac{w}{(1-w)^2}+\sum_{m\geq 1}\left(\frac{wq^m}{(1-wq^m)^2}+\frac{w^{-1}q^m}{(1-w^{-1}q^m)^2}-2\frac{q^m}{(1-q^m)^2}\right)\\
\overline{\wp}'(w,q) &= \sum_{m\in\mathbf{Z}}\left(\frac{wq^m}{(1-wq^m)^2}+2\frac{(wq^m)^2}{(1-wq^m)^3}\right).
\end{aligned}
$$

These functions satisfy

$$(\overline{\wp}')^2=4\overline{\wp}^3-\overline{g}_2\overline{\wp}-\overline{g}_3 \tag{9.2}$$

where

$$\overline{g}_2(q) = \frac{1}{3}\left(1 + 240\sum_{n\geq 1}\frac{n^3 q^n}{1-q^n}\right), \quad \overline{g}_3(q) = -\frac{1}{6^3}\left(1 - 504\sum_{n\geq 1}\frac{n^5 q^n}{1-q^n}\right).$$

The above equation has discriminant

$$\overline{\Delta}(q) = \overline{g}_2^3(q) - 27\overline{g}_3^2(q) = q\prod_{m\geq 1}(1-q^m)$$

and defines an elliptic curve with j-invariant $j(q) = q^{-1} + 744 + 196884q + \cdots$. Note that primes other than 2 and 3 do not occur in the denominators of the coefficients.

Lemma 9.4.1. *Let K be a non-Archimedean field of characteristic $\neq 2, 3$. Then the above formulas for $\overline{g}_2$, $\overline{g}_3$, $\overline{\Delta}$ and j define rigid-analytic functions on the punctured open disc $\Delta^* = \{0 < |q| < 1\}$ over K. For every $q \in \Delta^*$, the formulas for $\overline{\wp}$ and $\overline{\wp'}$ define meromorphic functions on $\mathbf{G}_m^{\mathrm{an}}$ with poles of order 2 and 3 respectively at the points q^m ($m \in \mathbf{Z}$) and no other poles. They satisfy equation (9.2) and*

$$\overline{\wp}(qw) = \overline{\wp}(w), \quad \overline{\wp}'(qw) = \overline{\wp}'(w), \quad \overline{\wp}(w) = \overline{\wp}(w^{-1}), \quad \overline{\wp}'(w) = \overline{\wp}'(w^{-1}).$$

9.5 The construction

Let $X = \mathbf{G}_m^{\mathrm{an}} = (\mathbf{A}_K^1 \setminus 0)^{\mathrm{an}}$ be the rigid punctured line with coordinate w, and let $q \in K$ be an element with $0 < |q| < 1$. The presentation $\mathbf{G}_m = \operatorname{Spec} K[w, v]/(wv - 1)$ gives rise to a "standard" affinoid open cover

$$X = \bigcup_n U_n, \quad U_n = \{|w|, |v| \leq |q|^{-n}\} = \{|q|^n \leq |w| \leq |q|^{-n}\}.$$

The algebra of functions on X can thus be identified with

$$\mathcal{O}_X(X) = \varprojlim_n \mathcal{O}_X(U_n) = \left\{ f = \sum_{n\in\mathbf{Z}} a_n w^n \,:\, \lim_{|n|\to\infty} |a_n|\rho^n = 0 \text{ for every } \rho > 0 \right\}.$$

We will need some finer affinoid covers of X. For rational numbers $a \leq b$, we let $X[a, b]$ be the open subset

$$X[a, b] = \{|q|^b \leq |w| \leq |q|^a\} \subseteq X.$$

Then $X[a, b]$ is an affinoid open of X, isomorphic to $\operatorname{Sp} A_{[a,b]}$ where

$$A_{[a,b]} = K\left\langle \frac{w}{q^a}, \frac{q^b}{w} \right\rangle = \left\{ f = \sum_{n\in\mathbf{Z}} a_n w^n \,:\, \lim_{n\to\infty} |a_n| \cdot |q|^{an} = 0, \lim_{n\to-\infty} |a_n| \cdot |q|^{bn} = 0 \right\}.$$

Then the $X[a, b]$ ($a, b \in \mathbf{Q}$, $a \leq b$) form an admissible cover of X, and so do $X[n, n+1]$ ($n \in \mathbf{Z}$). Note that $U_n = X[-n, n]$.

The cyclic group $q^{\mathbf{Z}}$ acts on $\mathbf{G}_m$ by multiplying the coordinate w by powers of q. We will denote by $t_q : X \to X$ the map induced by multiplication by q. The induced action on X is free and identifies $t_q^n(X[a, b])$ with $X[a+n, b+n]$. This action is *properly discontinuous* in the sense of the following definition.

Definition 9.5.1. Let X be a G-topological space satisfying (G_0), (G_1), and (G_2), and let Γ be a group acting freely and continuously on X (meaning that the maps $\gamma : X \to X$ are continuous maps of G-topological spaces for all $\gamma \in \Gamma$). We call this action *properly discontinuous* if X admits an admissible cover of the form $\{\gamma \cdot U_i\}_{i\in I, \gamma\in\Gamma}$ with $\gamma \cdot U_i \cap U_i = \emptyset$ for $\gamma \neq e$ and such that the sets $\bigcup_{\gamma\in\Gamma} \gamma \cdot U_i$ are admissible for all $i \in I$.

Indeed, we can take $I = \{-,+\}$, $U_- = X[-\frac{1}{2},0]$, $U_+ = X[0,\frac{1}{2}]$. In particular (see Problem 5 on Problem Set 6) we obtain a quotient rigid-analytic space $Y = X/q^{\mathbf{Z}}$ by identifying (using the setup of Remark 8.1.4)

$$U_{-+} = X[0,0] \sqcup X[-1/2,-1/2] \subseteq U_- \quad \text{and} \quad U_{+-} = X[0,0] \sqcup X[1/2,1/2] \subseteq U_+$$

via the isomorphism

$$\mathrm{id} \sqcup t_q : X[0,0] \sqcup X[-1/2,-1/2] \xrightarrow{\sim} X[0,0] \sqcup X[1/2,1/2].$$

In more intrinsic terms, the underlying set of Y is the set of orbits of $q^{\mathbf{Z}}$ in X, a subset $U \subseteq Y$ is an admissible open if and only if its preimage $\pi^{-1}(U)$in X is an admissible open, and similarly for admissible covers. The structure sheaf is defined as

$$\mathcal{O}_Y(U) = \mathcal{O}_X(\pi^{-1}(U))^{q^{\mathbf{Z}}} = \{f \in \mathcal{O}_X(\pi^{-1}(U)) : t_q^* f = f\}.$$

Lemma 9.5.2. *The rigid analytic space Y is proper.*

Proof. We first show that Y is separated. For this, it suffices to show that the map of affinoids

$$U_- \cap U_+ \to U_- \cap U_+$$

is a closed immersion. Identifying $U_- \cap U_+$ with $U_{-+} = X[0,0] \sqcup X[-1/2,-1/2]$, the map $U_{-+} \to U_-$ is the inclusion and $U_{-+} \to U_+$ is $U_{-+} \simeq U_{+-} \hookrightarrow U_+$, i.e. the map induced by $\mathrm{id} \sqcup t_q$. Turning to algebra, this is equivalent to saying that the map of affinoid algebras

$$(r \otimes r) \times (r \otimes t_q^*) \quad : \quad A_{[-1/2,0]} \widehat{\otimes} A_{[0,1/2]} \to A_{[0,0]} \times A_{[-1/2,-1/2]}$$

where $r^* w = w$ are the restriction maps and $t_q^* w = qw$, is surjective. Clearly, the two component maps

$$r \otimes r : A_{[-1/2,0]} \widehat{\otimes} A_{[0,1/2]} \to A_{[0,0]} \quad \text{and} \quad r \otimes t_q^* : A_{[-1/2,0]} \widehat{\otimes} A_{[0,1/2]} \to A_{[-1/2,-1/2]}$$

are surjective (note that the latter equals the composition of the isomorphism

$$\mathrm{id} \otimes t_q^* : A_{[-1/2,0]} \widehat{\otimes} A_{[0,1/2]} \xrightarrow{\sim} A_{[-1/2,0]} \widehat{\otimes} A_{[-1/2,0]}$$

and the restriction $r \otimes r : A_{[-1/2,0]} \widehat{\otimes} A_{[-1/2,0]} \to A_{[-1/2,-1/2]}$). Moreover, if $A = A_{[-1/2,0]} \widehat{\otimes} A_{[0,1/2]}$ denotes the source of our map, then we have $A_{[0,0]} \otimes_A A_{[-1/2,-1/2]} = 0$. The Chinese Remainder Theorem implies that the map $A \to A_{[0,0]} \times A_{[-1/2,-1/2]}$ is surjective.

To show that X is proper, we pick a rational $\varepsilon \in (0,1/4)$ and let V_- and V_+ be the image in Y of $X[-1/2-\varepsilon, \varepsilon]$ and $X[-\varepsilon, 1/2+\varepsilon]$, respectively. Then $Y = V_+ \cup V_-$ is an affinoid cover as well, and we have $U_+ \Subset V_+$ and $U_- \Subset V_-$. □

9.6 *Ample line bundle on Y*

Since $Y = X/q^{\mathbf{Z}}$ for a free and properly discontinuous action of $q^{\mathbf{Z}}$ on X, the category of sheaves on Y admits a simple description in terms of $q^{\mathbf{Z}}$-equivariant sheaves on X.

Lemma 9.6.1. *One has an equivalence of categories*

$$\mathrm{Sh}\, Y \simeq \mathrm{Sh}^{q^{\mathbf{Z}}} X := \{(\mathcal{F} \in \mathrm{Sh}\, X, \varphi : t_q^* \mathcal{F} \xrightarrow{\sim} \mathcal{F})\},$$

and similarly for coherent sheaves:

$$\mathrm{Coh}\, Y \simeq \mathrm{Coh}^{q^{\mathbf{Z}}} X := \{(\mathcal{F} \in \mathrm{Coh}\, X, \varphi : t_q^* \mathcal{F} \xrightarrow{\sim} \mathcal{F})\}.$$

Convention: f^* always denotes the left adjoint to f_* in whatever category of sheaves we work with. Thus for sheaves of sets or abelian groups this is the sheaf preimage (Hartshorne's f^{-1}), and for $\mathcal{O}_X$-modules this denotes the $\mathcal{O}$-module pullback.

In particular, this means that a trivial (constant, or locally free) sheaf $\mathscr{F}$ on X may give rise to a nontrivial sheaf on Y if endowed with a nontrivial action map $t_q^*\mathscr{F} \to \mathscr{F}$. For a simple example, let us take a unit $u \in \Gamma(X, \mathscr{O}_X^\times)$ and consider the object $\mathscr{L}_u \in \operatorname{Coh} Y$ corresponding to the equivariant coherent sheaf

$$(\mathscr{O}_X, \varphi\colon t_q^*\mathscr{O}_X = \mathscr{O}_X \xrightarrow{\cdot u} \mathscr{O}_X) \in \operatorname{Coh}^{q^{\mathbf{Z}}} X.$$

This is a line bundle (invertible sheaf) on Y, and we have $\mathscr{L}_u \otimes \mathscr{L}_v \simeq \mathscr{L}_{uv}$.

Lemma 9.6.2. *One has*

$$\operatorname{Hom}(\mathscr{L}_u, \mathscr{L}_v) = \{f \in \mathscr{O}_X(X) : v(w)f(qw) = u(w)f(w)\}.$$

In particular, $\mathscr{L}_u$ and $\mathscr{L}_v$ are isomorphic if and only if $u/v = (t_q^* f)/f$ for some $f \in \mathscr{O}_X^\times(X)$, and

$$\Gamma(Y, \mathscr{L}_u) = \{f \in \mathscr{O}_X(X) : f(qw) = u(w)^{-1} f(w)\}.$$

Proof. By definition of a map of equivariant sheaves, a map $(\mathscr{O}_X, \cdot u) \to (\mathscr{O}_X, \cdot v)$ in $\operatorname{Coh}^{q^{\mathbf{Z}}} X$ is a map $f : \mathscr{O}_X \to \mathscr{O}_X$ for which the square

$$\begin{array}{ccccc} t_q^*(\mathscr{O}_X) & = & \mathscr{O}_X & \xrightarrow{u} & \mathscr{O}_X \\ \downarrow{\scriptstyle t_q^* f} & & & & \downarrow{\scriptstyle f} \\ t_q^*(\mathscr{O}_X) & = & \mathscr{O}_X & \xrightarrow{v} & \mathscr{O}_X \end{array}$$

commutes, which amounts to the equation

$$v(w)f(qw) = u(w)f(w). \qquad \square$$

Compare with Chapter I of Mumford's *Abelian Varieties*.

Remark 9.6.3. Recall that for a group Γ acting on a module M, the *module of coinvariants* M_Γ is the quotient of M by the submodule generated by the elements $m - \gamma \cdot m$ for $m \in M$ and $\gamma \in Gamma$. The above lemma implies that $u \mapsto \mathscr{L}_u$ defines an injection

$$\mathscr{O}_X^\times(X)_{q^{\mathbf{Z}}} \hookrightarrow \operatorname{Pic} Y.$$

I don't have a reference, but it is quite likely that $\operatorname{Pic} X$ is trivial, and so the above map is an isomorphism.

To describe Y as an analytification of an elliptic curve, we seek a description of the line bundle $\mathscr{O}_Y(0)$ where $0 \in Y$ is the image of $1 \in X$. After a few tries, one checks that the function $u(w) = -w^{-1}$ does the job. In other words, we define $\mathscr{L} = \mathscr{L}_{-w^{-1}}$, and we will produce a section $\theta \in \Gamma(Y, \mathscr{L})$ with a zero of order 1 at 0 and no other zeroes. Set $V_k = \Gamma(Y, \mathscr{L}^k)$. By Lemma 9.6.2, we have

$$V_k = \left\{ f \in \mathscr{O}_X(X) : f(qw) = (-w)^k f(w) \right\}.$$

The direct sum $V = \bigoplus_{k \geq 0} V_k$ is a graded ring, and we aim to show that $E = \operatorname{Proj} V$ is an elliptic curve over K for which $Y \simeq E^{\mathrm{an}}$.

Lemma 9.6.4. *One has* $\dim V_k = k$ *for* $k \geq 1$.

Proof. Take $f = \sum_{n \in \mathbf{Z}} a_n w^n \in \mathscr{O}_X(X)$. Then $f \in V_k$ if and only if

$$a_n q^n = (-1)^k a_{n-k} \quad \text{for all } n. \tag{9.3}$$

It follows that $a_0, \dots, a_{k-1}$ determine f uniquely:

$$a_n = \pm q^{k\frac{m(m+1)}{2} + mr} a_r \quad \text{if } n = mk + r \text{ with } 0 \le r < k.$$

Conversely, given any values $a_0, \dots, a_{k-1}$, the above formula extends it to a sequence a_n ($n \in \mathbf{Z}$) satisfying (9.3). Since the power of q is quadratic in m, we have $\lim_{|n|\to\infty} |a_n|\rho^n = 0$ for every $\rho > 0$, and hence $\sum_{n\in\mathbf{Z}} a_n w^n$ defines an element of $\mathscr{O}_X(X)$. □

For example, for $k = 1$ the formulas in the above proof specify a unique up to scaling nonzero element of V_1, the *basic theta function*:

$$\theta(w) = \sum_{n\in\mathbf{Z}} (-1)^n q^{\frac{n(n+1)}{2}} w^n.$$

Lemma 9.6.5. *One has $\theta(w) = 0$ if and only if $w \in q^{\mathbf{Z}}$. Moreover, the zero at 1 is of order one.*

It is amusing to substitute $w = 1$ in $\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{n(n+1)}{2}} w^n$ and see that the n-th and the $(-n-1)$-st term cancel each other. This is the reason we chose $u(x) = -w^{-1}$ rather than $u(x) = w^{-1}$ in the definition of $\mathscr{L}$. To prove the lemma, we need the famous *Jacobi triple product* formula:

$$\theta(w) = (1 - w^{-1}) \prod_{m\ge 1} (1 + q^m)(1 - wq^m)(1 - w^{-1}q^m)$$

(see Problem 1 on Problem Set 8).

With the lemma at hand, we know that $\mathscr{L} = \mathscr{O}_Y(0)$. In principle, we could now proceed to study Proj V as in §9.1: choose $x \in V_2$ such that θ^2, x is a basis, then choose $y \in V_3$ such that $\theta^3, \theta x, y$ is a basis, note that in V_6 there is (after suitably rescaling x and y) an equation of the form $y^2 = x^3 + \cdots$, and so on. However, it is rather complicated to find natural candidates for x and y for which the equation can be explicitly computed. Instead, we shall use the integral formulas of § above.

See Mumford *Tata Lectures on Theta I* for a complete development of the theory of projective embeddings of abelian varieties using theta functions.

9.7 The Tate curve is an elliptic curve

We first prove the abstract result (which requires some nontrivial rigid geometry) and then proceed to calculate an explicit cubic equation for E afterwards.

Proposition 9.7.1. *There exists a unique elliptic curve E over K such that $Y \simeq E^{\mathrm{an}}$.*

Proof. Let $\varphi, \psi \in V_2$ be a basis. For example, we could take $\varphi = \theta^2$ and $\psi = \sum q^{n(n+1)} w^{2n}$. Note that φ and ψ do not vanish simultaneously: first, $\varphi = \theta^2$ has a double zero at q^m and no other zeroes. If $\psi(1) = 0$, then ψ/θ is a regular function and so $\psi/\theta \in V_1$, which is one-dimensional with basis θ, showing $\psi = c \cdot \theta^2$ for some $c \in K^\times$, contradiction. We obtain a map

$$(\varphi : \psi) \colon X \to \mathbf{P}^{1,\mathrm{an}}_K$$

which is $q^{\mathbf{Z}}$-invariant and hence factors through a map $\sigma \colon Y \to \mathbf{P}^{1,\mathrm{an}}_K$. Since Y is proper, σ is a proper map (Proposition 8.7.4) with $\sigma^*\mathscr{O}(1) \simeq \mathscr{L}^2$. For dimensional reasons, the map σ is quasi-finite, and hence finite (Corollary 8.7.11). By Remark 8.5.2(3) and GAGA (Theorem 8.7.12), the categories of finite morphisms to $\mathbf{P}^1_K$ in the category of K-schemes and of finite morphisms to $\mathbf{P}^{1,\mathrm{an}}$ in Rig_K are equivalent, and hence $Y \simeq E^{\mathrm{an}}$ for some finite map $E \to \mathbf{P}^1_K$. Again by GAGA, the group structure on Y induces a group structure on E, and hence E is an elliptic curve. □

Note that since $\sigma^*\mathcal{O}(1) = \mathcal{L}^2$, we must have $E \simeq \operatorname{Proj} V$ as claimed previously. We now seek a concrete description of the curve E, and for this we shall assume that $\operatorname{char} K \neq 2,3$. This is done only for simplicity, so that we can use the complex formulas (§9.6) without change (there we had some powers of 2 and 3 in the denominators). See [8, §5.1] for a characteristic-free approach.

Consider the meromorphic functions $\overline{\wp}$ and $\overline{\wp}'$ on X (Lemma 9.4.1). Then $x = \theta^3\overline{\wp}$, $y = \theta^3\overline{\wp}'$, and $z = \theta^3$ are regular and form a basis V_3. Moreover, they satisfy the equation

$$y^2 z = 4x^3 - \overline{g}_2 x z^2 - \overline{g}_3 z^3.$$

It follows that the map $(x : y : z) \colon X \to \mathbf{P}^{2,\mathrm{an}}_K$ induces a map

$$\tau \colon Y \to E_q^{\mathrm{an}}, \quad E_q = \{y^2 z = 4x^3 - \overline{g}_2 x z^2 - \overline{g}_3 z^3\} \subseteq \mathbf{P}^2_K,$$

which is an isomorphism. Indeed, the line bundle $\mathcal{L}^3$ corresponding to this map corresponds to a very ample line bundle on the elliptic curve E constructed previously, and hence τ is a closed immersion and must therefore be an isomorphism.

Corollary 9.7.2. *Suppose that* $\operatorname{char} K \neq 2,3$. *The Tate curve* $Y = \mathbf{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}$ *is isomorphic to the analytification of the elliptic curve* $E_q \subseteq \mathbf{P}^2_K$ *given by the equation*

$$\tau \colon Y \to E_q^{\mathrm{an}}, \quad E_q = \{y^2 z = 4x^3 - \overline{g}_2 x z^2 - \overline{g}_3 z^3\} \subseteq \mathbf{P}^2_K,$$

where

$$\overline{g}_2(q) = \frac{1}{3}\left(1 + 240\sum_{n\geq 1}\frac{n^3 q^n}{1-q^n}\right), \quad \textit{and} \quad \overline{g}_3(q) = -\frac{1}{6^3}\left(1 - 504\sum_{n\geq 1}\frac{n^5 q^n}{1-q^n}\right).$$

The j*-invariant of* E_q *is given by the power series with integral coefficients*

$$j(q) = \frac{1728\overline{g}_2^3}{q\prod_{m\geq 0}(1-q^m)} = q^{-1} + 744 + 196884q + \cdots$$

In particular, we have $|j(q)| > 1$.

(The last assertion is true in any characteristic and can be shown with a just a bit more computation.)

Lemma 9.7.3. *A Laurent series of the form* $f(q) = q^{-1} + \sum_{n\geq 0} a_n q^n \in K(\!(q)\!)$ *with* $|a_n| \leq 1$ *defines a bijection between* $\{0 < |q| < 1\}$ *and* $\{|w| > 1\}$.

Proof. See Problem 2 on Problem Set 8. □

Corollary 9.7.4. *Suppose that* K *is algebraically closed. Then for every elliptic curve* E *over* K *with* $|j(E)| > 1$ *there exists a unique* $q \in K$ *with* $0 < |q| < 1$ *such that* $E^{\mathrm{an}} \simeq \mathbf{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}$.

Proof. By Lemma 9.7.3, there is a unique such q with $j(E_q) = j(E)$. Since E and E_q have the same j-invariant and K is algebraically closed, we have $E \simeq E_q$. □

The proof shows that if K is not algebraically closed, then there exists a finite extension L/K such that $E \simeq E_q$. In fact, the extension L/K can be taken to have degree at most two, see [8, 5.1.18].

Remark 9.7.5. TODO: Something about split multiplicative reduction.

9.8 Applications

Proposition 9.8.1. *Let E be an elliptic curve with $|j(E)| > 1$. Then there exists a finite extension L/K such that after replacing E/K with E_L/L we have:*

1. $\mathrm{End}\, E \simeq \mathbf{Z}$,
2. *for every $m \geq 1$, the m-torsion of E sits in an extension*
$$0 \to \mu_m \to E[m] \to \mathbf{Z}/m\mathbf{Z} \to 0,$$
3. *for every prime ℓ invertible in K, the Tate module $T_\ell E = \varprojlim_n E(\overline{K})[\ell^n]$ sits in a short exact sequence of* $\mathrm{Gal}(\overline{K}/K)$*-modules*
$$0 \to \mathbf{Z}_\ell(1) \to T_\ell(E) \to \mathbf{Z}_\ell \to 0$$
where $\mathbf{Z}_\ell(1) = \varprojlim_n \mu_{\ell^n}(\overline{K})$.

Proof. See Exercises 3 and 4 on Problem Set 8. □

Corollary 9.8.2. *Let K be a number field and let E be an elliptic curve with complex multiplication (i.e.* $\mathrm{End}\, E \neq \mathbf{Z}$*). Then $j(E) \in \mathcal{O}_K$.*

Proof. Otherwise there exists a prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$ such that $v_\mathfrak{p}(j(E)) < 0$. Let $\hat{K}$ be the local field of $\mathfrak{p}$ (i.e. the fraction field of $\varprojlim \mathcal{O}_K/\mathfrak{p}^n$, or the completion of K with respect to $v_\mathfrak{p}$). This is a non-Archimedean field and the base change $E_{\hat{K}}$ is an elliptic curve with $|j(E_{\hat{K}})| < 1$. Therefore it becomes a Tate curve over a finite extension $\hat{L}$ of $\hat{K}$, which implies that $\mathrm{End}\, E_{\hat{L}} = \mathbf{Z}$. Thus $\mathrm{End}\, E \simeq \mathbf{Z}$ too. □

Corollary 9.8.3. *Let K be a number field and let E be an elliptic curve with $j(E) \notin \mathcal{O}_K$. Then $E(K)_{\mathrm{tors}}$ is finite.*

Proof. Take $\hat{L}$ as in the previous proof, which is a finite extension of $\mathbf{Q}_p$ for the unique prime p with $v_\mathfrak{p}(p) > 0$. Changing notation, it suffices to show that if Y is a Tate curve over a finite extension K of $\mathbf{Q}_p$ then the torsion subgroup of $Y(K) = K^\times/q^{\mathbf{Z}}$ is finite.

We first show that K has only finitely many roots of unity. It suffices to show that (1) the group of roots of unity of order prime to p is finite, (2) the group of roots of unity of order a power of p is finite. To prove (1), use Hensel's lemma and the fact that the residue field of K is finite. To show (2), compute the ramification index of $\mathbf{Q}_p(\zeta_{p^n})$ over $\mathbf{Q}_p$ and check that it goes to infinity. Since K has finite ramification index over $\mathbf{Q}_p$, it can contain only finitely many of these fields.

Let v be the valuation on K (normalized so that $v(\text{uniformizer}) = 1$) and let $m = v(q) > 0$. Suppose that $y \in K^\times$ gives an n-torsion point on Y for some prime ℓ, i.e. $y^n = q^k$ for some k. Write $n = dn'$ and $k = dk'$ with $(n', k') = 1$. Multiplying y by a power of q, we may assume that $0 \leq k < n$. We then have $y^{n'} = q^{k'} \cdot \zeta$ for some $\zeta \in \mu_d(K)$. On the other hand, $v(y) = k'm/n'$ must be an integer, forcing $n' \leq m$. Since $\mu(K)$ is finite, we also have $d \neq N$ for some N, and hence $n \leq Nm$. □

9.A Mumford curves

9.B Raynaud's uniformization of abelian varieties

Part I

Formal schemes

10
Formal schemes

11
Admissible blow-ups

12 Riemann–Zariski spaces

The description of the category of (quasi-paracompact and quasi-separated) rigid-analytic spaces in terms of formal models is rather formal (no pun intended). In this chapter, we explain how it gives rise to a natural topological space $\langle X \rangle$, called the *Riemann–Zariski space*, functorially attached to every such rigid-analytic space X. The space $\langle X \rangle$ contains X as a dense subset, and has the property of "filling in the missing points," thereby fixing the admissible topology.

We show that admissible sheaves on X correspond to sheaves on $\langle X \rangle$. For $X = \operatorname{Sp} A$ affinoid, we give an interpretation of the points of $\langle X \rangle$ in terms of continuous valuations on A, giving a link with Huber's theory of adic spaces. Finally, we explain (without proofs) how the theory is related with Berkovich's theory of analytic spaces: the Berkovich space attached to X is none other but a "universal separated quotient" of the topological space $\langle X \rangle$.

12.1 The Riemann–Zariski space

Let X be a quasi-compact and quasi-separated rigid-analytic space over K. Recall that the category $M(X)$ of admissible formal models $(\mathfrak{X}, \iota \colon \mathfrak{X}_{\mathrm{rig}} \simeq X)$ is a cofiltering poset.

Definition 12.1.1 (Riemann–Zariski space)**.** The limit of the obvious functor $(\mathfrak{X}, \iota) \mapsto \mathfrak{X}$ from $M(X)$ to locally ringed spaces

$$\langle X \rangle = \varprojlim_{M(X)} \mathfrak{X}$$

is called the *Riemann–Zariski space* of X.

Its underlying topological space, also denoted $\langle X \rangle$, is the inverse limit of the spaces $|\mathfrak{X}| = |X_0|$. The structure sheaf $\mathcal{O}_{\langle X \rangle}$ will be denoted $\mathcal{O}^+$. It has the property that for $x = (x_{\mathfrak{X}})_{M(X)} \in \langle X \rangle$, we have

$$\mathcal{O}^+_x = \varinjlim_{M(X)} \mathcal{O}_{\mathfrak{X}, x_{\mathfrak{X}}}.$$

In this chapter, we will be mostly dealing with $\langle X \rangle$ as a topological space, or even as a set. For a formal model $\mathfrak{X}$ of X, we denote by

$$\pi_{\mathfrak{X}} \colon \langle X \rangle \to \mathfrak{X}$$

the projection map. Since the poset $M(X)$ is cofiltering, open subsets of the form $\pi_{\mathfrak{X}}^{-1}(U)$ where $\mathfrak{X}$ is a formal model of X and $U \subseteq \mathfrak{X}$ is an affine open subset form a basis for the topology on $\langle X \rangle$. Moreover, the subcategory $\mathbf{B}_{/\mathfrak{X}} \subseteq M(X)$ of admissible blowups of a fixed formal model $\mathfrak{X}$ of X is cofinal in $M(X)$ by ■, and hence

■ Reference to Chapter 11

$$\langle X\rangle = \varprojlim_{(\mathfrak{X}'\to\mathfrak{X})\in\mathbf{B}_{/\mathfrak{X}}} \mathfrak{X}'.$$

The construction of $\langle X\rangle$ is functorial in X. Namely, if $f : Y \to X$ is a map between quasi-compact and quasi-separated rigid-analytic spaces, then by ■ we can find a formal model $f : \mathfrak{Y} \to \mathfrak{X}$ of f. Since by ■ for every $(\mathfrak{X}' \to \mathfrak{X}) \in \mathbf{B}_{/\mathfrak{X}}$ we can find a $(\mathfrak{Y}' \to \mathfrak{Y}) \in \mathbf{B}_{/\mathfrak{Y}}$ fitting in a commutative square

■ Reference to Chapter 11
■ Reference to Chapter 11

$$\begin{array}{ccc} \mathfrak{Y}' & \longrightarrow & \mathfrak{X}' \\ \downarrow & & \downarrow \\ \mathfrak{Y} & \longrightarrow & \mathfrak{X}, \end{array}$$

we obtain an induced map

$$\langle Y\rangle = \varprojlim_{(\mathfrak{Y}'\to\mathfrak{Y})\in\mathbf{B}_{/\mathfrak{Y}}} \mathfrak{Y}' \to \varprojlim_{(\mathfrak{X}'\to\mathfrak{X})\in\mathbf{B}_{/\mathfrak{X}}} \mathfrak{X}' = \langle X\rangle$$

which does not depend on the choice of the formal model $f : \mathfrak{Y} \to \mathfrak{X}$. We obtain a functor

$$X \mapsto \langle X\rangle : \mathbf{Rig}_K^{\mathrm{qcqs}} \to \mathbf{LRS}_{\mathcal{O}}.$$

For every formal model $(\mathfrak{X}, \iota)$ of X, we have the continuous specialization map ■

■ Reference to Chapter 10

$$\mathrm{sp}_{\mathfrak{X}} : X \to \mathfrak{X}.$$

If $f : \mathfrak{X}' \to \mathfrak{X}$ is a morphism in $M(X)$, then the triangle

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{sp}_{\mathfrak{X}'}} & \mathfrak{X}' \\ & \searrow^{\mathrm{sp}_{\mathfrak{X}}} & \downarrow f \\ & & \mathfrak{X} \end{array}$$

commutes. Consequently, we obtain a specialization map

$$\mathrm{sp}_X : X \to \langle X\rangle,$$

the unique continuous map (for the canonical topology on X) such that $\mathrm{sp}_{\mathfrak{X}} = \pi_{\mathfrak{X}} \circ \mathrm{sp}_X$ for every formal model $\mathfrak{X}$.

12.2 *The Riemann–Zariski space is a spectral space*

Definition 12.2.1 (Spectral space)**.** A topological space X is *spectral* if it satisfies the following conditions:

1. X is sober (Definition 5.A.1),
2. X is quasi-compact and quasi-separated (i.e. the intersection of two quasi-compact opens is quasi-compact),
3. X has a basis consisting of quasi-compact open subsets.

Remark 12.2.2. 1. Recall that the sobriety of X means that X can be naturally recovered from its associated topos $\mathrm{Sh}\,X$ (Proposition 5.A.4).

2. For any ring A, the space $\mathrm{Spec}\,A$ is spectral, and affine open subsets form a basis of topology consisting of quasi-compact open subsets.

3. Conversely, a theorem of Hochster [11] states that if X is a spectral topological space, then X is homeomorphic to Spec$A$ for some ring A. This explains the name.

Theorem 12.2.3. *Let $(X_i)_{i\in I}$ be an inverse system of spectral spaces, indexed by a cofiltering poset I, whose transition maps $\varphi_{ij} : X_i \to X_j$ $(i \geq j)$ are continuous and quasi-compact (i.e. the preimage of a quasi-compact open is quasi-compact). Then the inverse limit*

$$X = \varprojlim_n X_i$$

is spectral.

About the proof. By definition, X has a basis of topology consisting of $\pi_i^{-1}(U)$ where $\pi_i : X \to X_i$ is the projection and $U \subseteq X_i$ is a quasi-compact open, and hence condition (c) of Definition 12.2.1 is satisfied. This also shows that X is quasi-separated.

Let us give a few indications about the proof that X is quasi-c. Recall that if the X_i are also Hausdorff (and therefore compact), then X is compact Hausdorff. Indeed, it is a subspace of the compact Hausdorff $\prod_{i\in I} X_i$ cut out by the conditions $\varphi_{ij}(x_i) = x_j$, which are closed because X_i is compact and X_j is Hausdorff.

The main idea of the proof is to make the spaces X_i Hausdorff by considering the *constructible topology*.

Definition 12.2.4. Let X be a topological space. We denote by X_{cons} the set X endowed with the topology generated by subsets of the form U and $X \setminus U$ for $U \subseteq X$ a quasi-compact open subset. We call this topology the *constructible topology* on X.

Note that if X has a basis of topology consisting of quasi-compact opens, then the identity map $X_{\text{cons}} \to X$ is continuous. Moreover, if $f : X' \to X$ is a quasi-compact map, then $f : X'_{\text{cons}} \to X_{\text{cons}}$ is continuous. In particular, in the context of Theorem 12.2.3, we have an inverse system $\{X_{i,\text{cons}}\}_{i\in I}$ with a map of inverse systems $\{X_{i,\text{cons}} \to X_i\}$ which is levelwise bijective. Passing to the inverse limit, we obtain a bijective map

$$X' := \varprojlim_n{}' X_{i,\text{cons}} \to X.$$

Thus, to show that X is quasi-compact, it is enough to show that X' is. This in turn follows from what we said about inverse limits of compact Hausdorff spaces, thanks to the following key lemma:

Lemma 12.2.5. *Let X be a spectral space. Then X_{cons} is compact Hausdorff.*

Let us explain why X_{cons} is Hausdorff. Let $x, y \in X$ with $x \neq y$. Since X is sober, in particular T_0, and has a basis of quasi-compact opens, there exists a quasi-compact open $U \subseteq X$ containing one of the points, say x, but not the other. Then $X_{\text{cons}} = U \sqcup (X \setminus U)$ is a decomposition into two disjoint open subsets with $x \in U$ and $y \in X \setminus U$. (This shows moreover that X_{cons} is totally disconnected, and hence profinite.) □

Corollary 12.2.6. *Let X be a quasi-compact and quasi-separated rigid-analytic space over K. Then the Riemann–Zariski space $\langle X\rangle$ is spectral.*

Proof. Indeed, by definition we have $\langle X\rangle = \varprojlim_{M(X)} |\mathfrak{X}|$. The spaces $|\mathfrak{X}| = |X_0|$ are the underlying spaces of quasi-compact quasi-separated schemes, and hence are spectral. The maps $|X'_0| \to |X_0|$ in this system are quasi-compact (in fact every open subset of X'_0 is quasi-compact, since X'_0 is a Noetherian scheme). By Theorem 12.2.3, the inverse limit is spectral. □

12.3 Sheaves on the Riemann–Zariski space

Theorem 12.3.1. *Let X be a quasi-compact and quasi-separated rigid-analytic space over K. We have an equivalence of categories*

$$\mathrm{Sh}^{\mathrm{adm}}(X) \simeq \mathrm{Sh}\langle X\rangle$$

between the category of admissible sheaves (of sets) on X and the category of sheaves (of sets) on its Riemann–Zariski space $\langle X\rangle$.

Lemma 12.3.2. *Let $U \subseteq X$ be a quasi-compact admissible open subset. Then the induced morphism*

$$\langle U\rangle \to \langle X\rangle$$

is a quasi-compact open immersion. Open subsets of the form $\langle U\rangle$ for $U \subseteq X$ quasi-compact open form a basis of the topology on $\langle X\rangle$.

Proof. We showed in ■ that the open immersion $U \hookrightarrow X$ admits a formal model $\mathfrak{U} \hookrightarrow \mathfrak{X}$ which is an open immersion. We obtain a commutative square

■ Reference to Chapter 11

$$\begin{array}{ccc} \langle U\rangle & \longrightarrow & \langle X\rangle \\ {\scriptstyle \pi_{\mathfrak{U}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathfrak{X}}} \\ \mathfrak{U} & \longrightarrow & \mathfrak{X}. \end{array}$$

We claim that this square is Cartesian, i.e. $\langle U\rangle = \pi_{\mathfrak{X}}^{-1}(\mathfrak{U})$ (which implies the assertion of the lemma). Indeed, $\langle X\rangle$ is the inverse limit of admissible blow-ups $(\mathfrak{X}' \to \mathfrak{X}) \in \mathbf{B}_{/\mathfrak{X}}$ and likewise for $\mathfrak{U}$. We showed in ■ that every admissible blow-up $\mathfrak{U}' \to \mathfrak{U}$ extends to an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$. This shows that

■ Reference to Chapter 11

$$\mathfrak{X}' \mapsto \mathfrak{X}' \times_{\mathfrak{X}} \mathfrak{U} \quad : \quad \mathbf{B}_{/\mathfrak{X}} \to \mathbf{B}_{/\mathfrak{U}}$$

is cofinal. This formally implies the required assertion. □

Corollary 12.3.3. *The specialization map $\mathrm{sp}_X : X \to \langle X\rangle$ is a topological embedding when X is endowed with the canonical topology, i.e. is injective and the induced topology on X agrees with the canonical topology.*

Proof. The canonical topology has a basis consisting of quasi-compact admissible opens U, and we have $\mathrm{sp}_X^{-1}(\langle U\rangle) = U$. Thus the assertion follows from Lemma 12.3.2. □

To construct the desired equivalence $\mathrm{Sh}^{\mathrm{adm}} X \simeq \mathrm{Sh}\langle X\rangle$, we adopt the following notation.

- For an open subset $U \subseteq \langle X\rangle$, we write $U_* = U \cap X = \mathrm{sp}_X^{-1}(U)$, which is a subset of X, open for the canonical topology.
- For an open subset $U \subseteq X$ (for the canonical topology), we write U^* for the union of all open subsets $V \subseteq \langle X\rangle$ such that $V_* = U$ (equivalently, the largest such subset, by Corollary 12.3.3).

Note that we have $(U^*)_* = U$.

Lemma 12.3.4. *For $U \subseteq X$ a quasi-compact admissible open, we have $U^* = \langle U\rangle$.*

Proof. If $x \in U^* \setminus \langle U\rangle$, then picking a formal model $\mathfrak{U} \to \mathfrak{X}$ as in the proof of Lemma 12.3.2, so that $\langle U\rangle = \pi_{\mathfrak{X}}^{-1}(\mathfrak{U})$ then $\pi_{\mathfrak{X}}(x) \notin \mathfrak{U}$. Pick a closed point $y \in \mathfrak{X}$ in the closure of $\pi_{\mathfrak{X}}(x)$, so $y \notin \mathfrak{U}$ as well. Since $\mathrm{sp}_{\mathfrak{X}} : X \to \mathfrak{X}$ is surjective onto the set of closed points of $\mathfrak{X}$ ■, we can find $z \in X$ with $\mathrm{sp}_{\mathfrak{X}}(z) = z \notin \mathfrak{U}$. Consequently, $\mathrm{sp}_X(z) \notin \langle U\rangle$. ■ □

■ Reference to Chapter 11

■ Finish proof.

Lemma 12.3.5. *(a) Let* $U = \bigcup_{\alpha\in I} U_\alpha$ *be an admissible cover of an admissible open subset of* X*. Then* $U^* = \bigcup_{\alpha\in I} U_\alpha^*$.

(b) For every two admissible opens $U, V \subseteq X$*, we have* $(U \cap V)^* = U^* \cap V^*$.

(c) Let U*,* U_α *(*$\alpha \in I$*) be quasi-compact admissible opens of* X *such that* $U^* = \bigcup_{\alpha\in I} U_\alpha^*$*. Then* $U = \bigcup_{\alpha\in I}$ *is an admissible cover.*

Proof. aa □

With the three claims of the above lemma, we make the following constructions:

- Let $\mathscr{F}$ be a sheaf on $\langle X\rangle$. We define the presheaf $\varepsilon_*\mathscr{F}$ for the admissible topology on X by
$$(\varepsilon_*\mathscr{F})(U) = \mathscr{F}(U^*).$$
We check that $\varepsilon_*\mathscr{F}$ is a sheaf for the admissible topology: if $U = \bigcup_{\alpha\in I} U_\alpha$ is an admissible cover, then we have a diagram
$$\begin{array}{ccccc} (\varepsilon_*\mathscr{F})(U) & \longrightarrow & \prod_{\alpha\in I}(\varepsilon_*\mathscr{F})(U_\alpha) & \rightrightarrows & \prod_{\alpha,\beta\in I}(\varepsilon_*\mathscr{F})(U_\alpha\cap U_\beta) \\ \| & & \| & & \downarrow \\ \mathscr{F}(U^*) & \longrightarrow & \prod_{\alpha\in I}\mathscr{F}(U_\alpha^*) & \rightrightarrows & \prod_{\alpha,\beta\in I}\mathscr{F}(U_\alpha^*\cap U_\beta^*), \end{array}$$
where the bottom row is exact by Lemma 12.3.5(a) and the fact that $\mathscr{F}$ is a sheaf, and the right arrow is an isomorphism by Lemma 12.3.5(b), so the top row is exact and $\mathscr{F}$ is a sheaf.

- Let $\mathscr{G}$ be an admissible sheaf on X. We define a presheaf $\varepsilon_b^*\mathscr{G}$ on the base of $\langle X\rangle$ consiting of $U^* = \langle U\rangle$ for $U \subseteq X$ quasi-compact admissible open by
$$(\varepsilon_b^*\mathscr{G})(U^*) = \mathscr{G}(U).$$
We check that this is a sheaf on the base: if $U^* = \bigcup U_\alpha^*$, then $U = \bigcup U_\alpha$ is an admissible cover by Lemma 12.3.5(c). Moreover, $U_\alpha^* \cap U_\beta^* = (U_\alpha \cap U_\beta)^*$ by Lemma 12.3.5(b). It follows that $\varepsilon_b^*\mathscr{G}$ extends uniquely to a sheaf $\varepsilon_*\mathscr{G}$ on $\langle X\rangle$.

We arrive at the following more precise formulation of Theorem 12.3.1:

Proposition 12.3.6. *The constructions* $\mathscr{F} \mapsto \varepsilon_*\mathscr{F}$ *and* $\mathscr{G} \mapsto \varepsilon^*\mathscr{G}$ *define mutually inverse equivalences of categories between* $\mathrm{Sh}\langle X\rangle$ *and* $\mathrm{Sh}^{\mathrm{adm}} X$.

Proof. ■ □

■ Write something.

12.4 Examples of points of the Riemann–Zariski space

■

■ Example: types of points on the RZ space of the closed unit disc.

12.5 Points of the Riemann–Zariski space as valuations

The notion of a valuation on a field has a natural extension to general rings. Recall from §2.1 that for a linearly ordered abelian group Γ (written additively), we extend the addition from Γ to $\Gamma \cup \{\infty\}$ by $\infty + \gamma = \infty + \infty = \infty$ for $\gamma \in \Gamma$, and the order by $\infty \geq \gamma$ for all $\gamma \in \Gamma$.

Definition 12.5.1. Let A be a commutative ring. A *valuation* on A is a function

$$v: A \to \Gamma \cup \{\infty\}$$

where Γ is a linearly ordered abelian group (written additively) such that

(a) $v(0) = \infty$ and $v(1) = 0$,

(b) $v(xy) = v(x) + v(y)$,

(c) $v(x+y) \geq \min\{v(x), v(y)\}$.

The subset $\mathfrak{p}_v = v^{-1}(\infty)$ is a prime ideal of A, called the *support* of v, and the fraction field of $A/\mathfrak{p}_v$ is denoted by $k(v)$ and called the *residue field* of v. One defines equivalence of valuations as in the case of fields.

The valuation v factors through a valuation $\bar{v}: A/\mathfrak{p}_v \to \Gamma \cup \{\infty\}$ which extends to a unique valuation $\bar{v}: k(v) \to \Gamma \cup \{\infty\}$ on the residue field. We denote by $\mathcal{O}_v \subseteq k(v)$ the corresponding valuation ring. The pair $(\mathfrak{p}_v, \mathcal{O}_v)$ remembers v up to equivalence, and we have a bijection between equivalence classes of valuations on A and pairs consisting of a prime ideal of A and a valuation subring of its residue field.

Definition 12.5.2. Let A be a topological ring. A valuation $v: A \to \Gamma \cup \{\infty\}$ is *continuous* if $\{x \in A : v(x) > \gamma\}$ is open in A for every $\gamma \in \Gamma_v$.

As in the case of fields, it follows that $\{x \in A : v(x) \geq \gamma\}$ and $\{x \in A : v(x) = \gamma\}$ are open for all $\gamma \in \Gamma_v$. If $\Gamma_v \neq \{0\}$, then v is continuous if and only if $\{x \in A : v(x) \geq \gamma\}$ is open for all $\gamma \in \Gamma_v$.

Definition 12.5.3. Let A be an affinoid K-algebra. We denote by $\operatorname{Spa} A$ the set of equivalence classes of continuous valuations v on A such that $v(x) \geq 0$ for every powerbounded $x \in A$, and call it the *adic spectrum* of A.

In Huber's theory, this is denoted by $\operatorname{Spa}(A, A^\circ)$, but we use $\operatorname{Spa} A$ for brevity.

Lemma 12.5.4. *Let v be a valuation on an affinoid K-algebra A, and let $\mathscr{A} \subseteq A$ be an admissible $\mathcal{O}$-algebra with $A = \mathscr{A}[1/t]$. The following are equivalent:*

(a) $v \in \operatorname{Spa} A$,

(b) for every $x \in \mathscr{A}$, we have $v(x) \geq 0$, and for every $x \in \mathscr{A}$ with $v(x) < \infty$ there exists an $n \geq 1$ such that $v(t^n) > v(x)$.

Proof. For (a)⇒(b), we first note that $\mathscr{A} \subseteq A^\circ$, so $v \geq 0$ on $\mathscr{A}$. Second, since $\{t^n \cdot \mathscr{A}\}$ is a fundamental system of neighborhoods of $0 \in A$, continuity of v entails that for every $\gamma \in \Gamma_v$ there exists an n such that $v \geq \gamma$ on $t^n \cdot \mathscr{A}$. Taking $\gamma = v(x)$ gives the assertion.

For (b)⇒(a), recall that A° is the integral closure of $\mathscr{A}$ in A. Since valuation rings are integrally closed, $v \geq 0$ on $\mathscr{A}$ implies $v \geq 0$ on A°. For the continuity of v as in (b), we need to find for $\gamma \in \Gamma_v$ and n such that $v \geq \gamma$ on $t^n \cdot \mathscr{A}$. Write $\gamma = v(x)$ for $x \in A$, and write $x = t^n y$ with $y \in \mathscr{A}$, then the second part of (b) gives the assertion. □

Proposition 12.5.5. *Let X be a quasi-compact and quasi-separated rigid space, and let $x \in \langle X \rangle$. Then the local ring $\mathcal{O}_x^+$ has the property that every finitely generated ideal containing t^n for some $n \geq 1$ is invertible.*

Proof. We have $\mathcal{O}_x^+ = \varinjlim_{M(X)^{\mathrm{op}}} \mathcal{O}_{\mathfrak{X}, x_{\mathfrak{X}}}$, and since the $\mathfrak{X}$ are admissible, the rings $\mathcal{O}_{\mathfrak{X}, x_{\mathfrak{X}}}$ are t-torsion free. Thus t is a nonzerodivisor in $\mathcal{O}_x^+$.

Let $I \subseteq \mathcal{O}_x^+$ be a finitely generated ideal such that $t^n \in I$, say $I = (f_1, \ldots, f_r)$. By definition of inductive limit, the f_i come from some $\mathcal{O}_{\mathfrak{X}, x_{\mathfrak{X}}}$, and hence there exists an $\mathfrak{X}$ and a finitely generated ideal $I' \subseteq \mathcal{O}_{\mathfrak{X}, x_{\mathfrak{X}}}$ such that $I = I' \cdot \mathcal{O}_x^+$. We can also assume that $t^n \in I'$, and that there exists an admissible ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathfrak{X}}$ with $\mathcal{I}_{x_{\mathfrak{X}}} = I'$. Let $\mathfrak{X}' = \mathrm{Bl}_{\mathcal{I}} \mathfrak{X} \to \mathfrak{X}$ be the admissible blow-up of $\mathcal{I}$. Then $\mathcal{I} \cdot \mathcal{O}_{\mathfrak{X}'}$ is invertible, and hence so is $I' \cdot \mathcal{O}_{\mathfrak{X}', x_{\mathfrak{X}'}}$. Say $I' \cdot \mathcal{O}_{\mathfrak{X}', x_{\mathfrak{X}'}} = (g)$, then $I = g \cdot \mathcal{O}_x^+$ is principal. Now g divides t^n, which is a nonzerodivisor in $\mathcal{O}_x^+$, and hence g is a nonzerodivisor in $\mathcal{O}_x^+$, so I is invertible. □

Lemma 12.5.6 (See [9, 0 8.7.9] or [2, 1.9]). *Let A be a local ring containing an element $t \in \mathfrak{m}_A$ such that every finitely generated ideal containing t^n for some $n \geq 1$ is invertible, and let $J = \bigcap (t^n) \subseteq A$. Then $B = A[1/t]$ is a local ring whose maximal ideal is equal to J, and $V = A/J$ is a valuation subring of the residue field $k_B = B/J$. The ring A is the preimage of V under $B \to k_B$.*

Proof. First of all, (t) is invertible, thus t is a nonzerodivisor and $A \to B$ is injective. We also note that $tJ = J$, which implies that $J = t^{-1}J$, in particular J is an ideal in B. To show that B is local with maximal ideal J, it suffices to show that every element in $B \setminus J$ is invertible.

Let $f \in A \setminus J$, we will show that f is invertible in B. By assumption, we have $f \notin (t^n)$ for some $n \geq 1$. Since A is local, if an ideal $(f_1, \ldots, f_r) \subseteq A$ is principal, it must equal (f_i) for some i. Apply this to the ideal (f, t^n), we see that $(f, t^n) = (f)$ since $f \notin (t^n)$. Thus $t^n = fg$ for some $g \in A$, and f is invertible in B. Together with $J = t^{-1}J$, this shows that $B^\times = B \setminus J$.

Moreover, the above reasoning shows that every element of $B \setminus A$ has its inverse in A. This implies that $V = A/J$ is a valuation subring of k_B. □

Remark 12.5.7. Lemma 12.5.6 combined with Proposition 12.5.5 imply that the sheaf

$$\mathcal{O}_{\langle X \rangle} := \mathcal{O}_{\langle X \rangle}^+ \left[\frac{1}{t} \right]$$

is a sheaf of local rings on $\mathcal{O}_{\langle X \rangle}$. This sheaf corresponds to $\mathcal{O}_X$ under the equivalence $\mathrm{Sh}^{\mathrm{adm}} X \simeq \mathrm{Sh}\langle X \rangle$, while $\mathcal{O}_{\langle X \rangle}^+ \subseteq \mathcal{O}_{\langle X \rangle}$ corresponds to the subsheaf $\mathcal{O}_X^\circ = \{ f \in \mathcal{O}_X \, : \, |f| \leq 1 \}$ defined in Example 8.3.2. This gives $\langle X \rangle$ the structure of a "triple" (or a "doubly locally ringed space")

$$(\langle X \rangle, \mathcal{O}_{\langle X \rangle}, \mathcal{O}_{\langle X \rangle}^+).$$

As developed in [9], the category of such triples is a good target category for rigid-analytic spaces.

Let $X = \mathrm{Sp} A$ be an affinoid rigid space, let $\mathcal{A} \subseteq A$ be an admissible $\mathcal{O}$-algebra with $\mathcal{A}[1/t] = A$, and let $\mathfrak{X} = \mathrm{Spf} \mathcal{A}$, which is an admissible formal model of X. We construct a bijection between $\langle X \rangle$ and $\mathrm{Spa} A$.

Let $x \in \langle X \rangle$ and let $x_0 = \pi_{\mathfrak{X}}(x)$ be its image in $\mathfrak{X} = \mathrm{Spf} \mathcal{A}$. Then $\mathcal{O}_x^+$ satisfies the assumption of Lemma 12.5.5, and hence its t-adic completion $\widehat{V}_x$ is a valuation ring with $t \widehat{V}_x \neq \widehat{V}_x$. The composition

$$\mathcal{A} \to \mathcal{O}_{\mathfrak{X}, x} \to \mathcal{O}_x^+ \to \widehat{V}_x$$

induces a valuation v_x (up to equivalence) on $A = \mathcal{A}[1/t]$ which is non-negative on $\mathcal{A}$ and positive on t, and hence belongs to $\mathrm{Spa} A$ Lemma 12.5.4. This defines a map $x \mapsto v_x : \langle X \rangle \to \mathrm{Spa} A$.

Let now $v \in \mathrm{Spa} A$, let $\mathcal{O}_v \subseteq k(v)$ be the corresponding valuation ring, and let $\widehat{V}_v$ be its t-adic completion. This is a valuation ring because V_v is t-adically separated ■. The map

■ Explain this better.

$\mathscr{A} \to \mathscr{O}_v \to \widehat{V}_v$ induces a map of formal schemes

$$\operatorname{Spf} \widehat{V}_v \to \mathfrak{X} = \operatorname{Spf} \mathscr{A}.$$

If $\mathfrak{X}' \to \mathfrak{X}$ is an admissible blow-up, then the above map lifts uniquely to $\mathfrak{X}'$:

$$\begin{array}{ccc} & & \mathfrak{X}' \\ & \nearrow & \downarrow \\ \operatorname{Spf} \widehat{V}_v & \longrightarrow & \mathfrak{X}. \end{array}$$

Indeed, by the universal property of admissible blow-up, we only need to show that for an admissible ideal $I \subseteq \mathscr{A}$, the ideal $I \cdot \widehat{V}_v$ is invertible. This is true because $\widehat{V}_v$ is a valuation ring, $I \cdot \widehat{V}_v$ is finitely generated (because I is), and it is nonzero because $t^n \in I$ for some n and $t \neq 0$ in $\widehat{V}_v$. By the fact that such $\mathfrak{X}'$ are cofinal in $M(X)$ and universal property of the inverse limit $\langle X \rangle$, we obtain a map of locally ringed spaces

$$\operatorname{Spf} \widehat{V}_v \to \langle X \rangle.$$

Let $x_v \in \langle X \rangle$ be the image of the closed point of $\operatorname{Spf} \widehat{V}_v$ under this map. This defines a map $v \mapsto x_v \operatorname{Spa} A \to \langle X \rangle$.

Theorem 12.5.8. *The maps $x \mapsto v_x$ and $v \mapsto x_v$ described above establish a bijection*

$$\langle X \rangle \simeq \operatorname{Spa} A.$$

Proof. ■ □

■ Write something.

12.5.1 Berkovich spaces

Recall that if x and y are points of a topological space X, we say that y is a genneralization of x, or that x is a specialization of y, or that y specializes to x, and write $y \rightsquigarrow x$, if $x \in \overline{\{y\}}$. If X is T_0, this gives an order relation on X.

Definition 12.5.9. A spectral topological space X is *valuative* if for every $x \in X$, the set of all of its generalizations is totally ordered (by the specialization relation) and has a unique maximal element $x_{\max}$.

We denote by

$$\operatorname{sep}_X : X \to [X]$$

the quotient by the relation generated by $x \sim y$ if $y \rightsquigarrow x$ (that is, $x \sim x'$ if and only if $x_{\max} = x'_{\max}$), and call it the *universal separated quotient* of X.

Note that the set $X_{\max} = \{x \in X : x = x_{\max}\}$ maps bijectively onto $[X]$. However, this map will not be a homeomorphism in general.

Proposition 12.5.10. *Let X be a valuative topological space. Then the space $[X]$ is T_1, and every continuous map $X \to Y$ to a T_1 topological space Y factors uniquely through* sep_X.

Theorem 12.5.11. *Let X be a quasi-compact and quasi-separated rigid-analytic space over K. Then the following hold:*

(a) The Riemann–Zariski space $\langle X \rangle$ is a valuative topological space.

(b) Its universal separated quotient $[X] := [\langle X \rangle]$ is compact Hausdorff.

Definition 12.5.12 (Berkovich spectrum)**.** Let A be a Banach K-algebra with norm $|\cdot|_A$. A *multiplicative seminorm* on A is a map

$$|\cdot| : A \to [0,\infty)$$

such that $-\log|\cdot| : A \to \mathbf{R} \cup \{\infty\}$ is a valuation on A. We say that a multiplicative seminorm $|\cdot|$ on A is *bounded* if there exists a $C > 0$ such that $|f| \leq C \cdot |f|_A$ for all $f \in A$. We denote by $\mathscr{M}(A)$ the set of bounded multiplicative seminorms on A, endowed with the weakest topology such that the maps

$$|\cdot| \mapsto |f| \quad : \quad \mathscr{M}(A) \to [0,\infty)$$

are continuous for all $f \in A$. The topological space $\mathscr{M}(A)$ is called the *Berkovich spectrum* of A.

The notion of boundedness above depends only on the equivalence class of Banach norms. Since all norms on an affinoid K-algebra are equivalent, we can talk about $\mathscr{M}(A)$ for an affinoid K-algebra A without referring to a particular norm on A.

Berkovich showed that $\mathscr{M}(A)$ for an affinoid K-algebra A has remarkable properties, e.g. it is compact and Hausdorff and locally path connected, and its homotopy type is closely related to the combinatorial structure of special fibers of formal models of $X = \operatorname{Sp} A$.

Theorem 12.5.13. *Let A be an affinoid K-algebra and let $X = \operatorname{Sp} A$. For $|\cdot| \in \mathscr{M}(A)$, denote by $v_{|\cdot|} = -\log|\cdot| : A \to \mathbf{R} \cup \{\infty\}$ the associated valuation. Then:*

(a) For every $|\cdot| \in \mathscr{M}(A)$, the valuation $v_{|\cdot|}$ belongs to $\operatorname{Spa} A = \langle X \rangle$.

Warning: The map $\mathscr{M}(A) \to \langle X \rangle$ is usually not continuous.

(b) The composition

$$\mathscr{M}(A) \to \langle X \rangle \to [X]$$

is a homeomorphism.

■

■ Write something about the proofs of the above results.

Bibliography

[1] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*. Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[2] Ahmed Abbes. *Éléments de géométrie rigide. Volume I*, volume 286 of *Progress in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2010. Construction et étude géométrique des espaces rigides. [Construction and geometric study of rigid spaces], With a preface by Michel Raynaud.

[3] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[4] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry.

[5] Siegfried Bosch. *Lectures on formal and rigid geometry*, volume 2105 of *Lecture Notes in Mathematics*. Springer, Cham, 2014.

[6] Brian Conrad. Irreducible components of rigid spaces. *Ann. Inst. Fourier (Grenoble)*, 49(2):473–541, 1999.

[7] Brian Conrad. Several approaches to non-Archimedean geometry. In *p-adic geometry*, volume 45 of *Univ. Lecture Ser.*, pages 9–63. Amer. Math. Soc., Providence, RI, 2008.

[8] Jean Fresnel and Marius van der Put. *Rigid analytic geometry and its applications*, volume 218 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2004.

[9] Kazuhiro Fujiwara and Fumiharu Kato. *Foundations of rigid geometry. I*. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2018.

[10] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[11] M. Hochster. Prime ideal structure in commutative rings. *Trans. Amer. Math. Soc.*, 142:43–60, 1969.

[12] Jürgen Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1999.

[13] Martin Olsson. *Algebraic spaces and stacks*, volume 62 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2016.

[14] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2018.

[15] Yichao Tian. Introduction to rigid geometry. online lecture notes.

[16] Angelo Vistoli. Grothendieck topologies, fibered categories and descent theory. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 1–104. Amer. Math. Soc., Providence, RI, 2005.