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Introduction to non-Archimedean Geometry

Lecture course, Fall 2020

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NON-ARCHIMEDEAN or rigid-analytic geometry is an analog of complex analytic geometry over non-Archimedean fields, such as the field of p-adic numbers $\mathbf{Q}_p$ or the field of formal Laurent series $k(\!(t)\!)$. It was introduced and formalized by Tate in the 1960s, whose goal was to understand elliptic curves over a p-adic field by means of a uniformization similar to the familiar description of an elliptic curve over $\mathbf{C}$ as quotient of the complex plane by a lattice. It has since gained status of a foundational tool in algebraic and arithmetic geometry, and several other approaches have been found by Raynaud, Berkovich, and Huber. In recent years, it has become even more prominent with the work of Scholze and Kedlaya in p-adic Hodge theory, as well as the non-Archimedean approach to mirror symmetry proposed by Kontsevich. That said, we still do not know much about rigid-analytic varieties, and many foundational questions remain unanswered.

The goal of this lecture course is to introduce the basic notions of rigid-analytic geometry. We will assume familiarity with schemes.

Problem sets and other materials related to the course are available at

http://achinger.impan.pl/lecture20f.html

Our basic reference is the book *Lectures on Formal and Rigid Geometry* by Siegried Bosch. More references are found in the text.

This document uses the *tufte-book* LaTeXdocument class based on the design of Edward Tufte's books; some typesetting tricks were shamelessly stolen from Eric Peterson's lecture notes.

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1
Two interpretations of non-Archimedean geometry

THE p-ADIC NUMBERS $\mathbf{Q}_p$ are usually defined either as the completion of the rational numbers $\mathbf{Q}$ with respect to the p-adic absolute value

$$\left|\frac{a}{b}\right|_p = p^{\mathrm{ord}_p b - \mathrm{ord}_p a}, \tag{1.1}$$

or as the fraction field of the p-adic integers $\mathbf{Z}_p$ defined as the inverse limit

$$\mathbf{Z}_p = \varprojlim_n \mathbf{Z}/p^n\mathbf{Z}. \tag{1.2}$$

We can refer to (1.1) as the "metric" or "analytic" point of view, while (1.2) represents a more "algebraic" (or "formal") perspective. [1]

[1] We choose to ignore here the (rather useless) definition of p-adic numbers in terms of base-p digit expansions.

Both interpretations have their advantages and drawbacks. The metric approach is admittedly closer to one's intuition, and allows one to employ right away the powerful tools of topology and analysis. However, the topology of the p-adic numbers is quite pathological: $\mathbf{Q}_p$ is a totally disconnected topological space. This makes it difficult to proceed by analogy with real or complex analysis.

The algebraic approach allows us to reduce questions about $\mathbf{Q}_p$ to pure algebra over the rather simple rings $\mathbf{Z}/p^n\mathbf{Z}$. One therefore has commutative algebra and algebraic geometry at their disposal, which, in turn, allows one to more easily make sound and precise arguments. The downside: the relationship between objects over $\mathbf{Q}_p$ and over $\mathbf{Z}/p^n\mathbf{Z}$ can often be extremely convoluted.

TO ACHIEVE p-ADIC ENLIGHTENMENT, one needs a good grasp of both[2], as well as a means of switching between the two with ease. The goal of these lectures is to explain how to do p-adic geometry (or, more generally, non-Archimedean geometry[3]) by combining the analytic and the algebraic approaches. Roughly speaking, the first will be represented by Tate's notion of rigid analytic varieties, and the second by Raynaud's approach using formal schemes.

[2] *It seems as though we must use sometimes the one theory and sometimes the other, while at times we may use either. We are faced with a new kind of difficulty. We have two contradictory pictures of reality; separately neither of them fully explains the phenomena of light, but together they do.*

A. Einstein, L. Infeld *The Evolution of Physics*

[3] More precisely, *rigid (or rigid-analytic) geometry*, whose strange name we will justify later on.

WE WILL NOW GO BEYOND p-adic numbers and fix the notation which we will use most of the time. By a *non-Archimedean field* we mean a field K equipped with a non-Archimedean norm, which by definition is a function

$$|\cdot| \colon K \to [0, \infty)$$

such that

1. $|x| = 0$ if and only if $x = 0$,
2. $|xy| = |x| \cdot |y|$,
3. $|x + y| \leq \max(|x|, |y|)$.

We also assume that $|x| \neq 1$ for some $x \neq 0$ (i.e. that $|\cdot|$ is *nontrivial*), and that K is *complete* with respect to (the metric defined by) the norm. [4]

The third axiom, stronger than the triangle inequality $|x+y| \leq |x|+|y|$, is what makes the field non-Archimedean. It implies that the subset

$$\mathcal{O} = \{x \in K \text{ such that } |x| \leq 1\}$$

is a subring of K, called the *valuation ring*. It is local with maximal ideal

$$\mathfrak{m} = \{x \in K \text{ such that } |x| < 1\}.$$

We denote the residue field $\mathcal{O}/\mathfrak{m}$ by k.

Let $t \in m$ be a nonzero element.[5] Completeness of K is equivalent to the fact that the natural map

$$\mathcal{O} \to \varprojlim_n \mathcal{O}/t^n\mathcal{O}$$

is an isomorphism. The field K can be recovered as the fraction field of $\mathcal{O}$, in fact it is the localization $K = \mathcal{O}[\frac{1}{t}]$. The inverse limit above carries the inverse limit topology (with the $\mathcal{O}/t^n\mathcal{O}$ being equipped with the discrete topology), and the isomorphism is an isomorphism of topological rings if $\mathcal{O}$ has the metric topology induced by the norm $|\cdot|$. The topology on K is the unique one with respect to which $\mathcal{O}$ is an *open* subring. This implies that K is encoded *as a topological field* by the inverse system above.

The basic examples are complete discrete valuation fields (cdvf), which can be characterized as those K as above for which the maximal ideal $\mathfrak{m}$ is principal, so that $\mathcal{O}$ is a complete discrete valuation ring (cdvr) with maximal ideal $\mathfrak{m}$, residue field $k = \mathcal{O}/\mathfrak{m}$, and fraction field K. Naturally, our main example is

$$\mathcal{O} = \mathbf{Z}_p, \quad K = \mathbf{Q}_p, \quad \mathfrak{m} = (p), \quad k = \mathbf{F}_p,$$

and another one is the *Laurent series field* (over a base field k)[6]

$$\mathcal{O} = k[[t]] := \varprojlim_n k[t]/(t^n), \quad K = k((t)) := \mathcal{O}\left[\frac{1}{t}\right].$$

The characteristic of k is called the *residue characteristic* of K. If it is equal to the characteristic to K, we say that K is of *equal characteristic*, otherwise it is of *mixed characteristic*. In the latter case, K has characteristic zero. Thus $\mathbf{Q}_p$ and its normed extensions are of mixed characteristic, and the fields $k((t))$ have equal characteristic. In fact, every cdvf of equal characteristic is of the form $k((t))$.

In general, we will have to work with non-Archimedean fields K which are not cdvf's, in which case the valuation ring $\mathcal{O}$ is non-Noetherian. Indeed, it is often useful to consider K algebraically closed, while a complete discrete valuation field is never algebraically closed.[7]

[4] In some sources, non-Archimedean fields are not assumed to be complete and/or nontrivially valued.

[5] We call such a t a *pseudouniformizer*.

[6] Intuition: $k((t))$ is the field of functions on the "infinitesimal punctured disc"

$$\operatorname{Spec} k((t)) = \operatorname{Spec} k[[t]] \setminus \{t = 0\}.$$

[7] Consider a generator of $\mathfrak{m}$, i.e. an element of valuation one. Does it have a square root in K?

1.1 First example: the unit disc

The study of schemes begins with the case of the affine line over a base field k

$$\mathbf{A}^1_k = \operatorname{Spec} k[x],$$

from which one obtains $\mathbf{A}^n_k$ by direct product, then affine schemes of finite type over k by taking closed subschemes $X \subseteq \mathbf{A}^n_k$, and finally schemes locally of finite type over k by gluing. If k is algebraically closed, then by Hilbert's Nullstellensatz, closed points of $\mathbf{A}^1_k$ are in bijection with k.

In non-Archimedean geometry over an algebraically closed[8] non-Archimedean field K, similar role is played by the closed unit disc

[8] We make this assumption only for simplicity and only in this introduction.

$$\mathbf{D}^1_K = \{x \in K : |x| \leq 1\}.$$

Proceeding by analogy with scheme theory, we start with the algebra of functions on $\mathbf{D}^1_K$, which should consist of power series $f = \sum_{n\geq 0} a_n x^n$ which converge for $|x| \leq 1$. An easy check shows that a series in K converges if and only if its terms tend to zero. We conclude that we want the ring of "holomorphic functions" on $\mathbf{D}^1_K$ to be

$$K\langle X\rangle = \left\{ \sum_{n\geq 0} a_n X^n \in K[\![X]\!] \text{ with } a_n \to 0 \text{ as } n \to \infty \right\}.$$

Next, we would like to equip $\mathbf{D}^1_K$ with a *sheaf* of functions whose global sections is the above algebra $K\langle X\rangle$. The naive idea is to define, for an open subset $U \subseteq \mathbf{D}^1_K$, the ring $\mathcal{O}^{\text{wobbly}}(U)$ as the set of functions $U \to K$ which can be represented locally as a power series.

Indeed, this is trivially a sheaf, and we do obtain an injection

$$K\langle X\rangle \to \mathcal{O}^{\text{wobbly}}(\mathbf{D}^1_K).$$

However, this map is very far from being surjective. Indeed, $\mathbf{D}^1_K$ is highly disconnected, for example

$$\mathbf{D}^1_K = \{|x| = 1\} \cup \{|x| < 1\} \tag{1.3}$$

expresses $\mathbf{D}^1_K$ as a union of two disjoint open (!) subsets. The function $f \in \mathcal{O}(\mathbf{D}^1_K)$ equal to 1 on the first open and 0 on the second is not in the image of $K\langle X\rangle$. (This example justifies the adjective *wobbly*.) Clearly, something goes terribly wrong with analytic continuation in the nonarchimedean setting!

1.2 Tate's admissible topology on the unit disc

The first attempt at fixing this issue is due to Krasner, and is based on a non-Archimedean analog of Runge's theorem in complex analysis. A *Krasner analytic function* on $\mathbf{D}^1_K$ is a uniform limit of rational functions with no poles inside $\mathbf{D}^1_K$. This leads to a presheaf $\mathcal{O}$ for which $\mathcal{O}(\mathbf{D}^1_K) = K\langle X\rangle$, and which has the property that $\mathcal{O}(U)$ is a domain if U "should be" connected. Still, it is not a sheaf.

Let us explain, in a simple case, Tate's idea of fixing the issue. Consider the following covering of $\mathbf{D}^1_K$:

$$\mathbf{D}^1_K = \underbrace{\{|x| \le \rho\}}_{U} \cup \underbrace{\{\rho \le |x| \le 1\}}_{V} \tag{1.4}$$

with $0 < \rho < 1$, $\rho = |t|$ for some $t \in K$. The algebra of (Krasner analytic) functions $\mathcal{O}(U)$ on the smaller disc $U = \{|x| \le \rho\}$ consists of power series converging on this disc, i.e.

$$K\left\langle \frac{X}{t} \right\rangle = \left\{ f = \sum_{n\ge 0} a_n X^n \in K[[X]] : \lim_{n\to\infty} |a_n|\rho^n = 0 \right\}.$$

Similarly, for the annulus $V = \{\rho \le |x| \le 1\}$, $\mathcal{O}(V)$ consists of convergent Laurent series

$$K\left\langle X, \frac{t}{X} \right\rangle = \left\{ f = \sum_{n\in\mathbf{Z}} a_n X^n : \lim_{n\to\infty} |a_n| = 0, \lim_{n\to-\infty} |a_n|\rho^n = 0 \right\},$$

and functions $\mathcal{O}(U \cap V)$ on the intersection $U \cap V = \{|x| = \rho\}$ are

$$K\left\langle \frac{X}{t}, \frac{t}{X} \right\rangle = \left\{ f = \sum_{n\in\mathbf{Z}} a_n X^n : \lim_{|n|\to\infty} |a_n|\rho^n = 0 \right\}.$$

It turns out that we are lucky: the sequence

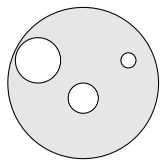
$$0 \to K\langle X\rangle \to K\left\langle \frac{X}{t} \right\rangle \times K\left\langle X, \frac{t}{X} \right\rangle \to K\left\langle \frac{X}{t}, \frac{t}{X} \right\rangle \tag{1.5}$$

is exact.[9] Thus $\mathcal{O}$ satisfies the sheaf condition with respect to the covering $U \cup V$.

[9] Check this!

TATE'S SOLUTION is now to identify a class of *admissible coverings* $U = \bigcup U_i$ of opens $U \subseteq \mathbf{D}^1_K$. For $U = \mathbf{D}^1_K$, these are the coverings admitting a *finite* refinement by subsets of the form

$$\{|x-a| \le |t|, |x-a_i| \ge |t_i|\}.$$



The covering (1.3) is not admissible in this sense, while (1.4) is. *Tate's acyclicity theorem* says that the presheaf $\mathcal{O}$ satisfies the sheaf condition for all admissible coverings. Exactness of (1.5) is a basic special case.

In particular, this implies that $\mathbf{D}^1_K$ is *quasi-compact* with respect to the admissible topology: every *admissible* cover admits a finite subcover. Moreover, it becomes *connected* in the sense that there is no admissible cover

$$U = \bigcup_{i\in I} U_i \cup \bigcup_{j\in J} V_j,$$

with both summands nonempty, such that $U_i \cap V_j = \emptyset$ for $(i,j) \in I \times J$, as reflected by the fact that $\mathcal{O}(\mathbf{D}^1_K) = K\langle X\rangle$ is a domain.

Formalizing the above requires the notion of a *G-topology* on a topological space X, which is the data of a class of *admissible* open subsets[10] and of *admissible* coverings of admissible open subsets satisfying some axioms. One has a natural notion of a sheaf with respect to a G-topology, which is a presheaf on the category of admissible opens which satisfies the

[10] For $\mathbf{D}^1_K$, we declare all open subsets admissible. The condition will however not be empty for $\mathbf{D}^n_K$ with $n > 1$.

sheaf condition with respect to admissible coverings. Thus $\mathcal{O}$ is a sheaf with respect to the admissible topology on $\mathbf{D}^1_K$.

In Tate's formalism, which we shall work out in the first part of the course, the basic geometric objects are *rigid-analytic varieties*. One uses as building blocks the *affinoid algebras*, which are quotients of the *Tate algebras*

$$K\langle X_1,\dots,X_r\rangle = \left\{ \sum_{n_1,\dots,n_r\geq 0} a_{n_1\dots n_r} X_1^{n_1}\dots X_r^{n_r} \,:\, a_{n_1\dots n_r} \to 0 \text{ as } n_1+\dots+n_r \to 0 \right\}.$$

To an affinoid algebra $A = K\langle X_1,\dots,X_r\rangle/I$ one associates the *affinoid* $\mathrm{Sp}A$. Its underlying topological space is the corresponding closed subset of

$$\mathbf{D}^r_K = \{(x_1,\dots,x_r)\in K^r \,:\, |x_i|\leq 1 \text{ for } i=1,\dots,r\}$$

cut out by the ideal I. One equips it with a G-topology (the admissible topology), and a sheaf of rings $\mathcal{O}$, similarly to the case of $\mathbf{D}^1_K$. A rigid-analytic variety is a topological space with a G-topology and a sheaf of rings with respect to that topology, which is locally (as a G-topologized space!) isomorphic to $\mathrm{Sp}A$ for some affinoid algebra A.

1.3 Raynaud's approach

The main drawbacks of Tate's theory are

- the admissible topology is counterintuitive and complicated to work with,
- and the underlying spaces do not have enough points (e.g. there exist nonzero abelian sheaves for the admissible topology whose stalk at every point is zero),
- one is bound to work over a fixed field; for a non-algebraic extension of nonarchimedean fields K'/K (e.g. $\mathbf{C}_p/\mathbf{Q}_p$) there is no map $\mathbf{D}^1_{K'} \to \mathbf{D}^1_K$,
- (why should there have to be a base field at all?)
- it is quite far from algebraic geometry (e.g. the opens are not defined by non-vanishing loci, but also be inequalities—not algebraic opens, but semi-algebraic opens).

There are several frameworks which address these issues in different ways, notably Huber's theory of *adic spaces*, Berkovich's theory of analytic spaces (usually called *Berkovich spaces*), and Raynaud's approach via *formal schemes*, worked out by Bosch and Lütkebohmert and recently developed further by Fujiwara–Kato and Abbes. In the second half of this course, we will become acquainted with all of these, mostly focusing on Raynaud's theory, as it is the closest to algebraic geometry.

THE STARTING POINT of Raynaud's theory is the following isomorphism (where $t\in K$ is a pseudouniformizer)

$$K\langle X\rangle = \left(\varprojlim_m \mathcal{O}[X]/(t^m)\right)\left[\frac{1}{t}\right]. \qquad (1.6)$$

We will prove this later, but you are welcome to try and check it yourself.

The isomorphism (1.6) expresses $K\langle X\rangle$ in terms of (0) the polynomial algebra $\mathcal{O}[X]$ through the algebraic operations of (1) t-adic completion, and (2) localization with respect to t. So, for example, if $\mathcal{O}$ is a discrete valuation ring, we immediately see that $K\langle X\rangle$ is Noetherian, because (0) the polynomial algebra $\mathcal{O}[X]$ is Noetherian, (1) the completion of a Noetherian ring with respect to an ideal is Noetherian, and (2) the localization of a Noetherian ring is Noetherian. (Unfortunately, our $\mathcal{O}$ will not always be Noetherian, so one needs to work harder.)

TO HAVE A GEOMETRIC PICTURE, we replace $\mathcal{O}[X]$ with its spectrum $X = \mathbf{A}^1_{\mathcal{O}}$. The projective system $\mathcal{O}/t^n\mathcal{O}[X]$ corresponds to a system of closed immersions

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots, \quad X_n = \mathbf{A}^1_{\mathcal{O}/t^{n+1}\mathcal{O}}.$$

Each of these immersions is defined a nilpotent ideal, and hence is a homeomorphism on the underlying spaces.

The above inductive system does not have a limit in the category of schemes. Instead, one can take its limit in the larger category of locally ringed spaces:

$$\mathfrak{X} = (|\mathfrak{X}|, \mathcal{O}_{\mathfrak{X}}) = \varinjlim_n X_n.$$

Since $|X_n| \hookrightarrow |X_{n+1}|$ are homeomorphisms, we can identify $|\mathfrak{X}|$ with $|X_0|$. Treating $\mathcal{O}_{X_n}$ as a sheaf on $|X_0| = |\mathfrak{X}|$, we have

$$\mathcal{O}_{\mathfrak{X}} = \varprojlim_n \mathcal{O}_{X_n} = \varprojlim_n \mathcal{O}_X/(t^{n+1}).$$

The locally ringed space $\mathfrak{X}$ is an example of a *formal scheme*, the *formal completion* of $X = \mathbf{A}^1_K$ with respect to the ideal $t\mathcal{O}_X$. In fact, in this context we could *define* formal schemes over $\mathcal{O}$ as systems of closed immersions $X_0 \hookrightarrow X_1 \hookrightarrow \cdots$ between $\mathcal{O}$-schemes, with X_n defined by the ideal $t^{n+1}\mathcal{O}_{X_{n+1}}$.

The final step, inverting t, is the hardest: in Raynaud's approach, one wants to define a rigid-analytic variety over $\mathcal{O}$ as the "generic fiber" of a formal scheme over $\mathcal{O}$. This is done purely formally by localizing the *category* of formal schemes over $\mathcal{O}$ with respect to *admissible blow-ups*, i.e. blowups along an ideal containing a power of t. In the words of Fujiwara and Kato, *rigid geometry is the birational geometry of formal schemes*.

1.4 *Why study rigid geometry?*

The goal of the course is not only to introduce the basic definitions and facts surrounding rigid-analytic varieties—we will see some important applications of the theory as well. I will now try to give a short preview without spoilers.

Disclaimer: There are many possible answers to the question above. The following is heavily influenced by my own perspective and expertise as an algebraic geometer interested in the topology of algebraic varieties.

The broad answer is:

Rigid geometry allows us to use methods of topology and analysis into an otherwise purely algebraic context.

For an explicit example, consider a complex algebraic curve, say a smooth plane curve X in $\mathbf{P}^2$ of degree d. As one learns in the basic algebraic geometry course, this curve has genus

$$g = \frac{(d-1)(d-2)}{2}.$$

Over the complex numbers, the underlying manifold (the *complex analytification*) of X is an oriented surface with g many handles. Can we make sense of the last sentence algebraically? The question sounds crazy at first: to begin with, the underlying topological space of X (with the Zariski topology) does not see the genus at all, so how can we try to decompose it into handles?

Rigid geometry allows us to break varieties into pieces and perform surgery.

The answer is to *degenerate* the curve until it breaks and becomes easier to manage.[11] Thus, let $\ell_1, \ldots, \ell_d$ be generically chosen linear forms on $\mathbf{P}^2$. If $\{f = 0\}$ is the homogeneous equation of our curve X, we consider the equation with an additional parameter t

[11] Can we study algebraic curves by putting them inside the Large Hadron Collider?

$$X_t = \{tf + (1-t)\ell_1 \cdot \ldots \cdot \ell_d = 0\} \quad \subseteq \quad \mathbf{P}^2_{k[t]}.$$

Thus $X_1 = X$, while X_0 is the union of d lines in $\mathbf{P}^2$ in general position.

The curve X_0, while much easier to understand than X, is singular. Its topology differs from that of X. The idea, made possible by rigid geometry, is to study the smooth fibers X_t which "infinitesimally close" to X_0. To make this precise, we first base change the above family to the field $K = k((t))$, obtaining a smooth algebraic curve X_K over K. Next, we turn it into a rigid-analytic variety $\mathscr{X} = (X_K)_{\mathrm{an}}$, its *rigid analytification*. It is cut out by the same equation in a rigid-analytic version of $\mathbf{P}^2_K$.

It turns out that $\mathscr{X}$ is "close enough" to X_0 that there exists a natural morphism of topological spaces (the *specialization map*)

$$\mathrm{sp}\colon |\mathscr{X}| \to |X_0|.$$

The preimage $U_i = \mathrm{sp}^{-1}(L_i)$ of the line $L_i = \{\ell_i = 0\} \subseteq |X_0|$ happens to be an *open* rigid subvariety of $\mathscr{X}$ which closely resembles a sphere with $d-1$ discs removed (the discs are the preimages of the points $L_i \cap L_j$ for $j \neq i$ under sp). This gives a combinatorial decomposition of $\mathscr{X}$ which one can use in place of the triangulation or handlebody decomposition on the complex analytification. For cubic curves (elliptic curves) one has the following picture:

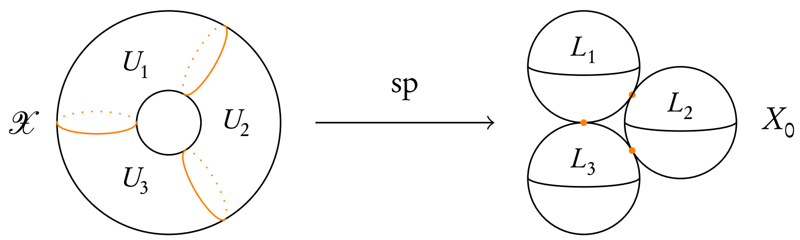


Figure 1.1: Intuitive picture of the specialization map ($d = 3$, so $g = 1$).

HERE ARE SOME EXAMPLES of serious applications of rigid geometry roughly along the above lines:

- Uniformization of curves and abelian varieties. (In fact, constructing a p-adic analytic analog of the expression of a complex elliptic curve as $\mathbf{C}$ modulo a lattice was Tate's original motivation for defining rigid-analytic varieties. We will see Tate's uniformization later in the course.)
- The approach to SYZ mirror symmetry proposed by Kontsevich.
- Raynaud's solution to Abhyankar's conjecture (constructing finite étale covers of $\mathbf{A}^1_{\mathbf{F}_p}$ with given Galois group).
- Study of moduli of curves (often done using tropical methods, which is philosophically similar).
- Semistable reduction.

Other extremely important applications belong to p-adic Hodge theory.

2
Non-archimedean fields

In this chapter, we learn some fundamentals about the kind of base fields we will work with — fields complete with respect to a nontrivial non-archimedean norm. We start with basic facts about general valuation rings; the extra generality is not needed for Tate's theory, but will prove useful later on.

In the appendix to this chapter, we review henselian local rings and Hensel's lemma.

2.1 Valuation rings and valuations

Definition 2.1.1. A subring $\mathcal{O}$ of a field K is a *valuation (sub)ring* of K if for every $x \in K^\times$, either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$.

The above condition implies that $K = \operatorname{Frac} \mathcal{O}$. This motivates the terminology: we will call a ring $\mathcal{O}$ a *valuation ring* if $\mathcal{O}$ is a domain and if it is a valuation ring of $K = \operatorname{Frac} \mathcal{O}$.

Lemma 2.1.2. *Every valuation ring is a local ring.*

Proof. It suffices to check that the set of non-units is closed under addition. If $x, y \in \mathcal{O}$ are nonzero non-units, then either $xy^{-1} \in \mathcal{O}$, in which case $x + y = y(xy^{-1} + 1)$ is a non-unit because y is a non-unit, or $yx^{-1} \in \mathcal{O}$, and we swap x and y. □

Lemma 2.1.3. *The relation*

$$x \leq y \quad \text{if} \quad yx^{-1} \in \mathcal{O} \tag{2.1}$$

induces a linear order on $\Gamma = K^\times/\mathcal{O}^\times$, *making* Γ *into a linearly ordered group.* [1]

[1] An *ordered abelian group* is an abelian group Γ with an order relation $\leq$ such that $a \leq b$ implies $a + c \leq b + c$. It is *linearly* or *totally* ordered if $\leq$ is a linear order.

Proof. First, if $x' = ux$ and $y' = vx$ with $u, v \in R^\times$, then $x \leq y \iff x' \leq y'$, so that $\leq$ induces a relation on $K^\times/\mathcal{O}^\times$. The fact that either $x \leq y$ or $y \leq x$ is the definition of a valuation ring. The rest is straightforward. □

The quotient homomorphism

$$K^\times \to K^\times/\mathcal{O}^\times$$

is a "valuation" on the field K, as we shall now define. First, we introduce the following convention: for an ordered abelian group Γ (written additively), we shall write $\Gamma \cup \{\infty\}$ for the ordered commutative monoid obtained by adding an element ∞ and declaring

$$\gamma \leq \infty \quad \text{and} \quad \gamma + \infty = \infty + \infty = \infty \quad (\gamma \in \Gamma).$$

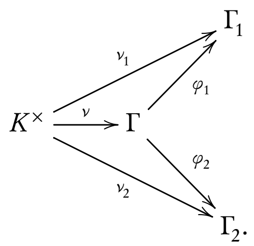
Definition 2.1.4. A *valuation* on a field K is a group homomorphism

$$v\colon K^\times \to \Gamma$$

into a linearly ordered group Γ (written additively, so that $v(xy) = v(x) + v(y)$), which, when extended to a map of monoids $v\colon K \to \Gamma \cup \{\infty\}$ by $v(0) = \infty$, satisfies

$$v(x+y) \geq \min\{v(x), v(y)\}.$$

The *value group* of a valuation $v\colon K^\times \to \Gamma$ is the image $v(K^\times)$. Thus v trivially induces a surjective valuation $v'\colon K^\times \to v(K^\times)$, and it is useful to identify v and v'. More generally, we will call two valuations $v_i\colon K^\times \to \Gamma_i$ ($i = 1, 2$) *equivalent* if there exists a third valuation $v\colon K^\times \to \Gamma$ and monotone homomorphisms $\varphi_i\colon \Gamma \to \Gamma_i$ ($i = 1, 2$) such that $v_i = \varphi_i \circ v$:



A valuation is *trivial* if it has trivial value group, i.e. $v(x) = 0$ for all $x \in K^\times$.

Proposition 2.1.5. *Let K be a field.*

(a) If $\mathcal{O} \subseteq K$ is a valuation ring and $\Gamma = K^\times/\mathcal{O}^\times$ is equipped with the linear order (2.1), *then the projection map $v\colon K^\times \to \Gamma$ is a valuation on K.*

(b) Conversely, if $v\colon K^\times \to \Gamma$ is a valuation, then

$$\mathcal{O} = \{x \in K \mid v(x) \geq 0\}$$

is a valuation ring of K, and its maximal ideal is $\mathfrak{m} = \{x \in K \mid v(x) > 0\}$.

(c) Constructions in (a) and (b) produce mutually inverse bijections

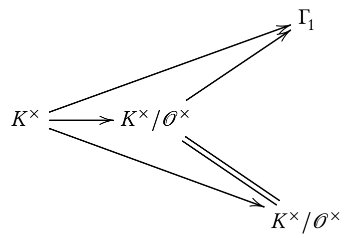
$$\{\textit{valuation rings of } K\} \simeq \{\textit{valuations on } K\}/\textit{equivalence}.$$

Proof. (a) We check the property $v(x+y) \geq \min\{v(x), v(y)\}$, which resembles the proof that a valuation ring is local. Let $x, y \in K^\times$, and suppose $xy^{-1} \in \mathcal{O}$, then

$$v(x+y) = v(y(xy^{-1}+1)) = v(y) + \underbrace{v(xy^{-1}+1)}_{\geq 0 \text{ since } xy^{-1}+1 \in \mathcal{O}} \geq v(y),$$

and similarly if $yx^{-1} \in \mathcal{O}$.

(b) Clearly for $x \in K$ either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$ and $\mathcal{O}$ is closed under multiplication. The fact that it is also closed under addition follows from $v(x+y) \geq \min\{v(x), v(y)\}$.



(c) Clearly, equivalent valuations define the same valuation ring. The only non-obvious assertion is that if $v_2\colon K^\times \to \Gamma_2 = K^\times/\mathcal{O}^\times$ is the valuation associated via (b) to the valuation ring $\mathcal{O}$ associated to a valuation $v_1\colon K^\times \to \Gamma_1$ via (a), then v_1 and v_2 are equivalent. We let $\Gamma = \Gamma_2 = K^\times/\mathcal{O}^\times$, φ_2 the identity, and $\varphi_2\colon \Gamma = K^\times/\mathcal{O}^\times \to \Gamma_1$ the map induced by v_1. □

2.2 *Valuations and norms*

If the value group is a subgroup of $\mathbf{R}$, one can turn a valuation into a "norm."

Definition 2.2.1. A *valuation of height one*[2] is a valuation $v\colon K^\times \to \mathbf{R}$.

Note that two valuations of height one $v_i\colon K^\times \to \mathbf{R}$ ($i = 1,2$) are equivalent if and only if $v_2(x) = c v_1(x)$ for some positive real c.[3]

Definition 2.2.2. A *nonarchimedean norm* on a field K is a map

$$|\cdot|\colon K \to [0,\infty)$$

such that

i. $|xy| = |x| \cdot |y|$,

ii. $|x| = 0$ if and only if $x = 0$,

iii. $|x+y| \leq \max\{|x|,|y|\}$.

Proposition 2.2.3. *Let K be a field.*

(a) If $v\colon K \to \mathbf{R}$ is valuation of height one, then[4]

$$|x| = \exp(-v(x))$$

(where $\exp(-\infty) = 0$) defines a nonarchimedean norm on K.

(b) Conversely, if $|\cdot|$ is a norm on K, then

$$v(x) = -\log|x|$$

(where $\log 0 = -\infty$) defines a valuation of height one. The corresponding valuation ring is the "closed ball" $\mathcal{O} = \{x \mid |x| \leq 1\}$.

(c) The constructions in (a) and (b) produce mutually inverse bijections

$$\{\textit{height one valuations on } K\} \simeq \{\textit{nonarchimedean norms on } K\}.$$

Proof. Clear. □

Proposition 2.2.4. *Let $|\cdot|$ be a nonarchimedean norm on a field K. Then*

$$d(x,y) = |x-y|$$

defines a metric on K, making K into a topological field. This metric and the induced topology have the following properties:

(a) Every triangle is isosceles, every point of an open ball is its center, and every two (open or closed) balls are either disjoint or one contains the other,

[2] This terminology is slightly nonstandard: what is usually meant by a valuation of height one is a nontrivial valuation whose value group *embeds* in $\mathbf{R}$.
More generally, the *height* (or *rank*) of a valuation is the order type of the set of all convex subgroups of the value group, (linearly) ordered by inclusion, where a subgroup $A \subseteq \Gamma$ is *convex* if $a \leq x \leq b$ and $a,b \in A$ implies $x \in A$.
As it turns out, and is easy to show, this is just the Krull dimension of the corresponding valuation ring $\mathcal{O}$.

[3] Exercise 3 on Problem Set 1.

[4] The base e of the exponential is of course an arbitrary choice. Sometimes there exists a more natural one. For example, if K is p-adic, i.e. $|p| < 1$ for a prime p, then one usually considers the norm

$$|x| = p^{-v(x)}.$$

(b) *The open ball* $\{|x-a|<\rho\}$, *the closed ball* $\{|x-a|\leq\rho\}$, *and the sphere* $\{|x-a|=\rho\}$ *are both open and closed for* $\rho>0$. *In particular, the valuation ring* $\mathcal{O}=\{|x|\leq 1\}\subseteq K$ *is an open subring.*

(c) *The topological space* K *is totally disconnected,*

(d) *Suppose that* K *is complete (every Cauchy sequence converges). A series* $\sum_{n=0}^{\infty} a_n$ *with* $a_n\in K$ *converges if and only if* $\lim a_n=0$.

Proof. Continuity of addition, multiplication, and inverse is clear and left to the reader.

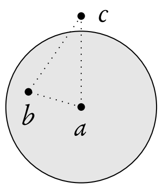
(a) The key observation is that if $|x|>|y|$, then $|x-y|=\max\{|x|,|y|\}=|x|$. Indeed, we have

$$|x|=|y+(x-y)|\leq\max\{|y|,|x-y|\}\leq\max\{|y|,|x|,|y|\}=|x|,$$

so the inequalities are equalities, implying $|x-y|=|x|$. Similarly, if $|y|>|x|$ then $|x-y|=|y|$, thus in general two of the numbers $|x|,|y|,|x-y|$ have to be equal.

If a triangle has vertices a,b,c, apply the above to $x=c-a$, $y=c-b$ to see that it is isosceles, with two longest sides being equal.

Now consider an open ball $B(a,\rho)=\{|x-a|<\rho\}$ and let $b\in B$, i.e. $|b-a|<\rho$. If $c\in K$, then consider the triangle with vertices a,b,c. The above observation shows that $|c-a|\geq\rho$ if and only if $|c-b|\geq\rho$, showing $B(a,\rho)=B(b,\rho)$.



If two open balls B and B' intersect at a point b, then taking b as the center of both balls shows that one is contained in the other.

(b) The open ball is of course open, and the closed ball is the union of the open ball and the sphere. It suffices to treat the sphere $S=\{|x|=\rho\}$ (centered at zero for simplicity). Let $a\in S$; we claim that the open ball $\{|x-a|<\rho\}$ is contained in S. Indeed, if $|x-a|<\rho$ then $|x|=|a+(x-a)|$ and since $|x-a|<\rho=|a|$, we have $|x|=|a|=\rho$, so $x\in S$.

(c) Let $S\subseteq K$ be a subset and let $a,b\in S$ be two distinct points, $\rho=|a-b|>0$. Then

$$S=(S\cap\{|x-a|<\rho/2\})\cup(S\cap\{|x-a|\geq\rho/2\})$$

expresses S as a sum of two disjoint and non-empty open subsets. Thus S cannot be connected if it has more than one point.

(d) Clearly if $\sum a_n$ converges then $\lim a_n=0$. Conversely, suppose $\lim a_n=0$; we check that $b_n=a_1+\cdots+a_n$ forms a Cauchy sequence. Let $\varepsilon>0$, and let N be such that $|a_n|<\varepsilon$ for $n\geq N$. Then for $m>n>N$

$$|b_m-b_n|=|a_{n+1}+\cdots+a_m|<\max\{|a_{n+1}|,\ldots,|a_m|\}<\varepsilon. \qquad\square$$

2.3 Geometric examples of valuations

This section is a bit of a digression, but will become important later in the course.

Long long time ago, before schemes were invented by Grothendieck, varieties were studied (or even defined) using valuations on their function fields. E.g. Zariski's proof of resolution of singularities on surfaces heavily relied on the classification of valuations on their function fields. We will see some of these below.

Example 2.3.1. Let R be a Dedekind domain with field of fractions K, and let $\mathfrak{m} \subseteq R$ be a maximal ideal. Standard examples:

- $R = \Gamma(X, \mathcal{O}_X)$ for X a smooth affine algebraic curve, with $\mathfrak{m}$ corresponding to a closed point $x \in X$,
- $R = \mathcal{O}_K$ the ring of integers in a number field K, e.g. $R = \mathbf{Z}[i]$.

The local ring $\mathcal{O} = R_{\mathfrak{m}}$ is a discrete valuation subring of K. The corresponding valuation on K is $v(x) = \max\{k : x \in \mathfrak{m}^k\}$. Every valuation on K which is trivial on k is equivalent to exactly one of these. [5]

[5] Sound familiar? [2, Chapter I 6]

The remaining examples deal valuations on function fields of surfaces over a base field k, where the situation is much more complicated, essentially due to the existence of non-trivial blowups. [6] We only consider valuations whose restriction to k is trivial.

[6] See [2, Exercise II 4.12].

Example 2.3.2 (Divisorial valuation)**.** Let X be a normal surface with field of rational functions K and let $D \subseteq S$ be a prime divisor. Then [reference Hartshorne] D defines a function "order of zero along D"

$$v_D : K = k(S) \to \mathbf{Z} \cup \{\infty\}$$

which is a valuation. The corresponding valuation ring is $\mathcal{O}_{X,\xi}$ where ξ is the generic point of D. Its residue field is $k(D)$, the function field of D.

Example 2.3.3 (Valuation of height two)**.** In the situation of Example 2.3.2, let $p \in D$ be a closed point at which D is regular. Then x defines a valuation v_p on $k(D)$ as in Example 2.3.1. We can combine the valuations v_D on $K = k(S)$ and v_p on $k(D)$ into a height two valuation

$$v_{D,p} : K \to \mathbf{Z}^2_{\text{lex}} \cup \{\infty\},$$

where $\mathbf{Z}^2_{\text{lex}}$ is $\mathbf{Z}^2$ with the lexicographic order ($(x,y) \geq (x',y')$ if $x > x'$ or $x = x'$ and $y \geq y'$). To define $v_{D,p}$, we pick a uniformizer (generator of the maximal ideal) $\pi \in \mathcal{O}_{X,\xi} = \mathcal{O}_{v_D}$ without zero or pole at p and set

$$v_{D,p}(f) = (v_D(f), v_p(g)), \quad g = (\pi^{-v_D(f)} f)|_\xi,$$

where the restriction makes sense because $v_D(g) = 1$, so $g \in \mathcal{O}_{v_D}$.

The valuation ring $\mathcal{O}_{v_{D,p}}$ consists of rational functions with no pole along D and whose restriction to D has no pole at p. It has three prime ideals, is of Krull dimension two, and is non-Noetherian. Its residue field is k. See Figure 2.1 for the monoid of monomials in $\mathcal{O}_{v_{D,p}}$ for $S = \mathbf{A}^2$.

Example 2.3.4 (Valuations from formal curve germs)**.** Let again S be a normal surface with function field K, and let

$$\gamma : \operatorname{Spec} k[[t]] \to S$$

be a morphism of schemes (a "formal curve germ"). We say that γ is *nonalgebraic* if its image is not contained in a proper closed subscheme of S, equivalently if γ maps the generic point $\operatorname{Spec} k((t))$ of $\operatorname{Spec} k[[t]]$ to the generic point $\eta = \operatorname{Spec} K$ of S. [7] The composition of γ^* with the standard

[7] There is plenty of nonalgebraic curve germs on an algebraic surface. For example, consider $S = \operatorname{Spec} k[x,y]$ the affine plane and γ defined by

$$\gamma^*(x) = t, \quad \gamma^*(y) = \exp t = \sum_{n \geq 0} \frac{t^n}{n!}.$$

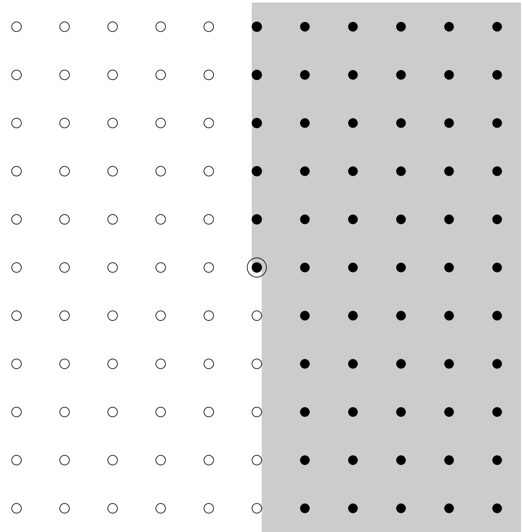


Figure 2.1: In Example 2.3.3, consider $S = \mathbf{A}^2$ with coordinates x, y, the divisor $D = \{x = 0\} \subseteq S$, and the point $p = \{y = 0\} \subseteq D$. The figure shows the monoid consisting of all $(m, n) \in \mathbf{Z}^2$ for which $v(x^m y^n) \geq 0$. Can you see why this monoid is not finitely generated? This is related to the fact that the valuation ring is non-Noetherian.

valuation on $k((t))$ gives a height one valuation

$$v_\gamma : K \to k((t)) \to \mathbf{Z} \cup \{\infty\}$$

with residue field k.

Example 2.3.5 (Height one valuation with dense value group)**.** Suppose that $K = k(x, y)$. Let λ be an irrational real number. Define the weight function on monomials in x and y by

$$\text{weight}_\lambda(x^m y^n) = m + \lambda n \quad \in \quad \mathbf{R}.$$

Define the valuation $v_\lambda : K \to \mathbf{R} \cup \{\infty\}$ by first defining it on polynomials:

$$v_\lambda\left(\sum_{m,n \geq 0} a_{mn} x^m y^n\right) = \min\{\text{weight}_\lambda(x^m y^n) : a_{mn} \neq 0\}$$

and extending to $k(x, y)$ by $v_\lambda(f/g) = v_\lambda(f) - v_\lambda(g)$. This gives a valuation on K which has height one but whose value group $\mathbf{Z} \oplus \lambda\mathbf{Z} \simeq \mathbf{Z}^2$ is dense in $\mathbf{R}$. See Figure 2.2 for the monoid of monomials in the valuation ring.

Remark 2.3.6. The valuation v_λ in Example 2.3.5 can be thought of as the valuation of the type considered in Example 2.3.4 induced by the "formal curve germ"

$$t \mapsto (t, t^\lambda).$$

In fact, for $\lambda' = a/b$ rational with $(a, b) = 1$, we can define the curve germ

$$\gamma_{a,b} : \operatorname{Spec} \mathbf{C}[[t]] \to \mathbf{A}^2_{x,y}, \quad \gamma^*_{a,b}(x) = t^b, \quad \gamma^*_{a,b}(x) = t^a.$$

Let $v_{a,b} = \frac{1}{b} v_{\gamma_{a,b}}$ where $\gamma_{a,b}$ is the valuation associated to the curve germ as in Example 2.3.4. If $a_n/b_n \to \lambda$, then the corresponding valuations v_{a_n, b_n} converge pointwise to v_λ.

2.4 *Nonarchimedean fields*

Definition 2.4.1. A *nonarchimedean field*[8] is a field K equipped with a nontrivial nonarchimedean norm $|\cdot|$ with respect to which it is complete.

[8] For many authors, "nonarchimedean field" is simply a field with a nonarchimedean norm.

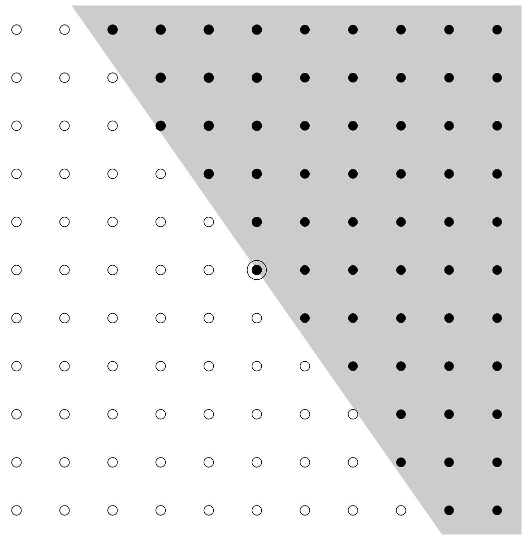


Figure 2.2: The monoid of all $(m,n)\in\mathbf{Z}^2$ for which $v(x^m y^n)\geq 0$ (Example 2.3.5). The boundary of the gray area is the line with slope $-1/\lambda$

$$x+\lambda y=0.$$

Since $\lambda\notin\mathbf{Q}$, this line contains no nonzero lattice points.

Proposition 2.4.2. *Let K be a field endowed with a nontrivial nonarchimedean norm $|\cdot|$. The ring operations on K extend uniquely to the completion $\widehat{K}$ of K with respect to $d(x,y)=|x-y|$, making $\widehat{K}$ into a nonarchimedean field.*

Definition 2.4.3. Let K be a field endowed with a nonarchimedean norm $|\cdot|$. A *pseudouniformizer* is an element $t\in K$ with $0<|t|<1$.[9]

Thus $|\cdot|$ is nontrivial if and only if K admits a pseudouniformizer.

[9] In other words, t is a *topologically nilpotent unit*, where topologically nilpotent means that $|t^n|\to 0$.

Proposition 2.4.4. *Let K be a field endowed with a nontrivial nonarchimedean norm $|\cdot|$, and let $t\in K$ be a pseudouniformizer. Let $\mathcal{O}=\{x\in K\,|\,|x|\leq 1\}$ be the valuation ring. Then K is complete (i.e. K is a nonarchimedean field) if and only if $\mathcal{O}$ is t-adically complete and separated, i.e. if the natural map*

$$\pi\colon \mathcal{O}\to\varprojlim_n \mathcal{O}/t^n\mathcal{O}$$

is an isomorphism. In this case, the map π is a homeomorphism, where the target is endowed with the inverse limit topology where each $\mathcal{O}/t^n\mathcal{O}$ is given the discrete topology.

Warning: if K is not discretely valued, then $\mathcal{O}$ will not be a complete local ring! In that case, the maximal ideal of $\mathcal{O}$ satisfies $\mathfrak{m}^2=\mathfrak{m}$, and hence $\mathcal{O}/\mathfrak{m}^n=k$ for all n, so that $\widehat{\mathcal{O}}\simeq k$. This is why we need to work with pseudouniformizers.

Proof. Set $\rho=|t|$; we have $0<\rho<1$. First, we note that

$$t^n\mathcal{O}=\{x\in K\,:\,|x|\leq\rho^n\}.$$

The kernel of π is $\bigcap_{n\geq 0}t^n\mathcal{O}=\{|x|\leq 0\}=\{0\}$. Thus π is always injective.

An element $\bar{f}$ of the inverse limit is a compatible system $(\bar{f}_n\in\mathcal{O}/t^n\mathcal{O})$. Let $f_n\in\mathcal{O}$ be elements mapping to $\bar{f}_n\in\mathcal{O}/t^n\mathcal{O}$. We claim that (f_n) is a Cauchy sequence. Indeed, we have $f_n-f_m\in t^n\mathcal{O}$ for $m>n$, so $|f_n-f_m|\leq\rho^n$ for $m>n$. Thus if K is complete, then (f_n) has a limit $f\in\mathcal{O}$. Now for every n, we have

$$|f-f_n|=|f_n-f_m|\leq\rho^n\quad\text{for}\quad m\gg 0,$$

which shows that $f-f_n\in t^n\mathcal{O}$. Thus $\pi(f)=\bar{f}$, i.e. π is surjective if K is complete.

Conversely, suppose that π is surjective. We will show that $\mathcal{O}$ is complete with respect to $|\cdot|$ (this easily implies that K is complete). Let $(f_n) \in \mathcal{O}$ be a Cauchy sequence. For every m, the images of f_n in $\mathcal{O}/t^m\mathcal{O}$ have to stabilize for $n \gg 0$. Let $\bar{f}_m \in \mathcal{O}/t^m\mathcal{O}$ be the stable value (i.e. $\bar{f}_m = \lim_n (f_n \bmod t^m)$ for the discrete topology on $\mathcal{O}/t^n\mathcal{O}$). It is easy to see that $\bar{f} = (\bar{f}_m)$ is an element of the inverse limit of $\mathcal{O}/t^n\mathcal{O}$. Let $f \in \mathcal{O}$ be an element with $\pi(f) = \bar{f}$, then $f = \lim f_n$.

The claim about the topologies follows from the fact that $t^n\mathcal{O} = \{|x| \leq \rho^n\}$ is a basis of neighborhoods of zero in $\mathcal{O}$. □

2.5 *Extensions of nonarchimedean fields*

The treatment here follows [1, Appendix A] and [3, II §4 and §6].

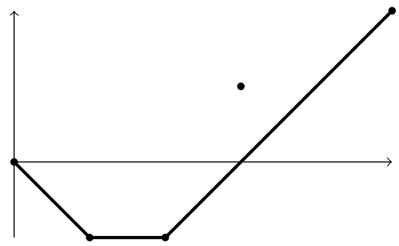


Figure 2.3: Newton polygon of the polynomial

$$1 + \pi^{-1}X - \pi^{-1}X^2 + \pi X^3 + \pi^2 X^5$$

Theorem 2.5.1. *Let K be a nonarchimedean field and let L/K be a finite extension. Then there exists a unique norm $|\cdot|$ on L extending K. The field L endowed with this norm is a nonarchimedean field.*

For $f = \sum_{i=0}^n a_i x_i \in K[X]$, we define its *Newton polygon* $\mathrm{NP}(f)$ as the lower convex envelope of the set $\{(0, v(a_0)), \ldots, (n, v(a_n))\}$ in $\mathbf{R}^2$. Its basic property is that $\mathrm{NP}(fg) = \mathrm{NP}(f) + \mathrm{NP}(g)$ (Minkowski sum, i.e. sort the segments of both polygons by slope and concatenate). In particular, if f is reducible, then $\mathrm{NP}(f)$ contains a point of the form (m, γ) with $0 < m < \deg f$ an integer and γ an element of the value group. One form of Hensel's lemma[10] states the converse:

Lemma 2.5.2 (Variant of Hensel's lemma)**.** *Let $f \in K[X]$ be a nonzero polynomial with $f(0) \neq 0$. Then f is irreducible if and only if $\mathrm{NP}(f)$ is a single segment without interior points of the form (m, γ) with $m \in \mathbf{Z}$ and $\gamma \in v(K^\times)$.*

[10] In the appendix to this lecture, we shall discuss different formulations of Hensel's lemma.

Proposition 2.5.3. *In the situation of Theorem 2.5.1, let $\mathcal{O} = \{|x| \leq 1\}$ be the valuation ring of K. An element $x \in L$ is integral over $\mathcal{O}$ if and only if $\mathrm{Nm}_{L/K}(x) \in \mathcal{O}$.*

Proof. Let $f \in K[X]$ be the minimal polynomial of x. Since f is irreducible, by Lemma 2.5.2 its Newton polygon has to be the line segment with endpoints $(\deg f, 0)$ and $(0, c)$ where $c = v(a_0)$ is the valuation of the constant term of f (Figure 2.4). But $c = (-1)^n \mathrm{Nm}_{L/K}(x)$, so if $\mathrm{Nm}_{L/K}(x) \in \mathcal{O}_K$ then $\mathrm{NP}(f)$ lies entirely above the line $y = 0$, which implies that $f \in \mathcal{O}[X]$, so that x is integral over $\mathcal{O}$.

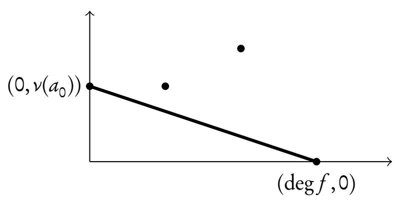


Figure 2.4: Newton polygon of an irreducible monic polynomial f (Proof of Proposition 2.5.3)

Conversely, if x is integral, then in fact its minimal polynomial f belongs to $\mathcal{O}[X]$; in particular, $\mathrm{Nm}_{L/K}(x) = (-1)^{\deg f} f(0) \in \mathcal{O}$. To see this, let $g \in \mathcal{O}[X]$ be monic with $g(x) = 0$. We have $g = fh$ for some (also monic) $h \in K[X]$. Then $\mathrm{NP}(g) = \mathrm{NP}(f) + \mathrm{NP}(h)$ lies above the line $y = 0$ and ends on it (because it is monic), and hence all of its slopes are non-positive. However, $\mathrm{NP}(f)$ is a single segment (connecting $(0, c)$ and $(\deg f, 0)$), and its slope is one of the slopes of $\mathrm{NP}(g)$ and hence is non-positive. Thus $c \geq 0$, i.e. $f \in \mathcal{O}[X]$. □

Proof of Theorem 2.5.1. Let $\mathcal{O} = \{|x| \leq 1\} \subseteq K$ be the valuation ring of K and let $\mathcal{O}' \subseteq L$ be the integral closure of $\mathcal{O}$ inside L. By Proposition 2.5.3,

$x \in \mathcal{O}'$ if and only if $|\mathrm{Nm}_{L/K}(x)| \le 1$. Since the norm is multiplicative, this shows that $\mathcal{O}'$ is a valuation ring of L. Moreover, $\mathcal{O}' \cap K = \mathcal{O}$ because $\mathcal{O}$ is integrally closed.[11]

Define $|x| = |\mathrm{Nm}_{L/K}(x)|^{1/d}$ for $x \in L$, where $d = [L : K]$. This restricts to the norm on K, is multiplicative, and $|x| \ne 0$ for $x \ne 0$. To show $|x + y| \le \max\{|x|, |y|\}$, we use the fact that $\{|x| \le 1\} = \mathcal{O}'$ is a valuation ring.

If $|\cdot|'$ is some other norm extending $|\cdot|$ to L, then since the corresponding valuation ring $\{|x|' \le 1\}$ is integrally closed, it contains $\mathcal{O}'$. This implies that $|\cdot| \le |\cdot|'$, and by Exercise BLAH, we have $|\cdot|' = |\cdot|^c$ for some constant c. But $c = 1$ since the two agree on K. □

[11] Easy exercise: show that every valuation ring is integrally closed.

Theorem 2.5.4 (Krasner). *Let K be a nonarchimedean field, and let $\overline{K}$ be an algebraic closure of K, which we endow with the unique extension of $|\cdot|$. Let $\widehat{\overline{K}}$ denote the completion of $\overline{K}$ with respect to this norm. Then $\widehat{\overline{K}} = \overline{K}^\wedge$ is algebraically closed.*

Proof. Let L be a finite extension of $\widehat{\overline{K}}$. By Theorem 2.5.1, there exists a unique norm on L extending the norm on $\widehat{\overline{K}}$ and L is complete with respect to that norm. To show $L = \widehat{\overline{K}}$, it therefore suffices to prove that $\widehat{\overline{K}}$ is dense in L.

Let $x \in L$ and let $1 > \rho > 0$. We shall find a $y \in \widehat{\overline{K}}$ with $|x - y| < \rho$. Without loss of generality, we may assume that $|x| \le 1$. Let $f = \sum_{i=0}^n a_i X^i \in \widehat{\overline{K}}[X]$ be its minimal polynomial (with $a_n = 1$). Since $\overline{K}$ is dense in $\widehat{\overline{K}}$, we can find $b_i \in \overline{K}$ ($i = 0, \ldots, n$) with $|a_i - b_i| < \rho$ (and again $b_n = 1$). This implies that

$$|g(x)| = |g(x) - f(x)| = \left| \sum_{i=0}^n (a_i - b_i) x^i \right| < \rho.$$

Now, the polynomial $g = \sum_{i=0}^n b_i X^i$ splits completely in $\overline{K}$:

$$g = \prod_{i=1}^n (X - y_i), \quad y_1, \ldots, y_n \in \overline{K}.$$

Evaluating at x and taking absolute value, we obtain

$$\rho > |g(x)| = \prod_{i=1}^n |x - y_i|.$$

Therefore one of the factors is less than ρ. □

2.A Henselian rings

Hensel's lemma played an important in the proof of Theorem 2.5.1. The first goal of this section is to elucidate its role by introducing the notion of a *henselian local ring*. Roughly speaking, it is a local ring in which the assertion of Hensel's lemma holds. There are however many equivalent characterizations of this class of local rings, reviewed in Proposition 2.A.1 below, and the reader familiar with the étale topology will surely appreciate the topological flavor of some of them. The second goal is to prove Hensel's lemma in its general form: *a local ring complete with respect to a* $\mathfrak{m}$*-primary ideal is henselian.*

Our treatment follows the Stacks Project [4, Tag 04GE] .

The ultimate reference is Raynaud's book *Anneaux locaux henseliens*.

Proposition 2.A.1. *Let A be a local ring with maximal ideal $\mathfrak{m}$. We set $k = A/\mathfrak{m}$, $x = \operatorname{Spec} k$, $X = \operatorname{Spec A}$, $i\colon x \to X$ the inclusion. The following conditions are equivalent:*

[4, Tag 04GG]

(a) If $f \in A[T]$ is monic and $t_0 \in k$ is a root of $\bar{f} = f$ mod $\mathfrak{m} \in k[T]$ such that $\bar{f}'(t_0) \neq 0$, then there exists a unique root $t \in A$ of f such that t mod $\mathfrak{m} = t_0$.

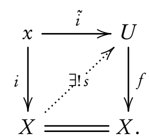
(b) If $f \in A[T]$ is monic and $\bar{f} = gh$ is a factorization of $\bar{f} = f$ mod $\mathfrak{m} \in k[T]$ with $g, h \in k[T]$ coprime, then there exists a factorization $f = \tilde{g}\tilde{h}$ with $\tilde{g}, \tilde{h} \in A[T]$ such that $\tilde{g}$ mod $\mathfrak{m} = g$, $\tilde{g}$ mod $\mathfrak{m} = g$, and $\deg \tilde{g} = \deg g$.

(c) Every finite A-algebra is a product of local rings.

(d) For every étale A-algebra B and every prime $\mathfrak{p} \subseteq B$ lying over $\mathfrak{m}$ and such that $k(\mathfrak{p}) = k$, there exists a section $s\colon B \to A$ of $A \to B$ with $\mathfrak{p} = s^{-1}(\mathfrak{m})$.

(e) For every étale morphism $f\colon U \to X$ and every lifting $\tilde{i}\colon x \to U$ of i (i.e. $i = f \circ \tilde{i}$) there exists a unique section $s\colon X \to U$ such that $s \circ i = \tilde{i}$.[12]

[12] Useful to picture this condition as a lifting problem:



Proof. Maybe I'll write something here later. □

Definition 2.A.2. (a) A local ring A is *henselian* if the equivalent conditions of Proposition 2.A.1 hold.

(b) A local ring A is *strictly henselian* if it is henselian and its residue field k is separably closed.[13]

(c) A valued field (K, v) is *henselian* if the valuation ring $\mathcal{O} = \{x \mid v(x) \geq 0\}$ is henselian.

[13] Equivalently: every étale cover of $\operatorname{Spec} A$ admits a section.

Remark 2.A.3. Condition (d) of Proposition 2.A.1 allows one to construct the *henselization* of a local ring A as the direct limit

$$A^h = \varinjlim_{(B,s)\in\mathscr{C}_A} B$$

where $\mathscr{C}_A$ is the category of pairs (B, s) with B an étale A-algebra and $s\colon B \to k$ a homomorphism extending $A \to k$. (This category is filtering and essentially small.)

Universal property: $A \to A^h$ is a local homomorphism into a henselian local ring which is initial among such (in the category of local rings and local homomorphisms).

Similarly, given a separable closure k^{sep} of k, we can construct the *strict henselization* A^{sh} by considering the category of étale A-algebras endowed with a homomorphism to k^{sep} extending $A \to k^{\mathrm{sep}}$. (Using the algebraic closure $\bar{k}$ instead of k^{sep} gives the same result.)

Remark 2.A.4. The strict henselization of a local ring is the local ring for the étale topology. To make this precise, we reformulate everything in terms of geometry. Recall that a *geometric point* of a scheme X is a map $\bar{x} \to X$ with $\bar{x} = \operatorname{Spec} k(\bar{x})$ for some separably closed field $k(\bar{x})$. (Again, one can use algebraically closed fields instead.) An *étale neighborhood* of a geometric point $\bar{x}$ of X is an étale morphism $U \to X$ endowed with a lifting $\bar{x} \to U$ of $\bar{x} \to X$. Étale neighborhoods of $\bar{x}$ in X form a cofiltering category $N(X, \bar{x})$, and the colimit

$$\mathcal{O}_{X,\bar{x}} = \varinjlim_{U \in N(X,\bar{x})} \Gamma(U, \mathcal{O}_U)$$

is isomorphic to the strict henselization $\mathcal{O}_{X,x}^{\mathrm{sh}}$ of $\mathcal{O}_{X,x}$ where x is the image of $\bar{x}$ in X (and where we use the separable closure of $k(x)$ in $k(\bar{x})$ as $k(x)^{\mathrm{sep}}$). [14]

Proposition 2.A.5 (Hensel's lemma)**.** *Every local ring A which is J-adically complete and separated for an $\mathfrak{m}$-primary*[15] *ideal $J \subseteq A$ is henselian. In particular, every complete local ring is henselian.*

For fans of the étale topology, we give a geometric proof:

Proof. We prove condition (e). Let $X = \operatorname{Spec} A$ and $x = \operatorname{Spec} k$ as before, and let

$$\begin{array}{ccc} & & U \\ & \overset{\tilde{i}}{\nearrow} & \downarrow f \\ x & \xrightarrow{i} & X \end{array}$$

be an étale neighborhood of $x \to X$. Set $X_n = \operatorname{Spec} A/J^{n+1}$ for $n \geq 0$. First, consider the diagram

$$\begin{array}{ccc} x & \xrightarrow{\tilde{i}} & U \\ \downarrow & \overset{s_0}{\nearrow} & \downarrow f \\ X_0 & \longrightarrow & X. \end{array}$$

Since $x \to X_0$ is an immersion defined by the nil ideal[16] $\mathfrak{m}/J \subseteq A/J$, by the infinitesimal criterion for étaleness[17] there exists a unique diagonal arrow s_0 making the square commute.

Starting from s_0, we shall successively build maps $s_n : X_n \to U$ lifting $X_n \to X$ along f. It suffices to apply the infinitesimal criterion to the squares

$$\begin{array}{ccc} X_n & \xrightarrow{s_n} & U \\ \downarrow & \overset{s_{n+1}}{\nearrow} & \downarrow f \\ X_{n+1} & \longrightarrow & X. \end{array}$$

[14] Similarly, the henselization is related in the same way to local rings for the Nisnevich topology.

[15] This means that for $x \in \mathfrak{m}$ we have $x^N \in J$ for $N \gg 0$ depending on x.

[16] An ideal in a commutative ring is *nil* (*locally nilpotent* in [4]) if it consists of nilpotent elements.

[17] **Infinitesimal criterion for étale maps**: A morphism $f : X \to Y$ locally of finite presentation is étale if and only if for every ring A and nil ideal $I \subseteq A$ (equivalently, every square zero ideal), and every commutative square of solid arrows

$$\begin{array}{ccc} \operatorname{Spec} A/I & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} A & \longrightarrow & Y \end{array}$$

there exists a unique dotted arrow making the diagram commute.

Since A is J-adically complete, in the limit, the maps give the desired section $s: X \to U$.[18] □

[18] If you are confused with the last step, set $U = \operatorname{Spec} B$ and temporarily revert to commutative algebra.

Remark 2.A.6. The most common proof uses condition (a) of Proposition 2.A.1, and uses "Newton's method" to iteratively construct the desired root t using explicit induction steps. Proofs in [1, Appendix A] and [3] use condition (b), which gives a more direct approach to proving Theorem 2.5.1, but makes for a messier and less illuminating argument.

Corollary 2.A.7. *Every nonarchimedean field is henselian.*

Proof. Let K be a nonarchimedean field, let $\mathcal{O} \subseteq K$ be its valuation ring, and let $t \in \mathcal{O}$ be a pseudouniformizer. Apply Proposition 2.A.5 with $A = \mathcal{O}$ and $J = (t)$. □

Lemma 2.A.8. *The following are equivalent for a field K endowed with a height one valuation v.*

(a) K is henselian.

(b) The assertion of Lemma 2.5.2 holds.

Proof. Left as exercise. □

The universal property of henselization induces a map $A^h \to \widehat{A}$.

Proposition 2.A.9. *For a valued field* (K, v)*, the following are equivalent:*

(a) K is henselian,

(b) every finite extension L of K admits a unique extension of the valuation v.

Proof. Suppose that K is henselian. Given Lemma 2.A.8, we can repeat the proof of Proposition 2.5.3 word for word. The first paragraph of the proof of Theorem 2.5.1 shows that we can extend the valuation ring of K to L, which gives an extension of the valuation, easily seen to be unique. For the reverse direction, see [3, Theorem II 6.6]. □

Henselian rings will appear later in the course: the local ring $\mathcal{O}_{X,x}$ of a point x on a rigid analytic space X is not complete, but it is henselian.[19]

[19] The same holds for complex analytic spaces, e.g. the local ring $\mathbf{C}\{t\}$ of power series with positive radius of convergence is henselian.

3
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