1. Blowups (Office hours Nov 10)

References: [1, I 4, pp. 28-30 and II 7, pp. 163-169], [3, Tag 010F]

Let X be a scheme and let $Y \subseteq X$ be a closed subscheme which is locally defined by finitely many equations in $\mathcal{O}_X$. We set $X^\circ = X \setminus Y$, denote by $i\colon Y \to X$ the closed immersion, and denote by

$$I_Y = \ker\left(\mathcal{O}_X \to i_*\mathcal{O}_Y\right)$$

the quasicoherent ideal sheaf corresponding to Y. To this data, one associates the *blowing-up* $\tilde{X} = \mathrm{Bl}_Y X$ of X along Y, which is a scheme sitting inside a diagram

$$\begin{array}{ccccc} \pi^{-1}(X^\circ) & \longrightarrow & \tilde{X} & \longleftarrow & \pi^{-1}(Y) \\ \simeq\downarrow & & \downarrow\pi & & \downarrow \\ X^\circ & \longrightarrow & X & \underset{i}{\longleftarrow} & Y \end{array}$$

where $\pi\colon \tilde{X} \to X$ is a proper map and where $\pi^{-1}(Y) \subseteq \tilde{X}$ is a divisor on X, called the *exceptional divisor*.

There are (at least) three perspectives for looking at the blow-up construction:

(1) For $X = \mathbf{A}^n_k$ (the affine n-space) and $Y = \{0\}$ the origin, $\tilde{X}$ parametrizes pointed lines through the origin. For $X \subseteq \mathbf{A}^n_k$, the blow-up of X along $\{0\}$ is the closure of the preimage of $X \setminus \{0\}$ in $\mathrm{Bl}_{\{0\}} \mathbf{A}^n_k$ (called the *strict transform*).

(2) One can define the blow-up of X along Y as the relative Proj

$$\tilde{X} = \mathrm{Proj}_X \bigoplus_{n \geq 0} I^n$$

of the *Rees algebra* $\bigoplus_{n\geq 0} I^n$.

(3) The exceptional divisor $\pi^{-1}(Y)$ is an effective Cartier divisor, meaning that the ideal

$$\tilde{I} = \ker(\mathcal{O}_{\tilde{X}} \to \tilde{i}_*\mathcal{O}_E)$$

is an invertible sheaf on $\tilde{X}$. One can show that $\tilde{X}$ satisfies (and is defined by) the following universal property: *for every morphism $f\colon Z \to X$ such that $f^{-1}(Y) \subseteq Z$ is an effective Cartier divisor there exists a unique map $\tilde{f}\colon Z \to \tilde{X}$ such that $f = \pi \circ \tilde{f}$.*

Our first goal is to see why these constructions describe the same object. The idea is that (1), while seemingly very special, becomes completely general when phrased correctly.

We will mostly stick to affine schemes $X = \operatorname{Spec} B$, with the closed subscheme $Y \subseteq X$ defined by an ideal $I = (f_1, \ldots, f_n) \subseteq B$.

1.1. Blow-up of the origin in $\mathbf{A}^n$.

Let k be a base ring. One can take k an algebraically closed field for simplicity, and then one can ignore some of the functorial nonsense below. We will however need the extra generality later. Let

$$\mathbf{A} = \mathbf{A}^n_k = \operatorname{Spec} A, \quad A = k[X_1, \ldots, X_n]$$

be the affine n-space over k, and let $I = (X_1, \ldots, X_n) \subseteq A$ be the "ideal of the origin," $0 = V(I) = \operatorname{Spec} k$ the "origin." Note that $0 \subseteq \mathbf{A}$ is not a single point scheme for a general k; rather, it is the "zero section" of the projection $\mathbf{A} \to \operatorname{Spec} k$.

By definition, a *line through the origin* in $\mathbf{A}$ is a closed subscheme $\ell \subseteq \mathbf{A}$ which (locally on $\operatorname{Spec} k$) is defined by a system of $n-1$ linear equations in $X_1, \ldots, X_n$ of maximal rank, i.e. whose maximal minors generate the unit ideal. Such an ℓ corresponds to a direct summand $L \subseteq k^n$ which is a projective module of rank one. Further, let

$$\mathbf{P} = \mathbf{P}^{n-1}_k = \operatorname{Proj} k[Y_1, \ldots, Y_n], \quad \deg Y_i = 1$$

be the projective $(n-1)$-space over k. Then $\mathbf{P}$ parametrizes lines through the origin in $\mathbf{A}$, in the sense that we have an isomorphism of functors

$$\mathbf{P}(k') \simeq \{\text{lines through origin in } \mathbf{A}_{k'}\} \quad : \quad \{k\text{-algebras}\} \to \mathbf{Sets}$$

The *blow-up* $\tilde{\mathbf{A}} = \mathrm{Bl}_0\, \mathbf{A}$ is defined as

$$\tilde{\mathbf{A}} = \{(x, \ell) \in \mathbf{A} \times \mathbf{P} \,:\, x \in \ell\}.$$

To be completely precise, it is the k-scheme representing the functor

$$\tilde{\mathbf{A}}(k') = \{(x \in \mathbf{A}(k'), \ell \in \mathbf{P}(k') \,:\, x \in \ell\} \quad : \{k\text{-algebras}\} \to \mathbf{Sets}$$

where by definition $x \in \ell$ means that the image of $x\colon \operatorname{Spec} k' \to \mathbf{A}$ is contained in the closed subscheme ℓ. In concrete terms, it is the closed subscheme of $\mathbf{A} \times_k \mathbf{P}$ cut out by the homogeneous equations (the 2×2 minors of the matrix $[X|Y]$)

$$X_i Y_j = X_j Y_i \quad (1 \le i, j \le n),$$

that is

$$\tilde{\mathbf{A}} = \operatorname{Proj} A[Y_1, \ldots, Y_n]/(X_i Y_j - X_j Y_i)_{i,j=1}^n.$$

We denote by $\pi\colon \tilde{\mathbf{A}} \to \mathbf{A}$ the projection. The restriction of π to the open subset $\mathbf{A}^\circ = \mathbf{A} \setminus \{0\}$ is an isomorphism.

Let $U_i = \tilde{\mathbf{A}} \cap D_+(Y_i)$ be the intersection of $\tilde{\mathbf{A}}$ with the standard affine open subset $D_+(Y_i) = \{Y_i \neq 0\} \subseteq \mathbf{P}$. In standard coordinates $Y_{ij} = Y_j/Y_i$ we have

$$U_i = \operatorname{Spec} A_i, \quad A_i = A[Y_{ij} \,:\, j \neq i]/(X_j - Y_{ij} X_i)_{j \neq i}.$$

In particular, since X_i divides all X_j in A_i, the closed subscheme $\pi^{-1}(0) \cap U_i = V(X_1, \ldots, X_n)$ of U_i is cut out by the single equation $X_i = 0$. Moreover, X_i is a nonzerodivisor in A_i, so $X_i A_i \simeq A_i$ as A_i-modules. Therefore the ideal defining $\pi^{-1}(0)$ in $\tilde{\mathbf{A}}$ is an invertible $\mathcal{O}_{\tilde{\mathbf{A}}}$-module. (In fact, it is isomorphic to the invertible sheaf $\mathrm{pr}_2^*(\mathcal{O}_{\mathbf{P}}(1))|_{\tilde{\mathbf{A}}}$ where $\mathrm{pr}_2\colon \mathbf{A} \times_k \mathbf{P} \to \mathbf{P}$ is the projection).

1.2. **Blow-up of the origin in a subscheme of $\mathbf{A}^n$.** With the previous notation, let $X \subseteq \mathbf{A}$ be a closed subscheme, and let $X^\circ = X \cap \mathbf{A}^\circ$, which we can regard as a subscheme of $\tilde{\mathbf{A}}$ via the isomorphism induced by π. We set

$$\tilde{X} = \text{the closure of } X^\circ \text{ in } \tilde{\mathbf{A}},$$

and call it the *blow-up at the origin of* $X \subseteq \mathbf{A}$.

Lemma 1.1. *The scheme $\tilde{X}$ is the closed subscheme of $\pi^{-1}(X)$ defined by the ideal of sections of $\mathcal{O}_{\pi^{-1}(X)}$ which are supported on $\pi^{-1}(0)$. One has*

$$\tilde{X} \cap U_i \simeq \operatorname{Spec} B_i/(f_i\text{-torsion}), \quad B_i = B[Y_{ij} \,:\, j \neq i]/(f_j - Y_{ij} f_i)_{j \neq i},$$

where f_i-torsion means elements annihilated by a power of f_i.

Proof. Clearly $\tilde{X}$ is the closure of the open subscheme $X^\circ \subseteq \pi^{-1}(X)$. We check the assertion locally, restricting to $U_i = \operatorname{Spec} A_i$ for $i = 1, \ldots, n$. In this situation, the open subscheme $X^\circ \cap U_i$ of the affine scheme $\pi^{-1}(X) \cap U_i = \operatorname{Spec} B_i$ is defined by the single equation $X_i = 0$, so $\pi^{-1}(X) \cap U_i = \operatorname{Spec} B_i[1/f_i]$. By the Sublemma below, $\tilde{X} \cap U_i = \operatorname{Spec} B_i/(f_i\text{-torsion})$. This shows the second assertion, which implies the first since an element of B_i is supported on $V(f_i)$ precisely when its image in $B_i[1/f_i]$ vanishes. □

Sublemma. *Let $X = \operatorname{Spec} B$ be an affine scheme, and let $U = \operatorname{Spec} B[1/f]$ be a standard open subscheme. Then the closure $\overline{U}$ of U in X is defined by the ideal*

$$\ker(B \to B[1/f]) = \{g \in B \,:\, \exists_{n \ge 1}\, f^n g = 0\}.$$

Note that with our general setup this construction allows one to blowup any closed subscheme $Y = V(I)$ defined by a finitely generated ideal $I = (f_1, \ldots, f_n) \subseteq B$ in any affine scheme $X = \operatorname{Spec} B$. Indeed, one can take $k = B$ (!) and consider the embedding $X \hookrightarrow \mathbf{A}$ defined by the k-algebra surjection

$$A = k[X_1, \ldots, X_n] \to B, \quad X_i \mapsto f_i.$$

Then $Y \subseteq X$ is the preimage of $0 \subseteq \mathbf{A}$, and the above construction gives us a blow-up $\tilde{X}$ of X along Y. (Though we do not see from this that the result is independent of the choice of generators of I.)

Moreover, as we observed in Lemma 1.1, the scheme $\tilde{X}$ is obtained by gluing the affine schemes $\operatorname{Spec} B_i/(f_i\text{-torsion})$ where

$$B_i = A_i/(X_i - f_i) = B[Y_{ij} : j \neq i]/(f_j - Y_{ij} f_i).$$

Again, since f_i divides all f_j in B_i, the preimage of Y in $\tilde{X} \cap U_i$ is cut out by the single equation $f_i = 0$. *Since we divided out by the f_i-torsion,* f_i is a nonzerodivisor on $\tilde{X} \cap U_i$, and we see that *the ideal of $\pi^{-1}(Y)$ in $\tilde{X}$ is an invertible sheaf.*

1.3. **The Rees algebra.** Let us explicate the algebras $B_i/(f_i\text{-torsion})$ above. Consider the *Rees algebra* of the ideal I

$$R_I = \bigoplus_{n \geq 0} I^n$$

where $I^0 = B$ by convention. Since $I = (f_1, \ldots, f_n)$ is finitely generated, we have a surjection of graded B-algebras

$$B[Y_1, \ldots, Y_n] \to R_I, \quad Y_i \mapsto f_i.$$

Its kernel contains the homogeneous elements $r_{ij} = Y_i f_j - Y_j f_i$, but in general it will be bigger than the ideal they generate.

Recall that for a graded algebra $R = \bigoplus_{n \geq 0} R_n$ and a homogeneous element $f \in R_1$, one defines the graded localization $R_{(f)}$ as the subring of $R[1/f]$ generated by elements of the form g/f^n with $g \in R_n$. Then $\operatorname{Proj} R$ is covered by the affine subschemes $U_f = \operatorname{Spec} R_{(f)}$ for $f \in R_1$, and if R is generated by R_1 as an R_0-algebra, any set of $f \in R_1$ generating R_1 as an R_0-module will suffice.

For $f \in I \subseteq B$, we denote by $B[I/f]$ the graded localization $(R_I)_{(f_i)}$ where f_i is regarded as an element in degree one. We call $B[I/f]$ the *affine blow-up algebra* [3, Tag 052Q].

Lemma 1.2. *We have $B[I/f_i] \simeq B_i/(f_i\text{-torsion})$.*

Proof. The B-algebra map $B[Y_{ij} : j \neq i] \to R[I/f_i]$ sending Y_{ij} to f_j/f_i (treated as a ratio of elements of degree one) is clearly surjective and sends $f_j - Y_{ij} f_i$ to zero and hence factors uniquely through B_i. An fraction $g/f_i^n \in B[I/f_i]$ with $g \in I^n$ represents the zero element if and only if $f_i^m g = 0$ for some m. This shows that the kernel of $B_i \to B[I/f_i]$ consists precisely of the f_i-torsion elements. ☐

Since the spectra $\operatorname{Spec} B_i/(f_i\text{-torsion})$ (resp. $\operatorname{Spec} B[I/f_i]$) glue together to form $\tilde{X} = \operatorname{Bl}_Y X$ (resp. $\operatorname{Proj} R_I$), the lemma implies that we have

$$\operatorname{Bl}_Y X \simeq \operatorname{Proj} R_I.$$

The homogeneous ideal of $\pi^{-1}(Y) \subseteq \tilde{X}$ is described as

$$I \cdot R_I = \bigoplus_{n \geq 0} I^{n+1},$$

and we see that this corresponds to $\mathcal{O}(1)$ on $\operatorname{Proj} R_I$ [1, Proposition II 7.13]. We observe once again that it is invertible.

1.4. **Invertible ideals.** We have seen that for $I = (f_1, \ldots, f_n)$ the equations defining the standard open $U_i \cap \tilde{X}$ simply force the f_i to divide other f_j by adding an extra variable Y_{ij} for each j, and then force the resulting principal ideal to be invertible by dividing out by the f_i-torsion.[1] This suggest that in fact by blowing up we have *universally* made $\pi^{-1}(Y)$ into an effective Cartier divisor.

Lemma 1.3. *Let $g \colon Z \to X$ be a map for which $g^{-1}(Y) \subseteq Z$ is an effective Cartier divisor. Then f factors uniquely through $\pi \colon \operatorname{Bl}_Y X \to X$.*

The following proof is an alternative to [1, Proposition II 7.14].

[1] Note that dividing by $\operatorname{Ann}(f) = \{g \in B : fg = 0\}$ does not suffice to make a principal ideal (f) of a ring B invertible, one needs to divide by $\bigcup_{n \geq 1} \operatorname{Ann}(f^n) = \mathrm{f}$-torsion.

Proof. Since the affine opens $\operatorname{Spec} B[I/f]$ cover $\mathrm{Bl}_Y X$, it suffices to show: every map $g\colon Z \to X$ such that the ideal of $g^{-1}(Y) \subseteq Z$ is invertible and *generated by* f factors uniquely through $\operatorname{Spec} B[I/f]$. We can assume that $Z = \operatorname{Spec} C$ is affine. Then the assertion follows from the commutative algebra Sublemma below. □

Sublemma. *Let B a ring, $I \subseteq B$ an ideal, $f \in I$ an element. Then every homomorphism $\varphi\colon B \to C$ such that $\varphi(I)C = \varphi(f)C$ and $\varphi(f)$ is a nonzerodivisor in C factors uniquely through the affine blowup algebra $B[I/f]$.*

Proof. Let g/f^n for $g \in I^n$ be an element of $B[I/f]$. Since $\varphi(I)C = \varphi(f)C$, we also have $\varphi(I^n)C = \varphi(f^n)C$, and there exists an element $h \in C$ such that $\varphi(g) = \phi(f)^n h$. Since $\phi(f)$ is a nonzerodivisor in C, the element h is unique with this property. The map $B[I/f] \to C$ sending g/f^n to h is easily seen to be a B-algebra homomorphism, and uniqueness is also clear. □

1.5. **The strict transform.** Let $Y \subseteq X$ be a closed subscheme and let $f\colon X' \to X$ be a map; set $Y' = f^{-1}(X')$. The universal property of blowing up implies the existence of a unique dotted arrow $\tilde{f}$ making the square below commute

$$\begin{array}{ccc} \mathrm{Bl}_{Y'} X' & \overset{\tilde{f}}{\dashrightarrow} & \mathrm{Bl}_Y X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow[f]{} & X. \end{array}$$

We call $\mathrm{Bl}_{Y'} X'$ the strict transform of X' in $\mathrm{Bl}_Y X$.

Lemma 1.4. *Suppose that f is flat. Then the above diagram is Cartesian.*

Proof. Let $I \subseteq \mathcal{O}_X$ (resp. $I' \subseteq \mathcal{O}_{X'}$) be the ideal sheaf of Y (resp. $Y' = f^{-1}(Y)$). The pull-back $X' \times_X \mathrm{Bl}_Y X$ (resp. the blow-up $\mathrm{Bl}_{Y'} X'$) is described by

$$\operatorname{Proj} \bigoplus_{n \geq 0} f^*(I^n) \quad (\text{resp. } \operatorname{Proj} \bigoplus_{n \geq 0} (I')^n).$$

Since f is flat, the sequences $0 \to I^n \to \mathcal{O}_X \to \mathcal{O}_X/I^n \to 0$ stay exact after pull-back to X', and since $V(J^n) \simeq f^{-1}(V(I^n))$, we obtain

$$f^*(I^n) \simeq (I')^n \quad \text{for all } n \geq 0.$$ □

Lemma 1.5. *Suppose that f is a closed immersion. Then the map $\tilde{f}$ is a closed immersion, and $\mathrm{Bl}_{Y'} X'$ is the closed subscheme of $\pi^{-1}(X')$ cut out by the sections of $\mathcal{O}_{\pi^{-1}(X')}$ supported on $\pi^{-1}(Y)$. It is equal to the scheme-theoretic closure of $X' \setminus Y$ in $\mathrm{Bl}_Y X$.*

Proof. We have already proved this in case of the closed immersion $X \subseteq \mathbf{A}$ (Lemma 1.1). Now we have $X' \subseteq X \subseteq \mathbf{A}$ and comparing the two yields the assertion of the lemma. □

1.6. **Ubiquity of blow-ups.**

Lemma 1.6 ([3, Tag 080B]). *If $X' \to X = \operatorname{Spec} B$ and $X'' \to X'$ are blowups in finitely generated ideals, then so is their composition $X'' \to X$.*

The following result often lets us assume that a given proper and birational map is a blowup.

Theorem 1.7. *Let X be a Noetherian scheme and let $\pi\colon X' \to X$ be a proper birational map.*

(a) [1, Theorem II 7.17] *If X is a quasiprojective variety and f is projective, then π is isomorphic to the blowup of X at some closed subscheme $Y \subseteq X$.*

(b) [2, I 5.7.12], [3, Tag 081T] *In general, there exists a closed subscheme $Y \subseteq X$ whose complement is dense and a morphism $\mathrm{Bl}_Y X \to X'$ over X. If π is an isomorphism over an open $U \subseteq X$, then we can choose Y to be disjoint from U. Moreover, the morphism $\mathrm{Bl}_Y X \to X'$ can be chosen to be a blow-up as well.*

Part (b) is a rather difficult result based on the *flattening theorem* of Raynaud and Gruson.

Proposition 1.8. *Let* $f\colon X \dashrightarrow S$ *be a rational map, with* S *proper over* k*. Then there exists a blow-up* $\tilde{X} \to X$ *such that* f *extends to a morphism* $\tilde{X} \to S$*.*

Proof. See [3, Tag 0C4V]. ☐

References

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