Regular connections

(Deligne's Riemann–Hilbert correspondence and generalizations)

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I

Motivations from differential topology

How does differential geometry see the topology of a smooth manifold M?

1. De Rham's theorem

$$H^*(M, \Omega_M^\bullet) \simeq H^*(M, \mathbf{R})$$

2. Riemann–Hilbert correspondence

$$\left\{ \begin{matrix} \text{vector bundles with integrable} \\ \text{connection } (\mathcal{V}, \nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega_M^1) \end{matrix} \right\} \simeq \left\{ \begin{matrix} \text{local systems} \\ \mathbf{V} \text{ on } M \end{matrix} \right\}$$

$$(\mathcal{V}, \nabla) \mapsto \mathbf{V} = \mathcal{V}^{\nabla = 0}, \quad \mathbf{V} \mapsto (\mathcal{V} = \mathbf{V} \otimes_{\mathbf{R}} \mathcal{O}_M, \nabla(v \otimes f) = v \otimes df)$$

3. De Rham's theorem with coefficients

$$H^*(M, \Omega_M^\bullet \otimes \mathcal{V}) \simeq H^*(M, \mathbf{V})$$

Everything works for complex manifolds if we replace $\mathbf{R}$ with $\mathbf{C}$ and use holomorphic functions and differentials.

Grothendieck's algebraic de Rham theorem

$U = \operatorname{Spec} A$ smooth affine variety over $\mathbf{C}$

$$\Omega_A^\bullet = \left[A \xrightarrow{d} \Omega_A^1 \xrightarrow{d} \Omega_A^2 \to \cdots \right] \quad \text{algebraic de Rham complex}$$

Does it know anything about the topology of U_{an}?

Theorem (Grothendieck 1966)

$$H^*(\Omega_A^\bullet) \simeq H^*(U_{\mathrm{an}}, \mathbf{C})$$

Example. $U = \mathbf{G}_m = \operatorname{Spec} \mathbf{C}[x, x^{-1}]$, so $U_{\mathrm{an}} = \mathbf{C} \setminus \{0\}$

$$\Omega_A^\bullet = \left[\mathbf{C}[x, x^{-1}] \xrightarrow{f \mapsto f' dx} \mathbf{C}[x, x^{-1}] dx \right], \quad H^1(-) = \mathbf{C} \cdot x^{-1} dx$$

More generally, for $X/\mathbf{C}$ smooth, we have $H^*(X_{\mathrm{an}}, \mathbf{C}) \simeq H^*(X, \Omega_X^\bullet)$ (hypercohomology of the algebraic de Rham complex)

Is there an algebraic version of Riemann–Hilbert as well?

Answered by Deligne in 1969 (**LNM 163**)

Existence Theorem

X smooth complex algebraic variety

$$(\mathcal{V},\nabla) \mapsto \mathcal{V}_{\mathrm{an}}^{\nabla=0} : \left\{ \begin{array}{c} \text{vector bundles with integrable} \\ \text{connection } (\mathcal{V}, \nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_X) \\ \text{which are } \textbf{regular at infinity} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{local systems} \\ \mathbf{V} \text{ on } X_{\mathrm{an}} \end{array} \right\}$$

Comparison Theorem

$$H^*(X, \Omega^\bullet_X \otimes \mathcal{V}) \simeq H^*(X_{\mathrm{an}}, \mathbf{V}) \quad \text{for } (\mathcal{V},\nabla) \text{ regular at infinity}$$

Goal today: define regularity and explain the above theorems for X of dimension one, and mention some generalizations

Why do we need a condition like regularity?

Problem. The functor

$$(\mathcal{V},\nabla)\mapsto \mathcal{V}_{\mathrm{an}}^{\nabla=0}\colon \left\{\begin{matrix}\text{vector bundles with integrable}\\ \text{connection } (\mathcal{V},\nabla\colon \mathcal{V}\to\mathcal{V}\otimes\Omega_X^1)\end{matrix}\right\}\to\left\{\begin{matrix}\text{local systems}\\ \mathbf{V} \text{ on } X_{\mathrm{an}}\end{matrix}\right\}$$

is faithful but not full.

Example. On $X=\mathbf{A}^1$ consider

$$(\mathcal{V}_1,\nabla_1)=(\mathcal{O}_X,d) \quad \text{and} \quad (\mathcal{V}_2,\nabla_2)=(\mathcal{O}_X,d-dz)$$

We have an isomorphism $f\mapsto e^z f:\mathbf{V}_1\simeq\mathbf{V}_2$ which is not algebraic.

The idea is to find a full subcategory of connections on which the functor is an equivalence. We throw away $(\mathcal{V}_2,\nabla_2)$ but keep $(\mathcal{V}_1,\nabla_1)$.

How do we know which one to throw away? *Hint:* the global solution $\exp z$ of $(\mathcal{O}_X,d-dz)$ does not have moderate growth at ∞.

II

Regular connections

(after Deligne)

Regularity in dimension one

$$K = \mathbf{C}((t)) \quad \supseteq \quad \mathbf{C}[[t]] = \mathcal{O} \qquad \circlearrowleft \quad \tau = t\frac{d}{dt}$$

$$\mathrm{MIC}(K/\mathbf{C}) = \left\{ \begin{array}{c} M \text{ fin. dim. over } K \text{ with a } \mathbf{C}\text{-linear } \nabla_\tau : M \to M \\ \text{satisfying } \nabla_\tau(fm) = \tau(f)m + f\nabla_\tau(m) \end{array} \right\}$$

Definition

$M \in \mathrm{MIC}(K/\mathbf{C})$ is *regular* if it admits a ∇_τ-stable $\mathcal{O}$-lattice $\overline{M} \subseteq M$.

I.e. M admits a basis $\{e_i\}$ such that $\nabla_\tau(e_i) = \sum_j a_{ij}e_j$ with $a_{ij} \in \mathcal{O}$.

For a meromorphic connection M on $D = \{0 < |z| < 1\}$, the induced connection $M \otimes \mathbf{C}((t))$ is regular if and only if the solutions of M in sectors have moderate growth $O(|t|^{-N})$ for $|t| \to 0$.

Examples of regular connections

1. “$t^{-\lambda}$” $= (K, 1 \mapsto \lambda)$, $\lambda \in \mathbf{C}$ is regular,

2. “$e^{1/t}$” $= (K, 1 \mapsto \frac{1}{t})$ is not.

3. (K^n, ∇_τ) cyclic corresponding to a DE

$$\big(\tau^n + a_{n-1}(t)\tau^{n-1} + \cdots + a_0(t)\big)u = 0, \quad a_i(t) \in K$$

is regular iff all $a_i(t) \in \mathcal{O}$. *(N.B. Every $M \in \mathrm{MIC}(K/\mathbf{C})$ is cyclic.)*

Residue and monodromy

$M \in \mathrm{MIC}(K/\mathbf{C})$ regular, $\overline{M} \subseteq M$ a ∇_τ-stable $\mathcal{O}$-lattice

$$\nabla_\tau : \overline{M} \to \overline{M} \quad \rightsquigarrow \quad \overset{\textit{residue map}}{\rho : \overline{M}_0 \to \overline{M}_0, \quad \overline{M}_0 := \overline{M}/t\overline{M}}$$

Its eigenvalues* are the *exponents* of $\overline{M}$.

Theorem (Canonical extension)

For $M \in \mathrm{MIC}(K/\mathbf{C})$ regular, there is a unique $\overline{M} = \overline{M}_{\mathrm{can}}$ with exponents in

$$\{0 \leq \mathrm{Re}(z) < 1\}.$$

If M is obtained by base change from a meromorphic $\mathcal{M} \in \mathrm{MIC}_{\mathrm{mero}}(\Delta^*)$ on the punctured disc Δ^*, then the monodromy of $\mathcal{M}^\nabla$ is conjugate to

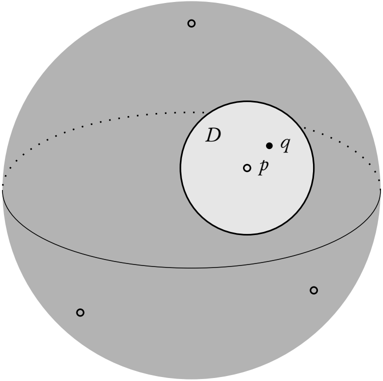
$$\exp(-2\pi i \rho_{\mathrm{can}}).$$

Proof of the Existence Theorem: essential surjectivity (1)

Setup

$\overline{X}$ smooth projective complex curve, $X \subseteq \overline{X}$ Zariski open
$\mathbf{V}$ local system on X_{an}, $\mathcal{V}$ the corresponding **analytic** connection

Fix $p \in \overline{X} \setminus X$, let $D \subseteq \overline{X}$ be a disc around p with coordinate t, $D^\circ = D \setminus \{p\}$
$q \in D^\circ$ a base point, $T \in \mathrm{GL}(\mathbf{V}_q)$ the monodromy around p.



Proof of the Existence Theorem: essential surjectivity (2)

Step 1: local canonical extensions

Pick $N \in \mathrm{End}(\mathbf{V}_q)$ with eigenvalues in $\{0 \le \mathrm{Re}\, z < 1\}$ such that $T = \exp(-2\pi i N)$. Consider the connection on D with pole at p:

$$\tilde{\mathcal{V}}_p = \mathbf{V}_q \otimes_{\mathbf{C}} \mathcal{O}_D, \quad \nabla = d - N t^{-1} dt \quad : \quad \mathbf{V}_q \otimes_{\mathbf{C}} \mathcal{O}_D \to \mathbf{V}_q \otimes_{\mathbf{C}} \Omega^1_D(p)$$

This also has monodromy T around p, so $\tilde{\mathcal{V}}_p|_{D^\circ} \overset{\text{can}}{\simeq} \mathcal{V}|_{D^\circ}$.

Step 2: global canonical extension

We glue these $\tilde{\mathcal{V}}_p$ $(p \in \overline{X} \setminus X)$ to $\mathcal{V}$ to obtain a log connection on $\overline{X}$

$$\tilde{\mathcal{V}} \to \tilde{\mathcal{V}} \otimes \Omega^1_{\overline{X}}(D), \quad D = \overline{X} \setminus X.$$

By GAGA$(+\varepsilon)$, this comes from an algebraic log connection $\tilde{\mathcal{V}}_{\mathrm{alg}}$ on $\overline{X}$, whose restriction $\mathcal{V}_{\mathrm{alg}}$ to X is regular by definition. We have $(\mathcal{V}_{\mathrm{alg}})_{\mathrm{an}} \simeq \mathcal{V}$.

Proof of the Existence Theorem: fullness

Let $f : \mathcal{V}_{\mathrm{an}} \to \mathcal{W}_{\mathrm{an}}$ be a horizontal morphism where $\mathcal{V}$, $\mathcal{W}$ are regular algebraic connections on X.

Consider f as a horizontal section of $\mathcal{E}_{\mathrm{an}}$ where $\mathcal{E} = \mathcal{V}^\vee \otimes \mathcal{W}$ (also regular).

Want to show that f is meromorphic at all $p \in \overline{X} \setminus X$. This follows from:

Lemma ("Gronwall's inequality")

Let f be holomorphic on $D^\circ = \{0 < |z| < 1\}$ satisfying a linear DE

$$\left(\tau^n + a_{n-1}(z)\tau^{n-1} + \cdots + a_0(z)\right) f = 0, \quad \tau = z\frac{d}{dz}$$

with $a_i(z)$ holomorphic on $D = \{|z| < 1\}$. Then f is meromorphic at zero.

III

Generalizations

Riemann–Hilbert correspondence for D-modules

X smooth over $\mathbf{C}$, $\mathcal{D}_X =$ sheaf of differential operators

Integrable connection ∇ on a $\mathcal{O}_X$-module $\mathcal{V} \longleftrightarrow$ str. of a (left) $\mathcal{D}_X$-module.

Can we allow non-constant constructible sheaves,
or D-modules with singularities?

Theorem (Kashiwara and Mebkhout, 1980's)

$$\left\{\begin{matrix}\text{regular holonomic} \\ D\text{-modules on } X\end{matrix}\right\} \simeq \{\text{perverse sheaves on } X_{\mathrm{an}}\}$$

Log schemes

$X/\mathbf{C}$ idealized log smooth log scheme **. . .** that is, X étale locally looks like

$$Y = \operatorname{Spec} \mathbf{C}[P]\,/\,\Sigma, \qquad \mathcal{M}_Y \text{ induced by } P \to \mathbf{C}[P]$$

monoid monomial ideal

Note: $\Omega^1_Y \simeq P^{\mathrm{gp}} \otimes \mathbf{C}[P]/\Sigma$ is free and spanned by $d\log p$'s.

Log strata are locally described as torus orbits ($T = \operatorname{Hom}(P, \mathbf{G}_m) \circlearrowright Y$)

Examples.

1. $\underline{X}$ smooth, $D \subseteq \underline{X}$ sncd $\rightsquigarrow$ log scheme X with $\Omega^1_X \simeq \Omega^1_{\underline{X}}(\log D)$.
2. $Z \subseteq D$ stratum $\rightsquigarrow$ induced log str. on Z with $\Omega^1_Z \simeq \Omega^1_{\underline{X}}(\log D)|_Z$.

Riemann–Hilbert correspondence for log schemes

$X/\mathbf{C}$ idealized log smooth log scheme

$$\mathrm{MIC}(X/\mathbf{C}) = \left\{ \begin{matrix} \text{coherent sheaves } \mathcal{V} \text{ with an integrable} \\ \text{log connection } \nabla \colon \mathcal{V} \to \mathcal{V} \otimes \Omega^1_X \end{matrix} \right\}$$

Imprecise definition: $(\mathcal{V}, \nabla) \in \mathrm{MIC}(X/\mathbf{C})$ is *regular* if its restriction to every stratum of X is regular in the classical sense.

Theorem (A. 2020)

$$\mathcal{V} \mapsto \mathcal{V}_{\mathrm{an}} \colon \mathrm{MIC}_{\mathrm{reg}}(X/\mathbf{C}) \simeq \mathrm{MIC}(X_{\mathrm{an}}/\mathbf{C})$$

The target category $\mathrm{MIC}(X_{\mathrm{an}}/\mathbf{C})$ admits a **topological description** in terms of certain constructible sheaves [Ogus 2003]

Non-Archimedean Riemann–Hilbert correspondence

X smooth rigid-analytic variety over the non-Archimedean field $\mathbf{C}((t))$

$\mathrm{MIC}(X/\mathbf{C}) = \{\mathbf{C}\text{-linear int. conn. on } X\}$ (so $\tau = t\frac{d}{dt}$ acts)

$\mathrm{MIC}_{\mathrm{reg}}(X/\mathbf{C}) \subseteq \mathrm{MIC}(X/\mathbf{C})$ *regular* connections

$\mathrm{LocSys}_{\mathbf{C}}(\Psi(X)) = \mathbf{C}$-local systems on the Betti realization $\Psi(X)$

[A.–Talpo 2019]

"Theorem" (in progress, relies on the RH for log schemes)

$$\mathrm{MIC}_{\mathrm{reg}}(X/\mathbf{C}) \simeq \mathrm{LocSys}_{\mathbf{C}}(\Psi(X)).$$