PURITY FOR NEWTON POLYGONS OF F-CRYSTALS (AFTER DE JONG–OORT)

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1. Statement of the results

Let S be a locally Noetherian scheme over $\mathbf{F}_p$. An *F-crystal* on S is a locally free crystal $\mathscr{E}$ of $\mathscr{O}^{\mathrm{cris}}_{S/\mathbf{Z}_p}$-modules on the crystalline site $(S/\mathbf{Z}_p)_{\mathrm{cris}}$, endowed with a homomorphism $F\colon F_S^*\mathscr{E} \to \mathscr{E}$; it is *nondegenerate* if both the kernel and cokernel of F are killed by a power of p.

If $S = \operatorname{Spec} k$ for an algebraically closed field k, then an F-crystal on S is a finite free $W(k)$-module E endowed with a Frobenius-semilinear endomorphism $F\colon E \to E$. If K is the fraction field of $W(k)$, then the Dieudonné–Manin theorem states that $(E \otimes K, F)$ is characterized up to isomorphism by its *Newton polygon* $\mathrm{NP}(E)$, which is the graph of the piecewise linear continuous function $\lambda\colon [0, r] \to \mathbb{R}$ $(r = \dim E \otimes K)$ with $f(0) = 0$ and slope λ_i on $[i-1, i]$, where $\lambda_1 \leq \ldots \leq \lambda_r$ are the p-adic valuations of the eigenvalues of some matrix representing F.

Thus if $\overline{s} = \operatorname{Spec} k \to S$ is a geometric point, we get the associated Newton polygon $\mathrm{NP}(\mathscr{E}_{\overline{s}})$, which depends only on the image s of $\overline{s}$ in S. We therefore get a *stratification by Newton polygons*

$$S = \bigsqcup_\beta S_\beta, \quad S_\beta = \{s \in S \,|\, \mathrm{NP}(\mathscr{E}_{\overline{s}}) = \beta\}.$$

Each S_β is a constructible subset of S. Grothendieck's theorem [4, §2.3] states that the Newton polygon only goes up by specialization: if $s \in \overline{S}_\beta$, then $\mathrm{NP}(\mathscr{E}_{\overline{s}}) \geq \beta$. Thus each S_β is in fact locally closed in S.

The goal of this talk is to sketch the proof of the following theorem of de Jong and Oort:

Theorem 1 ([3, Theorem 4.1]). *The stratification by Newton polygons jumps only in codimension one. More precisely, if η is the generic point of a component of $\overline{S}_\beta \setminus S_\beta$, then the local ring $\mathscr{O}_{\overline{S}_\beta, \eta}$ has dimension one.*

This theorem is deduced (using an earlier result of de Jong on extending homomorphisms between F-crystals [2]) from the following inconspicuously looking result.

Theorem 2 ([3, Theorem 3.2]). *Let $S = \operatorname{Spec} A$ be the formal germ of a normal surface singularity over an algebraically closed field k of characteristic p, i.e. A is a normal two-dimensional complete local k-algebra. Let $\widetilde{S} \to S$ be a resolution of singularities, let $U = S \setminus 0$ where $0 \in S$ is the closed point, and let $j\colon U \to \widetilde{S}$ be the inclusion. Then the induced map*

$$j^*\colon H^1(\widetilde{S}, \mathbf{Q}_p) \longrightarrow H^1(U, \mathbf{Q}_p)$$

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is an isomorphism.

As we shall explain first, this theorem is not so difficult (and known much earlier) if we replace $\mathbf{Q}_p$ with $\mathbf{Q}_\ell$ ($\ell \neq p$) as coefficients.

2. Two notions of purity and the easy version of Theorem 2

The term *purity* is used in algebraic geometry in (at least) two different ways.

2.1. Zariski–Nagata purity. Zariski–Nagata purity is the statement that the locus over which a given map fails to be étale has (pure) codimension one. To state it formally, consider a cartesian square of the form

$$\begin{array}{ccc} V & \longrightarrow & Y \\ {\scriptstyle f_U}\downarrow & \square & \downarrow{\scriptstyle f} \\ U & \xrightarrow{\;j\;} & X \end{array}$$

where

- X is a regular scheme,
- j is an open immersion whose complement has codimension > 1,
- Y is normal,
- f is finite,
- f_U is étale.

Theorem (Zariski–Nagata purity, [SGA2, Exp. X, Thm. 3.4])**.** *Under these assumptions f is étale.*

In other words, the restriction map

$$j^* \colon \{\text{finite étale schemes over } X\} \to \{\text{finite étale schemes over } U\}$$

is an equivalence, or $j_* \colon \pi_1(U) \to \pi_1(X)$ is an isomorphism.

The proof proceeds in a few steps:

(1) Reduce to the case X local of dimension 2.
(2) Show that f is flat. By miracle flatness [Matsumura 23.1], it is enough to show that Y is Cohen–Macaulay. But, since Y is normal, it is S_2.
(3) In the case when f is finite and flat, the locus where f fails to be étale can be described as the discriminant locus

$$V\left(\det\left(\mathrm{Tr}(xy) \colon \mathscr{O}_Y \otimes_{\mathscr{O}_X} \mathscr{O}_Y \to \mathscr{O}_X\right)\right)$$

which is clearly a divisor in X.

2.2. Cohomological purity. There are several equivalent formulations, one of them concerns computing the cohomology of the complement of a snc divisor. Let X be again a regular scheme, let $D \subseteq X$ be an snc divisor, let $U = X \setminus D$, and let $j \colon U \to X$ be the inclusion. We are interested in the étale cohomology of U with coefficients $\Lambda = \mathbf{Z}/\ell^N\mathbf{Z}$ (ℓ a prime invertible on X). Since $H^*(U, \Lambda) = H^*(X, Rj_*\Lambda)$, we can write the spectral sequence

$$E_2^{ab} = H^a(X, R^b j_*\Lambda) \quad \Rightarrow \quad H^{a+b}(U, \Lambda).$$

The purity theorem explicates the sheaves $R^b j_*\Lambda$ appearing on the E_2-page.

(It is easy to guess what the answer should be looking at the situation over $\mathbf{C}$: if $P \in D$ and $D = V(x_1 \dots x_r)$ in local coordinates at P, then the stalk of $R^b j_* \mathbf{Z}$ at P is the cohomology of

$$j^{-1}(\text{small open ball around } x) = \{(x_1, \dots, x_n) \in \mathbf{C}^n \mid |x_i| < \varepsilon, x_1 \cdot \dots \cdot x_r \neq 0\}$$

which has the homotopy type of an r-dimensional torus.)

Let us try to compute $R^b j_* \Lambda$ using the Kummer sequence

$$1 \to \Lambda(1) \to \mathscr{O}_U^* \to \mathscr{O}_U^* \to 1, \quad \Lambda(1) := \mu_{\ell^N}$$

Applying Rj_*, we obtain a connecting map

$$\delta \colon j_* \mathscr{O}_U^* \otimes \Lambda \to R^1 j_* \Lambda(1)$$

The étale subsheaf $\mathscr{O}_X^* \subseteq j_* \mathscr{O}_U^*$ is n-divisible, so the source $j_* \mathscr{O}_U^* \otimes \Lambda = (j_* \mathscr{O}_U^*)/\mathscr{O}_X^* \otimes \Lambda$ is the sheaf of divisors supported on D with coefficients in Λ, which can be identified with $\eta_* \Lambda$ where $\eta \colon \widetilde{D} = \bigsqcup D_i \to D$ is the normalization map. Therefore δ induces a map

$$\overline{\delta} \colon \eta_* \Lambda \to R^1 j_* \Lambda(1) \quad \text{or} \quad \overline{\delta} \colon \eta_* \Lambda(-1) \to R^1 j_* \Lambda.$$

Theorem (Cohomological purity, Grothendieck)**.** *The maps induced by $\overline{\delta}$ by exterior product*

$$\overline{\delta} \colon \wedge^b (\eta_* \Lambda(-1)) \to R^b j_* \Lambda$$

are isomorphisms for $b \geq 0$.

The same statement with coefficients $\mathbf{Z}_\ell$ or $\mathbf{Q}_\ell$ follows formally.

2.3. We note here that both notions of purity are philosophically present in the paper by de Jong and Oort: Theorem 1 makes one think of Zariski–Nagata purity (even though the scheme is no longer assumed to be regular), while the ℓ-adic version of Theorem 2 is closely related to cohomological purity.

Theorem. *Let $S = \operatorname{Spec} A$ be the formal germ of a normal surface singularity over an algebraically closed field k. Let $\widetilde{S} \to S$ be a resolution of singularities, let $U = S \setminus 0$ where $0 \in S$ is the closed point, and let $j \colon U \to \widetilde{S}$ be the inclusion. Let ℓ be a prime invertible in k. Then the induced map*

$$j^* \colon H^1(\widetilde{S}, \mathbf{Q}_\ell) \longrightarrow H^1(U, \mathbf{Q}_\ell)$$

is an isomorphism.

Let $(S, 0)$ be a germ of a normal surface singularity, $\widetilde{S} \to S$ a resolution of singularities, $U = S \setminus \{0\}$, $j \colon U \to \widetilde{S}$ the inclusion. Trying to compute the first cohomology of U using the Leray spectral sequence for j, we obtain an exact sequence

$$0 \to H^1(\widetilde{S}, \mathbf{Q}_\ell) \to H^1(U, \mathbf{Q}_\ell) \to H^0(\widetilde{S}, R^1 j_* \mathbf{Q}_\ell) \xrightarrow{\partial} H^2(\widetilde{S}, \mathbf{Q}_\ell).$$

It is now enough to show that the map ∂ is injective. Let us explicate its source and target using cohomological purity:

- We have $R^1 j_* \mathbf{Q}_\ell = \eta_* \mathbf{Q}_\ell(-1) = \bigoplus \mathbf{Q}_{\ell, E_i}(-1)$ where $E = \pi^{-1}(0) = \bigcup E_i$ is the exceptional divisor and $\eta \colon \widetilde{E} = \bigsqcup E_i \to E$ is the normalization map. Thus the source $H^0(\widetilde{S}, R^1 j_* \mathbf{Q}_\ell) \simeq \mathbf{Q}_\ell(-1)^{\pi_0(\widetilde{E})}$.

- For the target, we note that $H^2(\widetilde{S}, \mathbf{Q}_\ell) = H^2(E, \mathbf{Q}_\ell)$ by proper base change. Moreover, $H^2(E, \mathbf{Q}_\ell) \simeq H^2(\widetilde{E}, \mathbf{Q}_\ell) \simeq \bigoplus H^2(E_i, \mathbf{Q}_\ell) = \mathbf{Q}_\ell(-1)^{\pi_0(\widetilde{E})}$, which looks the same as the source.
- In these bases, the map ∂ is not the identity, but can be easily seen to be the intersection matrix $(E_i \cdot E_j)$ of the exceptional divisor E.

This matrix is well-known to be negative-definite, and hence ∂ is injective as desired.

Lemma (Mumford)**.** *The matrix $M = (E_i \cdot E_j)$ is negative-definite.*

Proof. Let $f \in A$ be a nonzero element of the maximal ideal, and let $D = V(f) \subseteq \widetilde{S}$. We can write $D = D' + \sum m_i E_i$ where D' is a nonzero effective divisor which does not contain a component of E and where all $m_i > 0$. Note that

$$(\sum \alpha_i E_i)^2 = \sum_i \left(\sum_j (m_i E_i \cdot m_j E_j) \right) \left(\frac{\alpha_i}{m_i} \right)^2 - \sum_{i<j} (m_i E_i \cdot m_j E_j) \left(\frac{\alpha_i}{m_i} - \frac{\alpha_j}{m_j} \right)^2 .$$

Now $\sum_j (m_i E_i \cdot m_j E_j) = -(m_i E_i \cdot D')$, which is ≤ 0 for all i and < 0 for at least one i, and $(m_i E_i \cdot m_j E_j) \geq 0$ for $i \neq j$, so all terms above are non-positive. Thus if $(\sum \alpha_i m_i E_i)^2 = 0$, then $\alpha_i = 0$ for some i by looking at the first sum, but then $\alpha_j = 0$ for all j such that E_j intersects E_i by looking at the second sum. But E is connected (by Zariski's main theorem), so α_i must all be zero. □

3. Proof of Theorem 2

3.1. Preliminaries. The restriction map $H^1(\widetilde{S}, \mathbf{Q}_p) \to H^1(U, \mathbf{Q}_p)$ is injective (e.g. by looking at the Leray spectral sequence); we need to show surjectivity. To this end, let α be a nonzero element of $H^1(U, \mathbf{Q}_p)$, which we view as a continous homomorphism $\alpha\colon \pi_1(U) \to \mathbf{Q}_p$. Normalizing, we can assume that the image of α is $\mathbf{Z}_p$; in this case, one can view α as a system of (connected) étale $\mathbf{Z}/p^n$-coverings $U_n \to U$. We need to prove that α extends to $\widetilde{S}$.

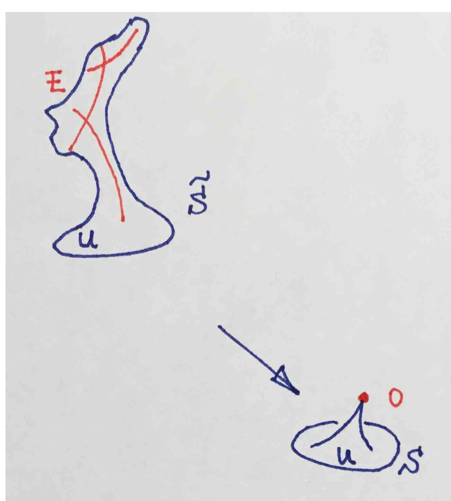

Figure 1. The situation in Theorem 2

By Zariski–Nagata purity, it is enough to show that α extends to generic points of the E_i. Since $\mathbf{Q}_p$ is torsion free, one shows easily that one can pass to a finite normal dominant $T \to S$.

3.2. Globalize S. Using algebraization (Artin) and alterations (de Jong), one can reduce to the following situation:

- $\pi\colon X \to \operatorname{Spec} k[[t]]$ is a projective semistable curve (i.e. X is regular, π is flat, and X_0 is a reduced snc divisor),
- $E \subseteq X_0$ is a union of connected components,
- $\rho\colon X \to X'$ is proper and birational, an isomorphism outside E, and maps E to a point $P \in X'$,
- $\hat{\mathscr{O}}_{X',P} \simeq A$ and $\widetilde{S} \simeq \rho^{-1}(\operatorname{Spec} A)$.

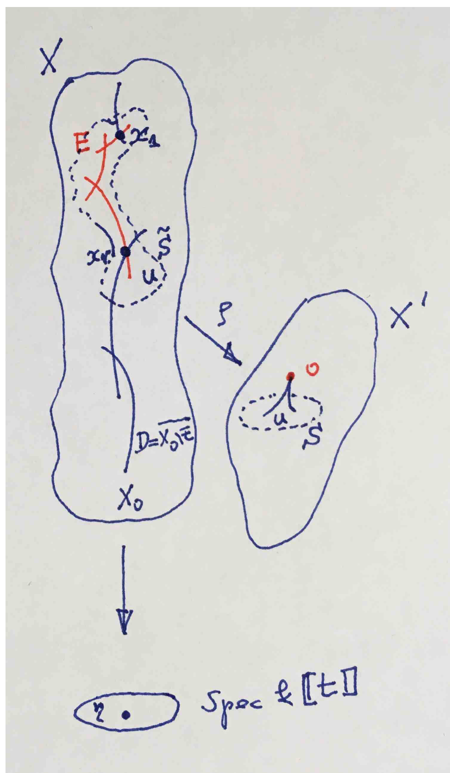

FIGURE 2. The situation in Theorem 2 after globalizing in §3.2

3.3. Globalize α (I). We wish to extend α from U (which can be viewed as a tiny neighborhood of E in X) to $X \setminus E$. First, we extend $\alpha|_{\widetilde{S}\setminus E}$ to all of $X_0 \setminus E$.

Let D be the closure of $X_0 \setminus E$ and let $D \cap E = \{x_1, \ldots, x_r\}$. Let $\mathscr{O}_i = \hat{\mathscr{O}}_{D,x_i}$, $K_i = \operatorname{Frac}(\mathscr{O}_i)$. Then $A \to \mathscr{O}_i$ is local, so α induces $\alpha_i\colon \operatorname{Gal}_{K_i} \to \mathbf{Z}_p$.

Lemma. *There exists an $\alpha_D\colon \pi_1(X_0 \setminus E) \to \mathbf{Z}_p$ inducing all α_i.*

Proof. We can assume that $D \setminus E$ is affine. First, we show the statement with $\mathbf{Z}/p$ coefficients using Artin–Schreier theory. The short exact sequence

$$0 \to \mathscr{O}_D \to j_* \mathscr{O}_{D\setminus E} \to \bigoplus K_i/\mathscr{O}_i \to 0$$

together with the Artin–Schreier sequences induce a diagram with exact rows and columns

$$\begin{array}{ccccccc}
H^0(D\setminus E, \mathscr{O}) & \longrightarrow & \bigoplus K_i/\mathscr{O}_i & \longrightarrow & H^1(D, \mathscr{O}_D) & \longrightarrow & 0 \\
\Big\downarrow {\scriptstyle F-1} & & \Big\downarrow {\scriptstyle F-1} & & \Big\downarrow {\scriptstyle F-1} & & \\
H^0(D\setminus E, \mathscr{O}) & \longrightarrow & \bigoplus K_i/\mathscr{O}_i & \longrightarrow & H^1(D, \mathscr{O}_D) & \longrightarrow & 0 \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
H^1(D\setminus E, \mathbf{Z}/p) & \longrightarrow & \bigoplus H^1(K_i, \mathbf{Z}/p) & \longrightarrow & * & \longrightarrow & 0 \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

where $* = 0$, being the cokernel of $F - 1$ on a finite dimensional k-vector space. This shows surjectivity of the bottom map as desired. Finally, the $\mathbf{Z}_p$-case can be deduced formally for the above. □

3.4. Globalize α (II). If we replace X with $X_m = X \otimes k[[t]]/(t^{m+1})$, then

- X_m is the union of $X_m \setminus E$ and $\hat{X}_{m,E}$ (the completion of X_m along E), glued along their intersection, which is (a thickening of) $\bigsqcup \operatorname{Spec} K_i$,
- α defines a system of flat $\mathbf{Z}/p^n$-covers of $\hat{X}_{m,E}$ (the completion of X_m along E),
- α_D defines a system of finite étale $\mathbf{Z}/p^n$-covers of $X_0 \setminus E$, and hence of $X_m \setminus E$ (by invariance of the étale site under nilpotent thickenings),
- α_D and α agree on $\operatorname{Spec} K_i$ by definition.

This suggest that we can glue α with α_D to obtain a global system of flat $\mathbf{Z}/p^n$-covers of X_m. The gluing is done by means of the following result:

Theorem ([1, §4.6])**.** *Let A be a noetherian ring, $I \subseteq A$ an ideal, $\hat{A}$ the I-adic completion, $X = \operatorname{Spec} A$, $Z = \operatorname{Spec} A/I$, $U = X \setminus Z$, $Y = \operatorname{Spec} \hat{A}$, $U' = U \setminus Z = U \times_X Y$. Then the natural functor*

$$\operatorname{Coh}(X) \to \operatorname{Coh}(U) \times_{\operatorname{Coh}(U')} \operatorname{Coh}(Y),$$

$$\mathscr{F} \mapsto (\mathscr{F}|_U, \mathscr{F}|_Y, \mathscr{F}|_U|_{U'} \simeq \mathscr{F}|_Y|_{U'})$$

is an equivalence of categories.

Passing to the limit and algebraizing (thanks to Grothendieck's existence theorem), we obtain a system of flat $\mathbf{Z}/p^n$-covers $Y_n \to X$ which are unramified away from E, i.e. a cohomology class $\beta \in H^1(X \setminus E, \mathbf{Z}_p)$. We need to show that β extends to X.

3.5. Use Neron models to view α geometrically. Let J be the Neron model of $J_\eta = \mathrm{Jac}(X_\eta)$ over $k[[t]]$. Since X is semistable, J is a semiabelian variety. Choose a $k[[t]]$-rational point of X which sends $\operatorname{Spec} k$ to a smooth point of $X_0 \setminus E$. This induces a map $X_\eta \to J_\eta$ and hence (by the universal property of a Neron model) a map $X^{\mathrm{sm}} \to J$.

The class $\beta_\eta \in H^1(X_\eta, \mathbf{Z}_p)$ corresponds to a homomorphism $\varphi\colon J_\eta[p^\infty] \to \mathbf{Q}_p/\mathbf{Z}_p$ in the following way: the $\mathbf{Z}/p^n$-covering $J_{n,\eta} \to J_\eta$ can be obtained as the pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_\eta[p^n] & \longrightarrow & J_\eta & \xrightarrow{p^n} & J_\eta & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \varphi[p^n]} & & \Big\downarrow & & \Big\| & & \\ 0 & \longrightarrow & \mathbf{Z}/p^n & \longrightarrow & J_{n,\eta} & \longrightarrow & J_\eta & \longrightarrow & 0. \end{array}$$

Let J_n be the Neron model of J. It is also semiabelian, as $J_{n,\eta}$ is isogenous to J_η. We claim that to show that β extends to X, it is enough to show that the maps $J_n \to J$ are étale. Indeed, if J° is the connected Neron model of J, then $J_n \to J$ is a $\mathbf{Z}/p^n$-torsor over J°. But there exists an N such that $[N]\colon J \to J$ lands in J°. This means that $N \cdot \beta$ extends to J. Now X^{sm} maps to J, so $N \cdot \beta$ extends to X^{sm} and hence to X, by Zariski–Nagata purity.

3.6. Prove $J_n \to J$ is étale. The following result of de Jong is an equicharacteristic version of an earlier result of Tate [6].

Theorem 3 ([2]). *Let R be a discrete valuation ring of characteristic $p > 0$ with fraction field K. Then the restriction functor*

$$\{p\text{-divisible groups on } R\} \to \{p\text{-divisible groups on } K\}$$

is fully faithful. If moreover R has a p-basis [5]*, then the restriction functor*

$$\{F\text{-crystals on } R\} \to \{F\text{-crystals on } K\}$$

is fully faithful as well.

For now, we only need the p-divisible group part. The p-divisible groups $J_\eta[p^\infty]$, resp. $J_{n,\eta}[p^\infty]$ admit filtrations

$$0 \to G^f_\eta \to J_\eta[p^\infty] \to E_\eta \to 0, \quad \text{resp. } 0 \to G^f_{n,\eta} \to J_{n,\eta}[p^\infty] \to E_{n,\eta} \to 0,$$

where G^f_η (resp. $G^f_{n,\eta}$) is the generic fiber of the finite part of $J[p^\infty]$ (resp. $J_n[p^\infty]$) and where E_η and $E_{n,\eta}$ extend to étale p-divisible groups over $k[[t]]$.

By Theorem 3, the restriction of $\varphi\colon J_\eta[p^\infty] \to \mathbf{Q}_p/\mathbf{Z}_p$ to G^f_η extends to a map $G^f \to \mathbf{Q}_p/\mathbf{Z}_p$. Suppose for simplicity that $G^f[p] \to \mathbf{Z}/p\mathbf{Z}$ is surjective (otherwise we replace φ by φ/p etc.). One then observes that $G^f_n \to G^f$ has kernel $\mathbf{Z}/p^n\mathbf{Z}$ and hence is étale.

4. Proof of Theorem 1

4.1. Preliminaries. It suffices to show that if $\mathscr{E}$ is an F-crystal on $\operatorname{Spec} A$ where A is a local noetherian $\mathbf{F}_p$-algebra of dimension > 1, and if $\mathscr{E}$ has constant Newton polygon on $\operatorname{Spec} A \setminus \{m\}$, then $\mathscr{E}$ has constant Newton polygon everywhere.

Standard reductions allow one to assume A is complete, normal, and 2-dimensional, with algebraically closed residue field (as in Theorem 2).

4.2. Slope filtration. Let K be the fraction field of A. One extend the arguments of Katz in [4] to show that $\mathscr{E} \otimes K$ admits a slope filtration

$$0 \subseteq \mathscr{E}_1 \subseteq \mathscr{E}_2 \subseteq \ldots \subseteq \mathscr{E} \otimes K$$

by sub-F-isocrystals. If r_i is the rank of $\mathscr{E}_i$, then the determinant $\wedge^{r_i}\mathscr{E}_i$ is an F-crystal of rank one with integral slope n_i. The Newton polygon of $\mathscr{E} \otimes K$ has the points (r_i, n_i) as vertices.

By Grothendieck's semicontinuity theorem for Newton polygons, $NP(\mathscr{E} \otimes k) \geq NP(\mathscr{E} \otimes K)$, thus it is enough to show that the points (r_i, n_i) lie on $NP(\mathscr{E} \otimes k)$. It is therefore enough to extend the rank one F-crystal $\wedge^{r_i}\mathscr{E}_i$ to $\widetilde{S}$ and extend the map $\wedge^{r_i}\mathscr{E}_i \to \wedge^{r_i}\mathscr{E} \otimes K$ to $\widetilde{S}$, or at least to a generic point of a component of E.

4.3. Extending the F-crystal $\wedge^{r_i}\mathscr{E}_i$. Let $\mathscr{L} = (\wedge^{r_i}\mathscr{E}_i)(-i)$. This is an F-crystal on K of rank one and slope zero (unit root). Consider the étale sheaf $\mathscr{L}^{F=1}$, this is a locally constant $\mathbf{Z}_p$-sheaf of rank one. As proved by Katz, for any $\mathbf{F}_p$-scheme this association gives an equivalence between unit root F-crystals and $\mathbf{Z}_p$-local systems.

Thus $\mathscr{L}^{F=1}$ extends to a representation $\pi_1(U) \to GL_1(\mathbf{Q}_p) \cong \mathbf{Z}_p \times \mathbf{Z} \times \mathbf{F}_p^\times \times (\mathbf{Z}/2 \text{ if } p = 2)$. By passing to a finite ramified covering of S, we can assume that the image of ρ is $\mathbf{Z}_p$. We are now in the situation when we can apply Theorem 2: we see that ρ extends to $\widetilde{S}$. By the aforementioned equivalence of categories, the rank one F-crystal $\wedge^{r_i}\mathscr{E}_i$ extends to an F-crystal on $\widetilde{S}$.

4.4. Extending the inclusion $\wedge^{r_i}\mathscr{E}_i \to \mathscr{E} \otimes K$. Let R be the completion of the local ring of the generic point η of a component of E. This is a discrete valuation ring admitting a p-basis [5]. We can now apply the second part of Theorem 3 to extend the inclusion $\wedge^{r_i}\mathscr{E}_i \to \mathscr{E} \otimes K$ to R, thus showing that the Newton polygon of $\mathscr{E}$ at η contains the point (r_i, n_i). This ends the proof.

5. Further developments

Vasiu [7] obtained a much more refined result: the strata of the Newton stratification are always affine (over any base scheme), using completely different methods.

Yang [8] proved that in the situation consider by de Jong and Oort, the set where the Newton polygons have a common breakpoint has complement of codimension > 1 in the closure of the stratum.

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