Problem Set 4

due May 28, 2025

Problem 1. Compute $\pi_1^{\text{et}}(X)$ for the following schemes:

1. A nodal cubic in $\mathbb{P}^2_{\mathbb{C}}$.
2. The "triangle" $X = \{x_0x_1x_2 = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$.
3. The punctured plane $X = \mathbb{A}^2_{\mathbb{C}} \setminus \{(0,0)\}$.

Problem 2. Let $(\mathcal{C}, F)$ and $(\mathcal{C}', F')$ be two pointed Galois categories and let $G\colon \mathcal{C} \to \mathcal{C}'$ be an exact functor such that $F = F' \circ G$. Prove that the following are equivalent:

(a) The induced homomorphism of profinite groups $\pi_1(\mathcal{C}', F') \to \pi_1(\mathcal{C}, F)$ is injective.

(b) For every object X of $\mathcal{C}$ there exists a diagram

$$X \longleftarrow Y \longrightarrow G(Z)$$

where $Y \to X$ is an epimorphism and $Y \to G(Z)$ is a monomorphism.

Problem 3. Let $f\colon Y \to X$ be a dominant map of normal varieties. Show that the image of $\pi_1(Y) \to \pi_1(X)$ is an open subgroup. Give an example showing that the normality assumption is necessary.

Problem 4. Let E be an elliptic curve over an algebraically closed field k. Compute $\pi_1(E)$ (the answer depends on the characteristic of k and on the Hasse invariant of E).

Hint: This is Hartshorne Ex. IV 4.8.

Problem 5. Let $k = \overline{\mathbb{F}}_p$ and let $U \subseteq \mathbb{A}^1_k$ be an open subset. Prove that there exists a finite étale morphism $U \to \mathbb{A}^1_k$.

Hint: Play with adding p-th powers.