Problem Set 3

due April 17, 2025

Problem 1. Let X be a smooth projective connected curve over $\mathbb{C}$ and let $U \subseteq X$ be a non-empty open subset. Let $f\colon V \to U$ be a finite étale covering, with V connected, and let Y be the unique smooth projective curve containing V as a dense open subset. Find a relation between the degree d of f, the genus g of X, the genus g' of Y, the cardinality n of $X \setminus U$, and the cardinality n' of $Y \setminus V$.

Hint: Use Riemann–Hurwitz.

Problem 2. A ring R is **perfect** if $\mathbb{F}_p \subseteq R$ and the Frobenius map $F\colon R \to R$ (defined by $F(x) = x^p$) is an isomorphism. *Example:* The perfection of the polynomial ring $\mathbb{F}_p[T^{1/p^\infty}] = \bigcup_{n\geq 0} \mathbb{F}_p[T^{1/p^n}]$.

1. Show that a perfect domain is noetherian if and only if it is a field.
2. Show that for every perfect $\mathbb{F}_p$-algebra R we have $\Omega^1_{R/\mathbb{F}_p} = 0$.
3. Find an example of a map of rings $A \to B$ which is formally étale but not flat.
4. Find an example of a map of rings $A \to B$ which is formally unramified and flat, but not formally étale.

Problem 3. Let $\mathcal{C}$ be a Galois category with fiber functor $F\colon \mathcal{C} \to$ **sets**, and let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$. Show that $\mathcal{C}'$ is a Galois category with fiber functor $F|_{\mathcal{C}'}$ if and only if the following conditions are satisfied:

Hint: You can assume that $\mathcal{C}$ is the category of finite Γ-sets for a profinite group Γ, with F the forgetful functor.

(a) if $X' \to X$ is an epimorphism and X' is isomorphic to an object of $\mathcal{C}'$, then so is X,

(b) an object of $\mathcal{C}$ is isomorphic to an object of $\mathcal{C}'$ if and only if all of its connected components are,

(c) the product of two objects in $\mathcal{C}'$ is isomorphic to an object of $\mathcal{C}'$.

Problem 4. Let $\{A_\lambda\}_{\lambda\in I}$ be an inductive system of rings indexed by a filtering poset I with smallest element 0, and let $A = \varinjlim A_\lambda$ be its direct limit. Let $A_0 \to B_0$ be a finitely presented homomorphism, and for every $\lambda \in I$ define $B_\lambda = B_0 \otimes_{A_0} A_\lambda$. Thus $\{B_\lambda\}_{\lambda\in I}$ forms an inductive system with direct limit $B = B_0 \otimes_{A_0} A$. Suppose that $A \to B$ is étale. Show that $A_\lambda \to B_\lambda$ is étale for some $\lambda \in I$.

Hint: First show that if $A \to B$ is standard étale, then $A_\lambda \to B_\lambda$ is standard étale for large enough λ.