Problem Set 2

due March 25, 2025

For $n, m \geq 1$ we define the **Baumslag–Solitar group** $\mathrm{BS}(n,m)$ by the following presentation

$$\mathrm{BS}(n,m) = \langle a,t \,|\, ta^n t^{-1} = a^m \rangle.$$

Problem 1. Show that for every homomorphism $\phi\colon \mathrm{BS}(2,3) \to G$ into a finite group G, we have

$$\phi([a,tat^{-1}]) = 1.$$

We will show later (or not) that $[a,tat^{-1}] \neq 1$ (can you prove it?). Thus the group $\mathrm{BS}(2,3)$ is not residually finite, and hence (by Malcev's theorem) does not admit a faithful linear representation.

Problem 2. Consider the group $\mathrm{BS}(1,2) = \langle a,t \,|\, tat^{-1} = a^2 \rangle$. Identify it with the group of affine linear transformations $f(x) = Ax + B$ with $A \in 2^{\mathbb{Z}}$ and $B \in \mathbb{Z}[1/2]$. Conclude that the subgroups generated by a and t are infinite cyclic and intersect trivially.

For Problem 3, you will need the following theorem (which does not follow from "general nonsense").

Theorem. Let $H \to G_i$ ($i = 1,2$) be injective group homomorphisms. Then the induced maps into their amalgamated product (pushout in the category of groups)

$$G_i \longrightarrow G_1 \star_H G_2$$

are injective for $i = 1,2$. In particular, the map $H \to G_1 \star_H G_2$ is injective.

Moreover, if $G_i' \subseteq G_i$ ($i = 1,2$) are two subgroups such that $G_i' \cap H = 1$, then the induced map

$$G_1' \star G_2' \longrightarrow G_1 \star_H G_2$$

is injective.

Problem 3. Consider the following group (the "Higman four group")

$$H = \langle x_1,x_2,x_3,x_4 \,|\, x_2x_1x_2^{-1} = x_1^2, x_3x_2x_3^{-1} = x_2^2, x_4x_3x_4^{-1} = x_3^2, x_1x_4x_1^{-1} = x_4^2 \rangle.$$

(a) Show that every homomorphism from H into a finite group is trivial; in other words, the profinite completion $\widehat{H}$ is the trivial group.

Hint: Look at the orders of $x_1, \ldots, x_4$ (and their smallest prime factor).

(b) Let $H_{123} = \langle x_1,x_2,x_3 \,|\, x_2x_1x_2^{-1} = x_1^2, x_3x_2x_3^{-1} = x_2^2 \rangle$. Show that x_1 and x_3 generate a free subgroup. (Use the theorem and Problem 2.)

(c) Write H as an amalgamated product of H_{123} and a similarly defined H_{341} along an F_2. Using the theorem again, conclude that H is nontrivial (and hence infinite by (a)).

(d) Show that every finitely generated group contains a maximal proper normal subgroup. Conclude that if $H_0 \subseteq H$ is such a subgroup, then $G = H/H_0$ is a finitely generated infinite simple group.