Term paper topics

Due date: June 6, 2025

Guidelines[1]

Each student should choose a topic based on their own interests, but all topics should be cleared with me in advance. Some possible topics, some more advanced than others, are suggested below. I am happy to help students choose a topic through individual consultations.

Format. Your paper should be written using Latex, and should be about 5–10 pages in length.

Style. Write your paper in the style of a research paper. That means that it has an introduction which explains the main result, and a clearly organized overall structure. The paper should be centered around one or two main results (as opposed to being a sequence of propositions and lemmas with no clear start or end).

Additional Requirement. Every paper must contain at least one nontrivial example worked out in detail.

Suggested topics

Project 1 (Non-quasiprojective groups)**.** Describe in detail an example of a finitely presented group which is not isomorphic to the fundamental group of a smooth quasi-projective complex variety. One such example is given in a recent paper of Esnault–de Jong, for earlier ones see the references in the paragraph after Theorem 1 in [12].

Suggested reading: [6]

Project 2 (The Nori fundamental group scheme)**.** Discuss finite flat group schemes and define Nori's fundamental group scheme. Give plenty of examples.

Suggested reading: [17], [21]

Project 3 (Noohi groups in topology)**.** Under what conditions is the category of covering spaces of a connected topological space a tame infinite Galois category (in the sense of Bhatt and Scholze)? This could be related to work of J. Brazas.

Suggested reading: [3], [5]

Project 4 (Simpson theory and Kähler groups)**.** Describe the restrictions on Kähler groups arising from Simpson's non-abelian Hodge correspondence.

Suggested reading: [20], [2]

Project 5 (Abhyankar's conjecture)**.** Abhyankar's conjecture, resolved by Raynaud and Harbater, describes the finite quotients of the étale fundamental group of the affine line in characteristic p. Outline some parts of the proof.

[1] Adapted from *https://math.berkeley.edu/˜molsson/Termpaper.pdf*

Suggested reading: [4]

Project 6 (The Grothendieck–Ogg–Shafarevich formula)**.** The GOS formula describes the Euler characteristic of an étale local system on a curve in terms of its wild ramification at infinity. Explain the proof of this formula and provide examples.

Suggested reading: [1]

Project 7 (The log étale fundamental group)**.** Define the log étale (or Kummer étale) fundamental group of a log scheme. Compare it to the tame fundamental group of the trivial locus (log Abhyankar's lemma).

Suggested reading: [11], [15]

Project 8 (Litt's finiteness theorem)**.** Explain the proof of Litt's theorem that on a given variety over a finitely generated field there exist only finitely many étale local systems of fixed rank which come from geometry.

Suggested reading: [13]

Project 9 (Toledo's example)**.** Describe the example, due to Toledo, of a Kähler group which is not residually finite.

Suggested reading: [22]

Project 10 (Finite presentation)**.** Discuss Lubotzky's criterion for a finitely generated profinite group to be finitely presented. Explain the proof of the recent theorem of Esnault–Shusterman–Srinivas that the étale fundamental group of a smooth projective variety in characteristic p is finitely presented.

Suggested reading: [14], [8]

Project 11 (The Narasimhan–Seshadri theorem)**.** The NS theorem is a prototope for Simpson's nonabelian Hodge correspondence, relating irreducible unitary local systems on a Riemann surface to stable vector bundles of degree zero. Explain the proof and provide examples.

Suggested reading: [16], [7]

Project 12 (The tame fundamental group and adic spaces)**.** Describe the discretely ringed adic space associated to a variety over a field and explain how it can be used to define the tame étale topology and the tame fundamental group. Prove that the resulting tame fundamental group is $\mathbb{A}^1$-invariant, assuming resolution of singularities.

Suggested reading: [9], [10]

Project 13 (Fundamental groups of conjugate varieties)**.** Describe examples, due to Serre, of a pair of complex varieties differing by an automorphism of $\mathbb{C}$ which have non-isomorphic topological fundamental groups.

Suggested reading: [18]

Project 14 (Construction of a variety with given π_1)**.** Following Simpson, show that every finitely presented group is the fundamental group of a *singular* projective variety. Mention the related result of Kapovich–Kollár.

Suggested reading: [19], [12]

References

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[5] Jeremy Brazas. Semicoverings: a generalization of covering space theory. *Homology Homotopy Appl.*, 14(1):33–63, 2012.

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