Étale fundamental group – cheat sheet

For every connected scheme X, the category $\mathbf{F\acute{E}t}_X$ of finite étale maps $Y \to X$ is a Galois category, 0BQ8 and every geometric point $\bar{x} \to X$ induces a fiber functor $F_{\bar{x}} \colon \mathbf{F\acute{E}t}_X \to \mathbf{FinSet}$. The **étale fundamental group** $\pi_1(X,\bar{x})$ is the Galois group of the pointed Galois category $(\mathbf{F\acute{E}t}_X, F_{\bar{x}})$, i.e. the automorphism group $\mathrm{Aut}(F_{\bar{x}})$. It is a profinite group. A morphism $X \to X'$ induces a continuous group homomorphism $\pi_1(X,\bar{x}) \to \pi_1(X',\bar{x})$.

The étale fundamental group enjoys the following properties.

1. **Topological invariance.** Let $i \colon X \to X'$ be a universal homeomorphism (for example, a nilpotent 0BQN closed immersion). Then $\pi_1(X,\bar{x}) \to \pi_1(X', i(\bar{x}))$ is an isomorphism.

2. **Comparison with Galois theory.** If $X = \mathrm{Spec}(K)$ for a field K, then finite étale K-algebras are finite 0BNE products of finite separable extensions of K. If $\bar{x} \to X$ is a geometric point, let K^{sep} be the separable closure of K in $k(\bar{x})$. Then $\pi_1(X,\bar{x}) \simeq \mathrm{Gal}(K^{\mathrm{sep}}/K)$.

3. **Comparison with topology.** Let X be a connected scheme locally of finite type over $\mathbb{C}$. Then SGA1 XII 5.1 analytification induces an equivalence between $\mathbf{F\acute{E}t}_X$ and the category of finite coverings of X^{an}. Consequently, for $\bar{x} \in X(\mathbb{C})$ we obtain a homomorphism $\pi_1^{\mathrm{top}}(X,\bar{x}) \to \pi_1(X,\bar{x})$ which induces an isomorphism $\pi_1^{\mathrm{top}}(X,\bar{x})^\wedge \xrightarrow{\sim} \pi_1(X,\bar{x})$ where $(-)^\wedge$ denotes profinite completion.

4. **Torsors.** Let X be a connected scheme and let G be a finite group. Then the pointed set of isomorphism SGA1 IX §5 classes of étale G-torsors on X is in bijection with the set $\mathrm{Hom}(\pi_1(X,\bar{x}),G)/G$ of conjugacy classes of continuous homomorphisms $\pi_1(X,\bar{x}) \to G$.

5. **Link with étale cohomology.** If G is abelian, then $\mathrm{Hom}(\pi_1(X,\bar{x}),G) \simeq H^1(X_{\mathrm{et}},G)$. In particular:

   (a) **Kummer theory.** For $n \geq 1$ invertible on X, we have an exact sequence SGA1 IX 6.4

   $$\mathcal{O}(X)^\times/\mathcal{O}(X)^{\times n} \longrightarrow \mathrm{Hom}(\pi_1(X,\bar{x}),\mathbb{Z}/n\mathbb{Z}) \longrightarrow \mathrm{Pic}(X)[n]$$

   (b) **Artin–Schreier theory.** If p is a prime and X is an $\mathbb{F}_p$-scheme, we have an exact sequence SGA1 IX 6.8

   $$\mathrm{cok}(1-F \colon \mathcal{O}(X) \to \mathcal{O}(X)) \longrightarrow \mathrm{Hom}(\pi_1(X,\bar{x}),\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X)^{F=1}.$$

6. **Criteria for surjectivity.** Let $f \colon X \to X'$ be a map of connected schemes and $\bar{x} \to X$ a geometric 0BN6 point. The following are equivalent: (a) $\pi_1(X,\bar{x}) \to \pi_1(X', f(\bar{y}))$ is surjective, (b) the pull-back functor $f^* \colon \mathbf{F\acute{E}t}_{X'} \to \mathbf{F\acute{E}t}_X$ is fully faithful, and (c) if $Y' \to X'$ is a connected finite étale covering, then $f^*(Y') = Y' \times_{X'} X$ is connected.

7. **π_1 of a normal scheme.** If X is an integral normal scheme, with generic point η, and $\bar{\eta} \to X$ is the SGA1 V 8.2 geometric generic point, then $\pi_1(X,\bar{\eta})$ is the quotient of $\mathrm{Gal}(k(\bar{\eta})/k(\eta))$ corresponding to the union of all finite separable extensions L of $k(\eta)$ contained in $k(\bar{\eta})$ such that the normalization of X in L is étale over X.

8. **Homotopy exact sequence.** let $f \colon X \to S$ be a be a flat proper morphism of finite presentation 0C0J between connected schemes whose geometric fibers are connected and reduced. Let $\bar{x} \to X$ be a geometric point and let $\bar{s} \to S$ be its image in S. Then the sequence of groups

   $$\pi_1(X_{\bar{s}},\bar{x}) \longrightarrow \pi_1(X,\bar{x}) \longrightarrow \pi_1(S,\bar{s}) \longrightarrow 1$$

   is exact.

9. **Fundamental exact sequence.** Let X be a qcqs scheme of finite type over a field k, let $\overline{k}$ be an algebraic closure of k and $\overline{x} \to X$ a $\overline{k}$-valued point. If $X_{\overline{k}}$ is connected, then the sequence of groups 0BTX

$$1 \longrightarrow \pi_1(X_{\overline{k}}, \overline{x}) \longrightarrow \pi_1(X, \overline{x}) \longrightarrow \mathrm{Gal}(\overline{k}/k) \longrightarrow 1$$

is exact.

10. **Zariski–Nagata purity of the branch locus.** Let X be a regular connected scheme and let $U \subseteq X$ be a non-empty open subset such that $\mathrm{codim}(X \setminus U) \geq 2$. Then U is connected and $\pi_1(U, \overline{x}) \to \pi_1(X, \overline{x})$ is an isomorphism. SGA1 X 3.3

11. **Lefschetz hyperplane theorem.** Let X be a connected projective scheme over a field k and let $Y \subseteq X$ be an ample effective Cartier divisor, $\overline{x} \to Y$ a geometric point. Suppose that $X \setminus Y$ is regular of dimension ≥ 2 (resp. ≥ 3). Then Y is connected and $\pi_1(Y, \overline{x}) \to \pi_1(X, \overline{x})$ is surjective (resp. an isomorphism). SGA2 X 2.2+3.10

12. **Birational invariance.** Let $f\colon Y \to X$ be a proper birational morphism of connected regular schemes and $\overline{y} \to Y$ a geometric point. Then $\pi_1(Y, \overline{y}) \to \pi_1(X, f(\overline{y}))$ is an isomorphism. SGA1 X 3.4

13. **Effective descent.** Let $f\colon Y \to X$ be surjective map which is either proper or flat+qcqs. Then $\mathbf{FÉt}_X \to \mathbf{DD}(Y/X, \mathbf{FÉt})$ is an equivalence. That is, finite étale maps to X are equivalent to finite étale maps to Y endowed with an isomorphism of the two pull-backs to $Y \times_X Y$ satisfying the cocycle condition on $Y \times_X Y \times_X Y$. SGA1 IX 6.8

14. **Specialization.** Let $\mathcal{O}$ be a complete discrete valuation ring, let $S = \mathrm{Spec}(\mathcal{O})$, and let s and η be its closed and generic point, respectively. Let X be a proper scheme over S. Then, $\mathbf{FÉt}_X \to \mathbf{FÉt}_{X_s}$ is an equivalence. As a result, if $X \to S$ has connected fibers, we obtain the specialization homomorphism 0BUQ

$$\pi_1(X_\eta) \longrightarrow \pi_1(X) \simeq \pi_1(X_s)$$

which is surjective if X is normal.

Suppose that $X \to S$ has connected geometric fibers. Then, we have a specialization homomorphism for the geometric fundamental groups SGA1 X 2.4

$$\pi_1(X_{\overline{\eta}}) \longrightarrow \pi_1(X_{\overline{s}})$$

which is surjective if $X_{\overline{s}}$ is reduced. If $X \to S$ is smooth, and p is the residue characteristic exponent of $\mathcal{O}$, the map induces an isomorphism on prime-to-p completions. SGA1 X 3.9

15. **Finite generation.** Let X be a connected scheme of finite type over an algebraically closed field k. Suppose that either $\mathrm{char}(k) = 0$ or that X is proper. Then $\pi_1(X)$ is topologically finitely generated (admits a dense finitely generated subgroup). SGA1 X 2.9

16. **Künneth formula.** Let X and Y be locally noetherian connected schemes over an algebraically closed field k. Suppose that either $\mathrm{char}(k) = 0$ and X is of finite type over k, or X is proper over k. Then $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$. SGA1 X 1.7

17. **Change of base field (π_1-properness).** Let X be a connected scheme of fintie type over an algebraically closed field k and let k' be an algebraically closed extension of k. Suppose that either $\mathrm{char}(k) = 0$ or X is proper over k. Then $\pi_1(X_{k'}) \to \pi_1(X)$ is an isomorphism. SGA1 X 1.8