Fundamental groups in algebraic geometry

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Foreword

This document contains lecture notes for a course taught at MIMUW in Spring 2025. The notes are under construction and will be (hopefully) frequently updated. Orange text and orange boxes ■ indicate things left to be filled in. Comments (including typos) are very welcome.

Sections and subsections marked with (A) contain advanced/supplementary material. Those marked with (B) review background material that would best fit a different course (perhaps one you already attended).

Further versions of the notes, together with other course material, will be posted at

https://achinger.impan.pl/lecture25s.html

Notation.

(a) Both the neutral element of a group and the trivial group are denoted by 1.

(b) The notation $\langle x_1, \ldots, x_n \,|\, w_1, \ldots, w_r \rangle$, where x_i are symbols and w_j are words in the x_i and x_i^{-1}, denotes the quotient of the free group on $\{x_1, \ldots, x_n\}$ by the smallest normal subgroup containing $w_1, \ldots, w_r$.

(c) References like [SP Tag 0ABC] refer to the Stacks Project.

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Part I
The topological fundamental group

1. Lecture 1: Fundamental groups of complex varieties

Summary. *We give a short overview of the goals of this lecture course in §1.1 and review some basic facts about the fundamental group of a topological space in §1.2. In §1.3 we discuss additional related material: descent, torsors, equivariant sheaves, group cohomology, and a characterization of $K(\pi, 1)$ spaces in terms of cohomology of local systems. In §1.4 we move to algebraic geometry; we define the topological fundamental group of a complex variety and give some examples. The final §1.5 discusses very basic corollaries of the Hodge decomposition.*

1.1. Goals and motivation

Our goal is to study the fundamental groups of algebraic varieties, and more generally of schemes. The definition from topology, using paths, really makes sense only over $\mathbb{C}$ and the first part of the lecture will focus on the properties of $\pi_1(X(\mathbb{C}))$ for a complex variety X. Here $X(\mathbb{C})$ is the set of closed points of X endowed with the analytic topology.

For more general schemes, a considerable effort is put into actually defining a suitable fundamental group in an algebraic way. This is a nontrivial task even for varieties defined over subfields $K \subseteq \mathbb{C}$, since (as first shown by Serre [Serre, 1964], see §3.3) the fundamental group $\pi_1(X(\mathbb{C}))$ might depend on the choice of the complex embedding $K \hookrightarrow \mathbb{C}$. The simplest (and maybe most useful) such definition is the étale fundamental group $\pi_1^{\text{ét}}(X)$, introduced by Grothendieck in SGA1 [Grothendieck, 1971]. Here X can be an arbitrary connected scheme, and $\pi_1^{\text{ét}}(X)$ is a profinite group (the inverse limit of finite groups); for example, if $X = \mathrm{Spec}(K)$ for a field K, then $\pi_1^{\text{ét}}(X)$ is the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$, and if X is locally of finite type over $\mathbb{C}$, then $\pi_1^{\text{ét}}(X)$ is the "profinite completion" (inverse limit of all finite quotients) of $\pi_1(X(\mathbb{C}))$. The goal of the second part of the course will be the development of the theory of the étale fundamental group.

There are (at least) two major reasons for studying fundamental groups of algebraic varieties:

(a) Classifying varieties up to homotopy is a first crude step in the classification process: in a family of complex algebraic varieties (say, smooth and proper), all the fibers are homeomorphic.

(b) In algebraic geometry, topological invariants such as the fundamental group or cohomology often come equipped with additional structures which actually vary within a family and carry important geometric and arithmetic information.

For an example of (2), a complex elliptic curve can be recovered from its first integral cohomology endowed with its Hodge decomposition.[1] In a different direction, if X is a variety over $\mathbb{Q}$, then the étale fundamental group of $X_{\mathbb{C}}$ (which is the profinite completion of $\pi_1(X(\mathbb{C}))$ and hence a rather concrete object) carries a continuous outer action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This action typically does not lift to an action on $\pi_1(X(\mathbb{C}))$. According to Grothendieck's anabelian conjectures, proved by Tamagawa and Mochizuki, a hyperbolic algebraic curve over $\mathbb{Q}$ (e.g. a smooth projective curve of genus $g \geq 2$, or $\mathbb{P}^1_{\mathbb{Q}} \setminus S$ with $|S| \geq 3$) can be reconstructed from this action. (Add references here)

1.2. Review of fundamental groups and covering spaces (B)

Even though we assume basic familiarity with the concept, we recall the definition of the fundamental group of a topological space. We advise the reader to pay attention to left/right actions which appear in the theory.

Definition 1.2.1. Let X be a topological space and let $x \in X$. The **fundamental group** of X with base point x, denoted $\pi_1(X,x)$, consists of homotopy class of loops in X based at x.

Here, a loop based at x is a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = \gamma(1) = x$, and two loops γ_0, γ_1 are homotopic if there exists a continuous map $H\colon [0,1] \times [0,1] \to X$ such that $H(0,y) = \gamma_0(y)$, $H(1,y) = \gamma_1(y)$, and $H(t,0) = H(t,1) = x$. The product $\gamma \cdot \gamma'$ is defined by first traversing γ' and then γ **(!)**, and the inverse γ^{-1} is represented by $t \mapsto \gamma(1-t)$.

If $x' \in X$ is another base point, and if $\lambda\colon [0,1] \to X$ is a path from x to x' (i.e. $\gamma(0) = x$ and $\gamma(1) = x'$), we obtain an isomorphism

$$\pi_1(X,x) \longrightarrow \pi_1(X,x')$$

defined by mapping γ to $\lambda\gamma\lambda^{-1}$.

We say that X is **path connected** if for every $x, x' \in X$ there exists a path from x to x'. We say that X is **simply connected** if it is path connected and $\pi_1(X,x) = 1$ (this does not depend on the choice of x).

The fundamental group admits a convenient description in terms of covering spaces.

Definition 1.2.2. Let X be a topological space. A **covering** of X is a map $Y \to X$ such that there exists an index set I, and for every $i \in I$ open subsets $U_i \subseteq X$ covering X, sets S_i, and homeomorphisms

$$\phi_i\colon Y \times_X U_i \xrightarrow{\sim} U_i \times S_i$$

over U_i (i.e. making the natural triangle commute). A morphism of coverings $(Y' \to X) \to (Y \to X)$ is a map $Y' \to Y$ over X. We denote the category of coverings of X by $\mathbf{Cov}_X$.

[1] Explicitly: E is the cokernel of $H^1(E,\mathbb{Z}) \to H^1(E,\mathbb{C}) \simeq H^0(E,\Omega^1_E) \oplus H^1(E,\mathcal{O}_E) \xrightarrow{\text{proj.}} H^0(E,\Omega^1_E)$.

Note that a continuous map $g\colon X' \to X$ induces a functor in the opposite direction

$$g^*\colon \mathbf{Cov}_X \longrightarrow \mathbf{Cov}_{X'}, \qquad g^*(Y \to X) = (Y \times_X X' \to X').$$

Covering spaces can also be characterized "internally" using sheaves.

Lemma 1.2.3. *Let X be a topological space. Denote by $\mathbf{Loc}_X$ the category of locally constant sheaves (of sets) on X. If $Y \to X$ is a covering space, then its sheaf of sections $\mathcal{F}$ defined by $\mathcal{F}(U) = \mathrm{Hom}_X(U,Y)$ is locally constant. The association $(Y \to X) \mapsto \mathcal{F}$ defines an equivalence of categories*

$$\mathbf{Cov}_X \xrightarrow{\sim} \mathbf{Loc}_X.$$

Proof. Worthwhile exercise. □

Coverings enjoy the following unique lifting property with respect to paths: if $f\colon Y \to X$ is a covering, $\gamma\colon [0,1] \to X$ is a path, and $y_0 \in Y$ is a point with $f(y_0) = \gamma(0)$, then there exists a unique path $\gamma'\colon [0,1] \to Y$ such that $\gamma'(0) = y_0$ and $\gamma = f \circ \gamma'$. Thus, if γ is a loop based at x ($\gamma(0) = \gamma(1) = x$), then it induces a map $f^{-1}(x) \to f^{-1}(x)$ sending y_0 to $\gamma'(1)$. This map depends only on the homotopy class of γ and is compatible with composition, thus defining a left action of $\pi_1(X,x)$ on the fiber $f^{-1}(x)$. In turn, we obtain a functor from $\mathbf{Cov}_X$ to left $\pi_1(X,x)$-sets sending $f\colon Y \to X$ to the fiber $f^{-1}(x)$ equipped with the action just described.

Theorem 1.2.4. *Let X be a path connected and locally simply connected topological space.*[2] *Then the functor*

$$F\colon \mathbf{Cov}_X \longrightarrow \pi_1(X,x)\text{-sets}, \qquad F(f\colon Y \to X) = f^{-1}(x) \tag{1.2.1}$$

is an equivalence of categories.

Corollary 1.2.5. *In the situation of the theorem, we obtain a bijection between the set of isomorphism classes of connected coverings of X and the set of subgroups of $\pi_1(X,x)$ up to conjugation. If Y corresponds to H then $\pi_1(Y) \simeq H$.*

The group $\pi_1(X,x)$ defines an object of the category of left $\pi_1(X,x)$-sets. The corresponding covering space $\tilde{X} \to X$ is called the **universal covering**. By the above corollary, it is the unique connected and simply connected covering of X.

Remarks 1.2.6. (a) The group $\pi_1(X,x)$ operates on the right **(!)** on $\tilde{X}$. Indeed, the right multiplication action of $\pi_1(X,x)$ on itself is an action in the category of left $\pi_1(X,x)$-sets. This action is free, and $X \simeq \tilde{X}/\pi_1(X,x)$ is the quotient space.

(b) In many contexts (e.g. topological spaces with bad local properties, or schemes...) the theory of coverings is more robust than the theory of paths.

[2] There are about ten possible definitions of "locally simply connected." Here we use the following: for every $x \in X$ and every open neighborhood U of x there exists an open neighborhood V of x contained in U with $\pi_1(V,x) = 1$.

(c) One can recover $\pi_1(X,x)$ from $\mathbf{Cov}_X$. To make this canonical (an extra choice is needed since $\mathbf{Cov}_X$ does not know of the base point x), we equip $\mathbf{Cov}_X$ with the fiber functor

$$F\colon \mathbf{Cov}_X \longrightarrow \mathbf{Sets}, \qquad F(f\colon Y\to X) = f^{-1}(x)$$

(via the equivalence (1.2.1) this corresponds to the forgetful functor from $\pi_1(X,x)$-sets to sets). Then $\pi_1(X,x)$ is canonically isomorphic to the automorphism group $\mathrm{Aut}(F)$ of the functor F.

(d) To avoid confusion we will consistently use the term *covering* as in *covering space* and the word *cover* as in *open cover*.

(e) The van Kampen theorem for an open cover $X = U\cup V$, $W = U\cap V$ with U,V,W path connected can be translated into (and in fact deduced from) the following gluing (descent) statement:

$$\mathbf{Cov}_X \simeq \mathbf{Cov}_U \times_{\mathbf{Cov}_W} \mathbf{Cov}_V,$$

where the right-hand side is the category of triples $(U'\to U, V'\to V, \varphi)$ where $U'\to U$ and $V'\to V$ are coverings and φ is an isomorphism

$$\varphi\colon U'\times_U W \longrightarrow V'\times_V W$$

in the category $\mathbf{Cov}_W$. We will discuss more general descent statements soon.

1.3. Complements: torsors, equivariant sheaves, and group cohomology

In this section we review some additional material that will be very useful throughout the semester. Not everything here has been discussed in the lecture.

Torsors. Let X be a space and let G be a sheaf of groups on X (for example, a constant sheaf). A ***G*-torsor** on X (compare [SP Tag 03AH]) is a sheaf of sets $\mathcal{F}$ on X endowed with a right action (in the category of sheaves):

$$\mu\colon \mathcal{F}\times G \longrightarrow \mathcal{F}$$

such that for every $x\in X$, the action of G_x on $\mathcal{F}_x$ is free and transitive. (In particular, this implies that $\mathcal{F}_x$ is non-empty for every $x\in X$.) In this case, for every open $U\subseteq X$ such that $\mathcal{F}(U)$ is non-empty, the action of $G(U)$ on $\mathcal{F}(U)$ is free and transitive as well (exercise). A morphism of G-torsors is a map of sheaves $\mathcal{F}'\to\mathcal{F}$ which is equivariant for the G-action (every such map must be an isomorphism). We say that $\mathcal{F}$ is **trivial** if $\mathcal{F}\simeq G$ (the sheaf G is a G-torsor for the right multiplication action), or equivalently if $\mathcal{F}(X)\neq\emptyset$.

Let $\mathcal{F}$ be a G-torsor on X and let $\{U_i\}_{i\in I}$ be an open cover of X such that $\mathcal{F}(U_i)\neq\emptyset$ (which exists since $\mathcal{F}_x\neq\emptyset$ for all $x\in X$). For each $i\in I$ choose a section $s_i\in\mathcal{F}(U_i)$. Since the action of $G(U_i\cap U_j)$ on $\mathcal{F}(U_i\cap U_j)$ is free and transitive, there exists a unique $g_{ij}\in G(U_i\cap U_j)$ such that

$$s_j|_{U_i\cap U_j} = (s_i|_{U_i\cap U_j})\cdot g_{ij}.$$

The system $\{g_{ij}\}$ forms a 1-cocycle: on $U_i \cap U_j \cap U_k$ we have $g_{ik} = g_{ij} \cdot g_{jk}$. Conversely, for any 1-cocycle $\{g_{ij}\}$ we can build a G-torsor $\mathcal{F}$: its restriction $\mathcal{F}_i$ to U_i will be just $G|_{U_{ij}}$, and the gluing isomorphism $\mathcal{F}_i|_{U_i \cap U_j} \simeq \mathcal{F}_j|_{U_i \cap U_j}$ is given by the right multiplication by g_{ij}. We define the **nonabelian first cohomology** of G as the filtered colimit

Suppose that G is the constant sheaf (with value G). In this case, using Lemma 1.2.3, G-torsors on X can be described as "principal G-bundles" or "G-coverings," i.e. covering spaces $Y \to X$ where Y is endowed with a free right G-action for which $Y \to X$ is invariant and induces an isomorphism $Y/G \xrightarrow{\sim} X$. If, in addition, X is path connected and locally simply connected, Theorem 1.2.4 implies that G-torsors on X are equivalent to left $\pi_1(X,x)$-sets endowed with a compatible transitive right G-action. Their isomorphism classes are in bijection with the set of group homomorphisms $\pi_1(X,x) \to G$ up to conjugation.

Equivariant sheaves. Let G be a group acting on a space Y on the right. A G-equivariant sheaf is a sheaf on X endowed with a "compatible G-action." A completely precise definition is given below, where we denote the action by $\mu \colon Y \times G \to Y$ and the projection to Y by $\mathrm{pr} \colon Y \times G \to Y$.

Definition 1.3.1. Let G be a group acting on a space Y on the right. A ***G*-equivariant sheaf** on Y is the data of a sheaf $\mathcal{F}$ on Y and an isomorphism

$$\phi \colon \mu^*\mathcal{F} \simeq \mathrm{pr}^*\mathcal{F} \qquad \text{on } Y \times G$$

satisfying the following cocycle condition on $Y \times G \times G$:

$$\mathrm{pr}_{01}^*\phi \circ (\mu \times \mathrm{id}_G)^*\phi \simeq (\mathrm{id}_X \times m)^*\phi$$

where $m \colon G \times G \to G$ is the multiplication map.

Thus concretely, a G-equivariant sheaf on Y is a sheaf $\mathcal{F}$ on Y together with the data of, for every $g \in G$, an isomorphism

$$\phi_g \colon g^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$$

satisfying the compatibility condition that the following triangle commutes for every $g, h \in G$

$$\begin{array}{ccc} (gh)^*\mathcal{F} & \xrightarrow{\phi_{gh}} & \mathcal{F} \\ \Vert & & \uparrow{\scriptstyle \phi_g} \\ g^*(h^*\mathcal{F}) & \xrightarrow[g^*(\phi_h)]{} & g^*\mathcal{F} \end{array}$$

The more abstract definition given above is useful for actions of non-discrete groups (in which case one also wants to require that "the ϕ_g vary continuously with g"), for example for a linear algebraic group acting on an algebraic variety.

Descent. In general by "descent" we mean that certain objects (sheaves, their sections, covering spaces etc.) can be defined locally, subject to "gluing isomorphisms on double overlaps" satisfying a certain cocycle condition. We will focus on sheaves of sets; the category of sheaves of sets on a space X will be denoted by $\mathbf{Sh}(X)$.

Recall that if $Y \to X$ is any map of spaces, we can build the simplicial space $Y_{/X}^{\bullet+1}$ whose n-th space is the $(n+1)$-fold fiber product of Y over X. We can picture it as a diagram

$$\cdots \qquad Y \times_X Y \times_X Y \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; Y \times_X Y \;\substack{\longrightarrow\\\longrightarrow}\; Y \longrightarrow X$$

If $\mathcal{F}$ is a sheaf on Y, its pullback $\mathcal{G} = \pi^*\mathcal{F}$ is equipped with additional structure: its further pullbacks $\mathrm{pr}_0^*\mathcal{G}$ and $\mathrm{pr}_1^*\mathcal{G}$ are both identified with $(\pi \circ \mathrm{pr}_0)^*\mathcal{F} = (\pi \circ \mathrm{pr}_1)^*\mathcal{F}$; call this isomorphism ϕ. The isomorphism ϕ satisfies the following "cocycle condition" on $Y \times_X Y \times_X Y$:

$$\begin{array}{ccc} \mathrm{pr}_0^*\mathcal{G} & \xrightarrow{\mathrm{pr}_{02}^*\phi} & \mathrm{pr}_2^*\mathcal{G} \\ & {\scriptstyle \mathrm{pr}_{01}^*\phi}\searrow \quad \nearrow{\scriptstyle \mathrm{pr}_{12}^*\phi} & \\ & \mathrm{pr}_1^*\mathcal{G} & \end{array} \tag{1.3.1}$$

Definition 1.3.2 (see [SP Tag 026B])**.** The category of descent data $\mathbf{DD}(Y/X)$ is the category of pairs $(\mathcal{G}, \varphi)$ where $\mathcal{G}$ is a sheaf on Y and $\varphi \colon \mathrm{pr}_1^*\mathcal{G} \xrightarrow{\sim} \mathrm{pr}_2^*\mathcal{G}$ is an isomorphism of sheaves on $Y \times_X Y$ satisfying the cocycle condition (1.3.1).

The above discussion shows that π^* can be upgraded to a functor

$$\pi^* \colon \mathbf{Sh}(X) \longrightarrow \mathbf{DD}(Y/X).$$

We say that π is of **effective descent** if the above functor is an equivalence of categories.

A typical example of effective descent is when $Y = \bigcup U_i$ is the disjoint union of a family of open subsets $U_i \subseteq X$ covering X. In this case, effective descent means that in order to define a sheaf on X it is enough to define sheaves on U_i together with isomorphism on double overlaps $U_i \cap U_j$ satisfying the cocycle condition on $U_i \cap U_j \cap U_k$. More generally, we have the following result.

Proposition 1.3.3. *Let $\pi \colon Y \to X$ be a surjective local homeomorphism. Then π is of effective descent.*

(A natural example of a local homeomorphism which does not come from an open cover is a (surjective) covering space. We will come back to this shortly.) In topology, proper surjective maps are also of effective descent [Reiterman and Tholen, 1994, Corollary 1.2]. This means in particular that sheaves can be defined on a locally finite *closed* cover. For us, another important example of effective descent comes from group quotients.

Descent along a free group quotient. Let G be a group and let $\pi \colon Y \to X$ be a G-torsor, so that $X = Y/G$. Then $Y \times_X Y$ consists of pairs of y, y' of points lying in the same G-orbit; since the action is free, we have $y' = yg$ for a unique $g \in G$, and hence the data of (y, y') is equivalent to the data (y,g). More generally, we have a commutative diagram (isomorphism of truncated simplicial objects), with the horizontal arrows being isomorphisms

$$\begin{array}{ccc} Y \times_X Y \times_X Y & \xrightarrow{\sim} & Y \times G \times G \\ \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\ Y \times_X Y & \xrightarrow{\sim} & Y \times G \\ \downarrow\downarrow & & \downarrow\downarrow \\ Y & \xrightarrow{\sim} & Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sim} & X \end{array}$$

This justifies the following proposition.

Proposition 1.3.4. *We have equivalences of categories*

$$\mathbf{Sh}_G(Y) \simeq \mathbf{DD}(Y/X) \simeq \mathbf{Sh}(X).$$

Remark 1.3.5. Let Y be a space with a right action of a group G. The category of G-equivariant sheaves $\mathbf{Sh}_G(Y)$ is a stand-in for the category of sheaves on X in case the action of G is not free. In other words, it can be thought of as the category of a possibly non-existent free quotient Y/G. One can make this precise using stacks. An extreme example is when $Y = \star$ is the one-point space, in which case $\mathbf{Sh}_G(X)$ is the category of G-sets. This category is the category of sheaves on the "classifying space BG." (In fact, in the theory of stacks one defines BG as the quotient stack $[\star/G]$.)

The following fundamental result allows one to relate the cohomology of Y and of $X = Y/G$.

Proposition 1.3.6 ([Grothendieck, 1957, II 5.2])**.** *Let $\pi\colon Y \to X$ be a G-torsor and let $\mathcal{F}$ be an abelian sheaf on X. There is a spectral sequence, functorial in the sheaf $\mathcal{F}$*

$$E_2^{pq} = H^p(G, H^q(Y, \pi^*\mathcal{F})) \quad \Rightarrow \quad H^{p+q}(X, \mathcal{F}).$$

Proof. We first check that we have a commutative diagram of categories and functors

$$\mathbf{Sh}(X,\mathbb{Z}) \xrightarrow[\pi^*]{\sim} \mathbf{Sh}_G(Y,\mathbb{Z}) \xrightarrow{\Gamma(Y,-)} \mathbf{Mod}(\mathbb{Z}[G]) \xrightarrow{(-)^G} \mathbf{Mod}(\mathbb{Z}), \qquad \Gamma(X,-)\colon \mathbf{Sh}(X,\mathbb{Z}) \to \mathbf{Mod}(\mathbb{Z})$$

Now, the functors $\Gamma(X,-)$, $\Gamma(Y,-)$ and $(-)^G$ above are left-exact and their derived functors are respectively $H^*(X,-)$, $H^*(Y,-)$, and $H^*(G,-)$. The desired spectral sequence will be the Grothendieck spectral sequence for the composition of functors (tacitly identifying $\mathbf{Sh}(X,\mathbb{Z})$ with $\mathbf{Sh}_G(Y,\mathbb{Z})$).

In order to have the spectral sequence, we need to check that the first functor $\Gamma(Y,-)$ sends injective abelian sheaves on X to $\mathbb{Z}[G]$-modules which are acyclic for the δ-functor $H^*(G,-)$. In fact, if $\mathcal{F}$ is an injective abelian sheaf on X, then the $\mathbb{Z}[G]$-module

$$M = \Gamma(Y, \pi^*\mathcal{F})$$

is injective. Indeed, let $A \hookrightarrow B$ be an injection of $\mathbb{Z}[G]$-modules; we want to show that

$$\mathrm{Hom}_G(B,M) \longrightarrow \mathrm{Hom}_G(A,M)$$

is surjective. But

$$\mathrm{Hom}_G(A,M) = \mathrm{Hom}_G(A,\Gamma(Y,\pi^*\mathcal{F})) = \mathrm{Hom}_{\mathbf{Sh}_G(Y,\mathbb{Z})}(\underline{A}_Y,\pi^*\mathcal{F}) = \mathrm{Hom}_{\mathbf{Sh}(X,\mathbb{Z})}(\underline{A}_X^G,\mathcal{F})$$

and similarly for B. Since $\underline{A}_X^G \to \underline{B}_X^G$ is injective, the map in question is surjective if $\mathcal{F}$ is injective. □

As a corollary of this spectral sequence, we get edge maps[3]

$$H^n(X,\mathcal{F}) \longrightarrow H^n(Y,\pi^*\mathcal{F})^G. \tag{1.3.2}$$

$$H^n(G,\Gamma(Y,\pi^*\mathcal{F})) \longrightarrow H^n(X,\mathcal{F}) \tag{1.3.3}$$

The following corollary is often very useful.

Corollary 1.3.7. *Let G be a finite group of order n such that $n\colon \mathcal{F} \to \mathcal{F}$ is an isomorphism (for example, $\mathcal{F}$ is a sheaf of $\mathbb{Q}$-vector spaces). Then the maps* (1.3.2)

$$H^n(X,\mathcal{F}) \longrightarrow H^n(Y,\pi^*\mathcal{F})^G$$

are isomorphisms for all $n \geq 0$.

Proof. It suffices to note that $E_2^{pq} = 0$ for $p > 0$, so that the spectral sequence collapses. □

Group cohomology and sheaf cohomology. $K(\pi,1)$ spaces. We will now focus on the other map (1.3.3). We have the following corollary.

Corollary 1.3.8. *Let X be a path connected and locally simply connected topological space, let $x \in X$, and let $\tilde{X} \to X$ be its universal cover (which is a $\pi_1(X,x)$-torsor by Remarks 1.2.6(1)). Then for every locally constant sheaf of abelian groups $\mathcal{F}$ on X, we have the maps*

$$\rho^n\colon H^n(\pi_1(X,x),\mathcal{F}_x) \longrightarrow H^n(X,\mathcal{F}). \tag{1.3.4}$$

These maps are isomorphisms for $n = 0,1$.

Proof. We apply Proposition 1.3.6 to the $\pi_1(X,x)$-torsor $\tilde{X}$, noting that $\pi^*\mathcal{F}$ is the constant sheaf on the connected space $\tilde{X}$ with value $\mathcal{F}_x$, and that the action of $\pi_1(X,x)$ on $\Gamma(\tilde{X},\pi^*\mathcal{F}) = \mathcal{F}_x$ agrees with the monodromy action of $\pi_1(X,x)$. The map ρ^n is then the edge map (1.3.3).

The spectral sequence trivially shows this is an isomorphism for $n = 0$. For $n = 1$, the obstruction to this is the group

$$E_2^{01} = H^0(\pi_1(X,x),H^1(\tilde{X},\pi^*\mathcal{F})).$$

But $\tilde{X}$ is simply connected, so that its first cohomology with values in any (locally) constant sheaf vanishes. One can see this (again) by noting that classes in $H^1(\tilde{X},\mathcal{F}_x)$ correspond to $\mathcal{F}_x$-torsors on $\tilde{X}$. □

[3]For a concrete construction of (1.3.3) without the use of spectral sequences see [Mumford, 2008, Chapter I, Appendix to §2].

Definition 1.3.9. Let X be a path connected and locally simply connected topological space. We say that X is (cohomologically) a $K(\pi,1)$ **space** if for every local system $\mathcal{F}$ the map (1.3.4) is an isomorphism for all $n \geq 0$.

Lemma 1.3.10. *Let X be a connected CW complex. Then X is a $K(\pi,1)$ space if and only if $\pi_i(X) = 0$ for $i \geq 2$.*

Proof. By the fibration homotopy exact sequence, it suffices to show that the universal covering $\tilde{X}$ is contractible. By Whitehead's theorem for cohomology [■], this will follow if we know that

$$H^n(\tilde{X},\mathbb{Z}) = 0 \quad \text{for } n \geq 1.$$

We have $H^n(\tilde{X},\mathbb{Z}) \simeq H^n(X,\pi_*\mathbb{Z})$. Since X is a $K(\pi,1)$, the latter group is isomorphic to $H^n(\pi_1(X,x),(\pi_*\mathbb{Z})_x)$. The $\pi_1(X,x)$-module $(\pi_*\mathbb{Z})_x$ is isomorphic to $\mathrm{Hom}(\pi^{-1}(x),\mathbb{Z})$ which can be identified with the group algebra $\mathbb{Z}[\pi_1(X,x)]$. Since this is a projective $\pi_1(X,x)$-module, we have $H^n(\pi_1(X,x),(\pi_*\mathbb{Z})_x) = 0$ for $n \geq 1$. □

1.4. The fundamental group of a complex variety

Let X be a scheme locally of finite type over $\mathbb{C}$. We denote by $X(\mathbb{C})$ or by X^{an} its set of complex points (by Nullstellensatz, this agrees with the set of closed points of X). We endow $X(\mathbb{C})$ with the **analytic topology** as follows. A subset $Z \subseteq X(\mathbb{C})$ is closed if and only if for every ideal $I \subseteq \mathbb{C}[T_1,\dots,T_n]$ and every map

$$W = \mathrm{Spec}(\mathbb{C}[T_1,\dots,T_n]/I) \longrightarrow X$$

the preimage of Z inside $W(\mathbb{C}) \subseteq \mathbb{C}^n$ is closed in the metric topology on $\mathbb{C}^n$. The natural map

$$\tau_X \colon X(\mathbb{C}) \longrightarrow X$$

is then continuous (as Zariski closed subsets of $\mathbb{C}^n$ are also closed in the metric topology). A morphism $X' \to X$ of schemes locally of finite type over $\mathbb{C}$ induces a continuous map $X'(\mathbb{C}) \to X(\mathbb{C})$ which commutes with τ_X and $\tau_{X'}$.

Later on, we shall equip $X(\mathbb{C}) = X^{\mathrm{an}}$ with a sheaf of holomorphic functions $\mathcal{O}_{X^{\mathrm{an}}}$, making it into a complex analytic space and upgrading τ_X to a map of locally ringed spaces. For now we will focus solely on the topological properties of $X(\mathbb{C})$.

Theorem 1.4.1. *Let X be a scheme locally of finite type over $\mathbb{C}$. The space $X(\mathbb{C})$ enjoys the following properties.*

(a) It is locally contractible (in particular, locally simply connected).

(b) The map τ_X induces a bijection on connected components

$$\pi_0(X(\mathbb{C})) \xrightarrow{\sim} \pi_0(X).$$

(c) If X is of finite type, then $X(\mathbb{C})$ has the homotopy type of a finite CW complex.

Definition 1.4.2. Let X be a scheme locally of finite type over $\mathbb{C}$ and let $x \in X(\mathbb{C})$. We set

$$\pi_1^{\mathrm{top}}(X,x) = \pi_1(X(\mathbb{C}),x)$$

and call it the **topological fundamental group** of X.

Corollary 1.4.3. *Let X be a scheme locally of finite type over $\mathbb{C}$ and let $x \in X(\mathbb{C})$. Then the topological fundamental group $\pi_1^{\mathrm{top}}(X,x)$ is finitely presented.*

Example 1.4.4. (a) Let $X = \mathbf{P}^1_{\mathbb{C}} \setminus S$ where S is a set of $n+1$ closed points. Then $\pi_1^{\mathrm{top}}(X)$ is a free group on n letters.

(b) Let X be a smooth projective curve of genus g. Then $\pi_1^{\mathrm{top}}(X) \simeq \Gamma_g$ is the genus g **surface group**

$$\Gamma_g = \langle a_1, b_1, \dots, a_g, b_g \,|\, [a_1,b_1]\cdots[a_g,b_g] = 1\rangle.$$

According to the uniformization theorem, the universal covering of X is $\mathbb{P}^1_{\mathbb{C}}$ if $g=0$, $\mathbb{C}$ if $g=1$, and the open disk $D \subseteq \mathbb{C}$ if $g \geq 2$.

(c) Let $X = \mathbb{A}^n \setminus V(f)$ where f is the polynomial

$$f = \prod_{i<j}(T_i - T_j).$$

Thus X is the complement of a hyperplane arrangement; its $\mathbb{C}$-points correspond tuples $(x_1,\dots,x_n)$ of pairwise distinct complex numbers. Then $\pi_1(X)$ is the pure braid group P_n on n strands.

(d) The space X in (3) admits a free action of the permutation group S^n. Let $X' = X/S_n$ be the quotient (whose $\mathbb{C}$-points are smooth length n subschemes of $\mathbb{A}^1_{\mathbb{C}}$). Then $\pi_1^{\mathrm{top}}(X')$ is the braid group B_n on n strands, sitting inside the extension

$$1 \longrightarrow P_n \longrightarrow B_n \longrightarrow S_n \longrightarrow 1$$

which corresponds to the S_n-torsor $X \to X'$.

(e) Let X be an abelian variety of dimension g. Then $\pi_1^{\mathrm{top}}(X) \simeq \mathbb{Z}^{2g}$. Indeed, $X(\mathbb{C})$ is homeomorphic to $(S^1)^{2g}$.

(f) More advanced example: let $\mathcal{A}_g$ be the moduli stack of principally polarized abelian varieties of dimension g (it is a smooth Deligne–Mumford stack). Then

$$\pi_1^{\mathrm{top}}(\mathcal{A}_g) \simeq \mathrm{Sp}(2g,\mathbb{Z}).$$

By adding a suitable level structure, we obtain a finite covering $\mathcal{A}_g(n) \to \mathcal{A}_g$ which is a smooth scheme; its fundamental group is the corresponding level n congruence subgroup

$$\Gamma(n) = \ker(\mathrm{Sp}(2g,\mathbb{Z}) \to \mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})).$$

(g) Let X be a smooth Fano variety (X is smooth projective and $-K_X$ is ample). Then X is simply connected. This is a consequence of Yau's solution of the Calabi conjecture. Using the Kodaira vanishing theorem, it is rather easy to show that X does not admit non-trivial finite coverings.

(h) Let X be a K3 surface. Then X is simply connected.

(i) Some K3 surfaces admit a fixed-point free involution $\iota\colon X \to X$. The quotient $Y = X/\iota$ is then an Enriques surface, and every Enriques surface (in characteristic $\neq 2$) arises this way. We thus have $\pi_1^{\mathrm{top}}(Y) \simeq \mathbb{Z}/2\mathbb{Z}$. We will later see that every finite group arises as the fundamental group of a smooth projective complex variety.

Definition 1.4.5. We say that a finitely presented group Γ belongs to **class** $\mathcal{P}$ if there exists a smooth projective complex variety X with $\pi_1^{\mathrm{top}}(X) \simeq \Gamma$.

1.5. Corollaries of the Hodge decomposition

While the proof of the Hodge decomposition theorem relies on analytic methods, it is in fact possible to pin down the decomposition itself without going too deep. In particular, the formulation below makes it clear that the Hodge decomposition is independent of the choice of a Kähler form, at the same time avoiding the mention of harmonic forms etc. (besides, it is more useful to say "the following defines a decomposition" instead of "there exists a decomposition.")

Theorem 1.5.1. *Let X be a complex manifold.*

(a) (Poincaré lemma) The complex of sheaves

$$0 \longrightarrow \mathbb{C} \xrightarrow{\text{incl.}} \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n \longrightarrow 0$$

is exact.

(b) (de Rham theorem) Consequently, we have an isomorphism

$$H^n(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq H^n(X,\mathbb{C}) \simeq H^n(X,\Omega_X^\bullet),$$

where $\Omega_X^\bullet = [\mathcal{O}_X \to \Omega_X^1 \to \cdots]$ is the holomorphic de Rham complex of X and $H^n(X,\Omega_X^\bullet)$ is its (hyper)cohomology.

(c) (Hodge–de Rham degeneration) Suppose that X is compact Kähler or proper algebraic. Let $F^p\Omega_X^\bullet = \Omega_X^{\bullet \geq p}$ be the complex

$$0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Omega_X^p \longrightarrow \Omega_X^{p+1} \longrightarrow \cdots.$$

Then $\{F^p\Omega_X^\bullet\}$ defines a decreasing filtration on the de Rham complex $\Omega_X^\bullet$, called the **Hodge filtration***, which satisfies*

$$\mathrm{gr}_F^p(\Omega_X^\bullet) = F^p\Omega_X^\bullet / F^{p+1}\Omega_X^\bullet \simeq \Omega_X^p[-p].$$

The associated spectral sequence of a filtered complex

$$E_1^{pq} = H^q(X, \Omega_X^p) \quad \Rightarrow \quad H^{p+q}(X, \Omega_X^\bullet) = H^{p+q}(X, \mathbb{C})$$

(called the **Hodge–de Rham spectral sequence***) degenerates at the E_2 page. Consequently, the cohomology group $H^n(X, \mathbb{C})$ is endowed with a decreasing filtration, again called the Hodge filtration,*

$$F^p H^n(X, \mathbb{C}) = \mathrm{im}(H^n(X, F^p\Omega_X^\bullet) \to H^n(X, \Omega_X^\bullet) = H^n(X, \mathbb{C})),$$

which satisfies

$$\mathrm{gr}_F^p H^n(X, \mathbb{C}) \simeq H^q(X, \Omega_X^p), \qquad p + q = n.$$

(d) *(The Hodge decomposition) Suppose that X is compact Kähler. Let $\overline{(-)}$ denote the complex conjugation on $H^n(X, \mathbb{C}) = H^n(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. Then for $p + q = n$*

$$H^n(X, \mathbb{C}) = (F^p H^n(X, \mathbb{C})) \oplus \overline{(F^{q+1} H^n(X, \mathbb{C}))}.$$

Consequently, if we define

$$H^{p,q}(X) = (F^p H^n(X, \mathbb{C})) \oplus \overline{(F^q H^n(X, \mathbb{C}))}$$

then $H^{p,q}(X) \subseteq F^p H^n(X, \mathbb{C})$ projects isomorphically onto $\mathrm{gr}_F^p H^n(X, \mathbb{C}) = H^q(X, \Omega_X^p)$, and we have a direct sum decomposition

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X) \simeq \bigoplus_{p+q=n} H^q(X, \Omega_X^p),$$

satisfying $\overline{H^{p,q}(X)} = H^{q,p}(X)$. We have $F^p H^n(X, \mathbb{C}) = \bigoplus_{p'+q'=n,\, p' \geq p} H^{p', n-q'}$.

(e) *(Hodge symmetry) Suppose that X is compact Kähler or proper algebraic. Then*

$$\dim(H^q(X, \Omega_X^p)) = \dim(H^p(X, \Omega_X^q)).$$

Remark 1.5.2. Let X be a smooth and proper complex variety. Assertion (c) of Theorem 1.5.1 is then equivalent (thanks to GAGA, see Lecture 3) to the analogous assertion about the algebraic de Rham complex $\Omega_{X/\mathbb{C}}^\bullet$. There exist entirely algebraic proofs of assertion (c), due to Faltings [Faltings, 1988] (using p-adic Hodge theory) and Deligne–Illusie [Deligne and Illusie, 1987] (relying on reduction modulo p). In contrast, as shown by Mumford and others (see [Deligne and Illusie, 1987, 2.6(i)]), the result is false over fields of positive characteristic, though the Deligne–Illusie paper gives useful criteria for it to hold.

Definition 1.5.3 (Betti numbers)**.** For a topological space X we define

$$b_n(X) = \dim H^n(X, \mathbb{Q}).$$

Corollary 1.5.4. *Let X be a smooth and proper complex algebraic variety. Then $b_n(X(\mathbb{C}))$ is even for n odd.*

Corollary 1.5.5. *Let X be a smooth and proper complex algebraic variety and let $\Gamma = \pi_1^{\mathrm{top}}(X)$. Then* $\dim H^1(\Gamma, \mathbb{Q})$ *is even.*

Note that $H^1(\Gamma, \mathbb{Q}) = \mathrm{Hom}(\Gamma, \mathbb{Q})$ and its dimension is the rank of the (free part of the) abelianization $\Gamma^{\mathrm{ab}} = \Gamma/[\Gamma, \Gamma]$.

Corollary 1.5.6. *The following groups are not isomorphic to the fundamental group of a smooth and proper complex algebraic variety: $\mathbb{Z}^n$ with n odd, F_n (free group on n letters) with n odd, the braid group B_n.*

Note that these groups appear as the fundamental groups of smooth quasi-projective varieties (see Example 1.4.4). One can ask if every finitely presented group can be realized in this way. The answer is no (see suggested term paper topics for some examples).

Moreover, the group $\mathbb{Z}$ is the fundamental group of a compact complex surface, the Hopf surface $H = (\mathbb{C}^2 \setminus 0)/q^{\mathbb{Z}}$ where $0 < |q| < 1$, which is homeomorphic to $S^1 \times S^3$ (see [■]).

The group $\mathbb{Z}$ is also the fundamental group of a singular projective scheme, namely the "triangle" $V(x_0x_1x_2) \subseteq \mathbb{P}^2_{\mathbb{C}}$. In fact, as we shall see in ■ , every finitely presented group can be realized as the topological fundamental group of a projective scheme over $\mathbb{C}$. (Simpson [Simpson, 2011] has shown that every finitely presented group is the topological fundamental group of a non-normal irreducible projective scheme. See also [Kapovich and Kollár, 2014].)

Remark 1.5.7. We say that a finitely presented group Γ is a **Kähler group** if there exists a connected compact Kähler manifold X with $\pi_1(X) \simeq \Gamma$. Thus every group of class $\mathcal{P}$ is a Kähler group. It is an open problem whether the converse holds.

2. Lecture 2: Lefschetz hyperplane theorem

Summary. *We first review Morse theory in §2.1 in order to study the homotopy type of a smooth affine complex variety in §2.2. This is used to prove the Lefschetz hyperplane theorem in §2.3. Finally, in §2.4. we discuss the Godeaux–Serre construction of a smooth projective algebraic variety with a given finite group as its fundamental group.*

In the additional section §2.5, we show that every finitely presented group is the fundamental group of a projective complex scheme (a union of linear subspaces of $\mathbb{P}^n$*).*

2.1. Morse theory

On our way to the Lefschetz hyperplane theorem, we need to review some basic Morse theory. The following discussion contains no improvement whatsoever over the excellent references: [Milnor, 1963, I], [Voisin, 2002, REF] (or [Voisin, 2007, REF]), and [Lazarsfeld, 2004, §3.1].

The goal of Morse theory is to describe the homotopy type of a smooth manifold X using an auxiliary function $f\colon X \to \mathbb{R}$ (called a **Morse function**), by analyzing how the homotopy type of

$$X_{\leq t} = f^{-1}((\infty, t])$$

changes as t crosses past critical values of f.

Example 2.1.1 (See Milnor's book)**.** Let $X \subseteq \mathbb{R}^3$ be the torus consisting of points at distance r $(0 < r < 1)$ from the unit circle in the xy-plane, and let $f(x,y,z) = x$. Then $X_{\leq t}$ is homotopy equivalent to

- the empty space if $t < -1-r$;
- a disc if $-1-r < t < -1+r$;
- a tube $S^1 \times [0,1]$ if $-1+r < t < 1-r$;
- a torus with an open disc removed if $1-r < t < 1+r$;
- a torus if $1+r < t$.

We note (draw a picture to see this!) that for $i = 2,3,4,5$ the space in (i) is homotopy equivalent to a space obtained from the space in $(i-1)$ by attaching a cell of dimension $k = 0,1,1,2$. In the local coordinates $(t_1,t_2) = (y,z)$ around each of the four points $(x_i,0,0)$, where $x_i = -1-r, -1+r, 1-r, 1+r$, the function f takes the form

$$f(t_1,t_2) = t_1^2 + t_2^2, \quad -t_1^2 + t_2^2, \quad t_1^2 - t_2^2, \quad -t_1^2 - t_2^2$$

up to higher order terms. We conclude that the number of negative signs in the quadratic part of the expansion of f at a critical point coincides with the dimension of the cell attached (e.g. crossing a saddle point produces a 1-cell).

In general, crossing a non-degenerate critical point of "index k" will produce a k-cell. Let us try to make it precise.

Definition 2.1.2. Let X and X' topological spaces and let $k \geq 0$. We say that X' is **obtained from X by attaching a k-cell** if there exists a continuous map

$$\phi \colon \partial D^k \longrightarrow X$$

from the boundary of the k-dimensional closed unit disc D^k and a pushout square

$$\begin{array}{ccc} \partial D^k & \longrightarrow & D^k \\ \downarrow & & \downarrow \\ X & \longrightarrow & X'. \end{array} \tag{2.1.1}$$

Thus, if X is a CW-complex, then so is X'. The following lemma clarifies some subtleties when we try to attach cells "up to homotopy."

Lemma 2.1.3. *Let X be a topological space.*

(a) (Whitehead, see [Milnor, 1963, Lemma 3.6]) Let $\phi_0, \phi_1 \colon \partial D^k \to X$ be two maps and let X'_0 and X'_1 be the corresponding pushouts (2.1.1)*. Then every homotopy between ϕ_0 and ϕ_1 gives rise to a homotopy equivalence between X'_0 and X'_1 restricting to the identity on X.*

(b) (P. Hilton, see [Milnor, 1963, Lemma 3.7]) Let $\phi \colon \partial D^k \to X$ be a map and let $f \colon Y \to X$ be a homotopy equivalence. Let X' be the pushout (2.1.1) *and let Y' be the corresponding pushout for $f \circ \phi \colon \partial D^k \to Y$. Then f extends to a homotopy equivalence $f' \colon X' \to Y'$.*

(c) (see [Milnor, 1963, proof of Theorem 3.5]) Suppose that X admits an open cover $X = \bigcup_{n \in \mathbb{N}} X_n$ with $X_n \subseteq X_{n+1}$ satisfying the following conditions:

(a) X_0 is empty;

(b) X_{n+1} is homotopy equivalent to X_n with a k_n-cell attached, for some $k_n \geq 0$;

(c) X has the direct limit topology.

Suppose in addition that X is dominated by a CW-complex[4] Then X is homotopy equivalent to a CW-complex with one k-cell for every n with $k = k_n$.

Definition 2.1.4. Let X be a smooth manifold and let $f \colon X \to \mathbb{R}$ be a smooth function. A point $x \in X$ is a **nondegenerate critical point** of f if the following conditions are satisfied

(a) x is a critical point of f, i.e. $df = 0$ at x;

(b) in some (equiv. every) local coordinate system $t_1, \ldots, t_n$ at x, the matrix of second derivatives (called the *Hessian*)

$$\left[\frac{\partial^2 f}{\partial t_i \partial t_j} \right]_{i,j=1}^{n}$$

is invertible (has rank n).

[4]For example, if X is a manifold in $\mathbb{R}^N$, it is a deformation retract of a tubular neighborhood, and hence dominated by a CW-complex.

The number of negative eigenvalues of the above square matrix (equal to the maximal dimension of a subspace on which the associated quadratic form is negative definite; also coordinate independent) is called the **index** of the critical point f.

Lemma 2.1.5 (Morse lemma). *Let $x \in X$ be a non-degenerate critical point of f with index k. Then there exists a local coordinate system $t_1, \ldots, t_n$ at x in which f takes the form*

$$f(t_1, \ldots, t_n) = f(x) - t_1^2 - \ldots - t_k^2 + t_{k+1}^2 + \ldots + t_n^2.$$

An important corollary:

Corollary 2.1.6. *Nondegenerate critical points are isolated.*

Later on we will need the following simple fact.

Corollary 2.1.7. *Suppose that $f(x) \neq 0$. Then $1/f$ has index $n-k$ at x.*

Morse theory setup. Let us now fix the following setup and notation. Let X be a fixed (Hausdorff) smooth manifold and $f\colon X \to \mathbb{R}$ a smooth function which satisfies the following properties

(a) all critical points of f are non-degenerate;

(b) for every $t \in \mathbb{R}$ the set
$$X_{\leq t} = f^{-1}((-\infty, t])$$
is compact;

(c) for every two critical points $x, x' \in X$ we have $f(x) \neq f(x')$.

A function f satisfying these properties will be called a **Morse function**. We also set $X_{\geq t} = f^{-1}([t, \infty))$ and $X_{[t,t']} = f^{-1}([t, t'])$. The latter is a compact set.

The above assumptions imply in particular that the set C of critical points of f is discrete and forms a (possibly finite) sequence $C = \{x_1, x_2, \ldots\}$ with

$$f(x_1) < f(x_2) < \cdots$$

The above sequence of real numbers is discrete, thus either finite or divergent to $+\infty$.

The main result of basic Morse theory is the following.

Theorem 2.1.8. *In the above setup, the following assertions hold:*

(a) Let $t \leq t'$ in $\mathbb{R}$ such that $X_{[t,t']} \cap C = \emptyset$. Then X_t is a deformation retract of $X_{t'}$, so that the inclusion $X_t \to X_{t'}$ is a homotopy equivalence. Moreover, the inclusion map $X_t \to X_{t'}$ is isotopic to a diffeomorphism $X_t \xrightarrow{\sim} X_{t'}$.

(b) Let $x \in C$ be a critical point of index k and $f(x) = t$, and let $\varepsilon > 0$ be such that $X_{[t-\varepsilon, t+\varepsilon]} \cap C = \{x\}$. Then for $\varepsilon \ll 1$, the space $X_{\leq t+\varepsilon}$ is homotopy equivalent to a space obtained from $X_{\leq t-\varepsilon}$ by attaching a k-cell.

Proof. (Write something?) □

Corollary 2.1.9. *In the situation of Theorem 2.1.8, X has the homotopy of a CW-complex with one k-cell for each critical point of index k.*

So far we have said nothing about how to construct a Morse function on a given X. In fact, for $X \subseteq \mathbb{R}^n$, the square distance function from a random point in $\mathbb{R}^n$ does the job.

Lemma 2.1.10. *Let $X \subseteq \mathbb{R}^n$ be a smooth submanifold of dimension d. Then for $p \in \mathbb{R}^n$ outside a nowhere dense subset of $\mathbb{R}^n$, the square-distance function*

$$f_p(x) = ||x - p||_2^2$$

on X is a Morse function.

Proof idea. The condition that $X_{\leq t}$ is compact is clearly satisfied. We ensure that f_p has only non-degenerate critical points.

The condition that x is a critical point of f_p means precisely that the vector $v = p - x$ is perpendicular to the tangent space $T_pX \subseteq T_p\mathbb{R}^n = \mathbb{R}^n$. Consider the manifold (the total space of the normal bundle of X in $\mathbb{R}^n$)

$$N = \{(x,v) \,:\, v \in (T_pX)^{\perp}\},$$

and the map

$$\Psi\colon N \longrightarrow \mathbb{R}^n, \qquad \Psi(x,v) = x + v$$

(which is a map between n-dimensional manifolds). Thus if $p = \Psi(x,v)$ then x is a critical point of f_p. One checks that this point is non-degenerate if and only if (x,v) is a critical point of Ψ. By Sard's theorem, the set of critical values of Ψ is nowhere dense, and we are done.

We omit the proof that one can ensure that every critical value is taken exactly once. (This is not seriously needed for Theorem 2.1.8 either.) □

2.2. Homotopy types of smooth affine varieties

We are going to use Morse theory to prove the following surprising result.

Theorem 2.2.1 (Andreotti–Frenkel, Bott, Thom)**.** *Let $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ be a smooth closed subscheme of dimension d. Then $X(\mathbb{C})$ has the homotopy type of a finite CW-complex of dimension at most d.*

This is striking since $X(\mathbb{C})$ is a manifold of dimension $2d$! According to Corollary 2.1.9, in order to show Theorem 2.2.1, we need to find a Morse function $f\colon X(\mathbb{C}) \to \mathbb{R}$ whose critical points all have indices $\leq d$.

Proposition 2.2.2. *Let $X \subseteq \mathbb{C}^n$ be a smooth complex submanifold of dimension d, let $x \in X$, and let $p \in \mathbb{C}^n \setminus X$ be such that x is a nondegenerate critical point of f_p of index k. Then $k \leq d$.*

Proof. This is an enlightening but direct computation, see [Lazarsfeld, 2004, Proposition 3.1.7]. A change of coordinates and basic order estimates allow one to assume that X is given by a single equation of degree two

$$z_{d+1} = 1 + \sum_{i,j=1}^{d} a_{ij} z_i z_j, \qquad a_{ij} = a_{ji}$$

with $x = (0,\dots,0,1,0,\dots,0)$ ($(d+1)$-st basis vector) and $p = (0,\dots,0)$. Thereupon, writing $z_i = x_i + \sqrt{-1}y_i$, the symmetric matrix of second derivatives of f takes the block form

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix}, \qquad A = \left[\frac{\partial^2 f}{\partial x_j \partial x_j}\right], \quad B = \left[\frac{\partial^2 f}{\partial x_j \partial y_j}\right]$$

(check this; write out in terms of a_{ij}) Then if (v,w) is a vector in block form with eigenvalue λ, then its "conjugate" $(v,-w)$ is an eigenvector for $-\lambda$, which shows the assertion. □

We are now ready to show the Andreotti–Frenkel theorem.

Proof of Theorem 2.2.1. Let $p \in \mathbb{C}^n \setminus X$ be such that f_p is a Morse function. Since X is algebraic (in particular real algebraic considered as a subset of $\mathbb{R}^{2n}$) and f_p is real algebraic, the set of critical points of f_p is real algebraic. But it is also discrete, and hence finite (by the Tarski–Seidenberg theorem). By Proposition 2.2.2, each of these critical points has index at most d. We conclude by Corollary 2.1.9. □

Remark 2.2.3. The same strategy shows that a Stein manifold of dimension d has the homotopy type of a (not necessarily finite) CW-complex of dimension $\leq d$.

2.3. Lefschetz hyperplane theorem

We will now use Morse theory to establish Lefschetz's theorem comparing the topology of a smooth projective variety to that of its ample divisor.

Theorem 2.3.1 (Lefschetz, Thom–Bott)**.** *Let X be a complex projective variety and let $Y \subseteq X$ be an effective Cartier divisor. Denote by $i\colon Y \hookrightarrow X$ the inclusion map. Suppose that the line bundle $\mathcal{L} = \mathcal{O}_X(Y)$ is ample (for example, that $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ and $Y = X \cap H$ for a hyperplane H), and that $U = X \setminus Y$ is smooth of dimension d.*

(a) The induced map on cohomology groups

$$i^*\colon H^n(X(\mathbb{C}),\mathbb{Z}) \longrightarrow H^n(Y(\mathbb{C}),\mathbb{Z})$$

is an isomorphism for $n < d-1$ and is injective for $n = d-1$.

(b) The induced map on homology groups

$$i_*\colon H_n(Y(\mathbb{C}),\mathbb{Z}) \longrightarrow H_n(X(\mathbb{C}),\mathbb{Z})$$

is an isomorphism for $n < d-1$ and is surjective for $n = d-1$.

(c) *Suppose that $n > 1$, and let $x \in Y$. Then Y is connected and the induced map on homotopy groups*

$$i_*\colon \pi_n(Y(\mathbb{C}),x) \longrightarrow \pi_n(X(\mathbb{C}),x)$$

is an isomorphism for $n < d-1$ and is surjective for $n = d-1$.

Proof. We will show the theorem under the additional assumption that Y is very ample, so that there exists an embedding $X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ and a hyperplane H such that $Y = H \cap X$. We set $U = X \setminus Y$, which is a smooth affine variety, embedded inside $\mathbb{A}^N_{\mathbb{C}}$.

We first prove (H^*) using Theorem 2.2.1 (then H_* follows similarly). The maps i^* fit inside a long exact sequence

$$\cdots \longrightarrow H^n_c(U(\mathbb{C}),\mathbb{Z}) \longrightarrow H^n(X(\mathbb{C}),\mathbb{Z}) \xrightarrow{i^*} H^n(Y(\mathbb{C}),\mathbb{Z}) \longrightarrow \cdots$$

where $H^n_c(U(\mathbb{C}),\mathbb{Z})$ are the cohomology groups with compact support (this is the cohomology exact sequence of the short exact sequence of sheaves $0 \to j_!\mathbb{Z}_U \to \mathbb{Z}_X \to \mathbb{Z}_Y \to 0$ where $j\colon U \to X$ is the inclusion and $j_!$ denotes extension by zero). We deduce that to show (H^*) it is enough to show that $H^n_c(U(\mathbb{C}),\mathbb{Z}) = 0$ for $n < d$. Since $U(\mathbb{C})$ is an orientable manifold of dimension $2d$, by Poincaré duality, this group is dual to the homology group $H_{2d-n}(U(\mathbb{C}),\mathbb{Z})$. But this is zero for $2d - n > d$ since by Theorem 2.2.1 the space $U(\mathbb{C})$ is homotopy equivalent to a CW-complex of dimension d.

In order to show (π_*), we need a variant of this argument, due to Thom and Bott. Let $p \in \mathbb{C}^N \setminus U(\mathbb{C})$ be such that the square distance function $f_p(x) = ||x-p||_2^2$ is a Morse function on $U(\mathbb{C})$ (Lemma 2.1.10). Let T be an open tubular neighborhood of $Y(\mathbb{C})$ in $X(\mathbb{C})$, so that $Y(\mathbb{C})$ is a deformation retract of T. (Find a reference for the tubular neighborhood.) Choose $R > 0$ large enough so that the ball $||x-p||_2^2 < R$ contains the compact set $U(\mathbb{C}) \setminus T$. Consider the function $1/f_p$, which is again Morse function on $U(\mathbb{C})$, but now its indices are $\geq d$ (Corollary 2.1.7). Thus $U(\mathbb{C})$ is obtained from the manifold $M = \{f_p \geq R\} \subseteq T$ by attaching cells of dimension $\geq d$.

By the exact sequence of relative homotopy groups

$$\cdots \longrightarrow \pi_n(Y(\mathbb{C}),x) \longrightarrow \pi_n(X(\mathbb{C}),x) \longrightarrow \pi_n(X(\mathbb{C}),Y(\mathbb{C})) \longrightarrow \cdots$$

we see that what we want is that the $\pi_n(X(\mathbb{C}),Y(\mathbb{C}))$ is trivial for $n < d$. Let $f\colon D^n \to X(\mathbb{C})$ be a continuous map with $f(\partial D^n) \subseteq Y$. Then f can be deformed (clarify this!) into a map $f'\colon D^n \to M \cup Y(\mathbb{C}) \subseteq T$. Using the deformation retraction of T onto $Y(\mathbb{C})$, we can then deform f' into a map which lands in $Y(\mathbb{C})$. □

Suppose that X is smooth. Using the Bertini theorem, one can always find a smooth hyperplane section of a smooth $X \subseteq \mathbb{P}^n_{\mathbb{C}}$. By induction, we deduce:

Corollary 2.3.2. *Let Γ be a group isomorphic to the fundamental group of a smooth projective complex variety. Then there exists a smooth projective complex surface of general type with fundamental group X. Moreover, there exists a $g \geq 0$ and a surjective homomorphism $\Gamma_g \to \Gamma$ where Γ_g is the genus g surface group.*

Remark 2.3.3. The "moreover" part of Corollary 2.3.2 is a bit silly, since the surface group Γ_g surjects onto the free group F_g (for example, via the map sending a_i and b_i to the i-th generator x_i), and hence every finitely generated group admits a surjection from Γ_g for some g. However, a concrete surjection from a surface group is very useful in practice. Moreover, as we shall see later in the course, in the case of the étale fundamental group, finite generation will be proved using an analog of this fact (so that the question can be reduced to finite generation for curves).

An almost-corollary of the above corollary is the following.

Corollary 2.3.4. *Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a smooth complete intersection of dimension at least two. Then X is connected and $\pi_1^{\mathrm{top}}(X) = 1$.*

This will be shown a degeneration argument from [Atiyah and Hirzebruch, 1962]. We first review Ehresmann's theorem.

Theorem 2.3.5 (Ehresmann)**.** *Let $f\colon X \to S$ be a proper submersion between smooth manifolds. Then f is a locally trivial fibration; more precisely, for every $s \in S$ there exists an open neighborhood $s \in U \subseteq S$ and an diffeomorphism*

$$X \times_S U \simeq X_s \times U$$

over U, where $X_s = f^{-1}(s)$. In particular, for every $s, s' \in S$ in the same connected component of S, the fibers X_s and $X_{s'}$ are diffeomorphic.

Corollary 2.3.6. *Let $f\colon X \to S$ be a smooth and proper morphism of schemes of finite type over $\mathbb{C}$, with connected fibers and S connected. Then for every $s, s' \in S(\mathbb{C})$ there exists an isomorphism*

$$\pi_1(X_s(\mathbb{C})) \simeq \pi_1(X_{s'}(\mathbb{C})).$$

We are now ready to prove Corollary 2.3.4.

Proof of Corollary 2.3.4. Write $Y = Y_1 \cap \ldots \cap Y_c$, with $Y_i \subseteq \mathbb{P}^n_{\mathbb{C}}$ a hypersurface of degree $d_i \geq 1$.

Suppose first that the intersections

$$Y_1, \quad Y_1 \cap Y_2, \quad \ldots, \quad Y_1 \cap \ldots \cap Y_c \tag{2.3.1}$$

are all smooth. Then the assertion follows by an iterated application of the Lefschetz hyperplane theorem.

We reduce to the above situation by a degeneration argument. Let $V_i = H^0(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(d_i))$ be the space of degree d_i homogeneous polynomials in $n+1$ variables, and let $P_i = \mathbb{P}V_i^*$ be the associated projective space parameterizing degree d_i hypersurfaces in $\mathbb{P}^n_{\mathbb{C}}$. Let

$$U \subseteq P_1 \times \cdots \times P_c$$

be the subscheme parameterizing tuples $(Y_1, \ldots, Y_c)$ for which the intersections (2.3.1) are smooth. Let $V \subseteq P_1 \times \cdots \times P_c$ be the subscheme defined by the condition $Y_1 \cap \cdots \cap Y_c$ is smooth.

We claim that U is a non-empty open subscheme. This follows by an iterated application of Bertini's theorem. Let $(Y_1',\dots,Y_c') \in V$ and let $Y' = Y_1' \cap \dots \cap Y_c'$ be the corresponding intersection. We have shown that $\pi_1(Y') = 1$.

Since $P_1 \times \cdots \times P_c$ is irreducible and V is a non-empty open, the scheme V is connected. Consider the "universal family" $\mathcal{Y} \to V$ over V:

$$\mathcal{Y} = \{((Y_1,\dots,Y_c),y) \in V \times \mathbb{P}^n_{\mathbb{C}} : y \in Y_1 \cap \dots \cap Y_c\} \subseteq V \times \mathbb{P}^n_{\mathbb{C}} \longrightarrow V.$$

The fiber over $(Y_1,\dots,Y_c)$ is thus $Y_1 \cap \dots \cap Y_c$, which is smooth. It is not difficult to see that in fact $\mathcal{Y} \to V$ is a smooth morphism.

By Corollary 2.3.6, the fibers of $\mathcal{Y}(\mathbb{C}) \to V(\mathbb{C})$ are diffeomorphic to one another. It follows from $\pi_1(Y') = 1$ that all intersections $Y_1 \cap \dots \cap Y_c$ are simply connected. $\square$

2.4. The Godeaux–Serre construction

We shall use Corollary 2.3.4 to prove the following theorem.

Theorem 2.4.1 (Serre[5])**.** *Let G be a finite group. Then there exists a smooth projective variety*[6] *X over $\mathbb{C}$ with $\pi_1^{\mathrm{top}}(X) \simeq G$.*

Proof. The proof will proceed in several steps.

Step 1. We construct a G-representation $\rho\colon G \to \mathrm{PGL}(V)$ such that the "non-free" locus in $P = \mathbb{P}(V)$ defined by

$$Z = \bigcap_{g \neq 1} \mathrm{Fix}(g) = \{x \in P : G_x \neq 1\} \subseteq P$$

is of codimension at least three. For example $V = \mathbb{C}[G] \oplus \mathbb{C}[G] \oplus \mathbb{C}[G]$ will do.

Step 2. Let $Q = P/G$. Then Q is a (possibly singular) projective variety. The quotient map $\pi\colon Q \to P$ is finite. If $U = P \setminus Z$ is the open subset on which G acts freely, and $V = \pi(U) \subseteq Q$, then the restricted map $\pi\colon V \to U$ is finite étale (and the covering $V(\mathbb{C}) \to U(\mathbb{C})$ is a G-torsor). The closed subset $Q \setminus V$ has codimension ≥ 3.

Step 3. Let $X \subseteq Q$ be a smooth complete intersection of dimension 2 such that $X \subseteq V$. It exists by the Bertini theorem [Jouanolou, 1983, 6.11]. Let $Y = \pi^{-1}(X)$. Then $Y \to X$ is finite étale, and in particular X is smooth. It is also a complete intersection (since $\mathrm{Pic}(P) = \mathbb{Z}$, so that every effective Cartier divisor is a hypersurface of some degree).

Step 4. By the Lefschetz hyperplane theorem, since $\dim(Y) > 1$, we have that Y is simply connected (Corollary 2.3.4), and hence $Y(\mathbb{C}) \to X(\mathbb{C})$ is the universal covering of $X(\mathbb{C})$. We conclude that $\pi_1(X,x) = \mathrm{Aut}(Y/X) = G$. $\square$

Remark 2.4.2. The same argument shows that for every finite group G and every algebraically closed field k there exists a smooth projective variety X over k whose étale fundamental group is isomorphic to G.

[5] The canonical reference seems to be [Serre, 1958, §16], but the exact statement is not there.

[6] One can take X to be a surface of general type, by taking intersections with high degree hypersurfaces.

Example 2.4.3 ([Serre, 1964]). Consider $G = \mathbb{Z}/p$ (p prime) acting on $\mathbb{P}(\mathbb{C}^p) = \mathbb{P}^{p-1}_{\mathbb{C}}$ by cyclically permuting the coordinates. Then the hypersurface Y given by the Fermat equation

$$x_1^p + \cdots + x_p^p = 0$$

is G-stable and the action of G on Y is free. Thus $X = Y/G$ has $\pi_1^{\mathrm{top}}(X) \simeq G$.

Remark 2.4.4 (Nonliftable variety). Serre [Illusie, 2005, Appendix] has used a related construction to produce an example of a smooth projective variety over a field of characteristic p which does not lift to characteristic zero. Here are some details of this lovely construction:

Let k be an algebraically closed field of characteristic $p > 0$, and let $n \geq 3$. The group $\mathrm{PGL}_{n+1}(k) = \mathrm{Aut}(\mathbb{P}^n_k)$ contains a large finite p-group G, namely the group of strictly upper-triangular matrices with entries in $\mathbb{F}_p$. As in the proof of Theorem 2.4.1, we construct a complete intersection Y in $\mathbb{P}^n_k$ of dimension ≥ 3 on which G acts freely, then define $X = Y/G$. Suppose that A is an Noetherian local ring with residue field k and $\tilde{X} \to \mathrm{Spec}(A)$ a smooth and proper morphism with $\tilde{X} \otimes_A k \simeq X$. Serre proves that we must have $pA = 0$.

To show this, we may assume that A is Artinian. Using topological invariance of the étale fundamental group (to be reviewed later in the course) one sees that the G-torsor $Y \to X$ lifts to a G-torsor $\tilde{Y} \to Y$. Thus $\tilde{Y} \to \mathrm{Spec}(A)$ is an infinitesimal deformation of Y. Since $\dim(Y) \geq 3$, we have $H^2(Y, \mathcal{O}_Y) = 0 = H^1(Y, \mathcal{O}_Y)$, and hence $\mathrm{Pic}(\tilde{Y}) \to \mathrm{Pic}(Y)$ is an isomorphism. The line bundle $L = \mathcal{O}_Y(1)$ lifts to a line bundle $\tilde{L}$ on $\tilde{Y}$. Since $H^1(Y, L) = 0$, the map $H^0(\tilde{Y}, \tilde{L}) \to H^0(Y, L)$ is surjective, and hence we obtain an embedding $\tilde{Y} \hookrightarrow \mathbb{P}^n_A$. One shows rather easily that the action of G on $\tilde{Y}$ extends to an action on $\mathbb{P}^n_A$ and hence produces a map in the diagram below

$$\begin{array}{ccc} & & G \\ & \swarrow^{\cdots} & \downarrow \\ \mathrm{PGL}_{n+1}(A) & \longrightarrow & \mathrm{PGL}_{n+1}(k). \end{array}$$

One shows that such a dotted arrow cannot exist, for linear algebra reasons. (Check this)

2.5. Projective schemes with given fundamental group (A)

Our goal is to construct, for every finitely presented group Γ, a projective scheme (not a variety!) with topological fundamental group Γ. We first prove a descent statement (see §1.3 for a discussion of effective descent).

Lemma 2.5.1. *Consider a commutative square of topological spaces*

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

Assume that

(a) the maps $Y \to X$ and $Y' \to X'$ are of effective descent (at least for locally constant sheaves);

(b) the map $Y' \to Y$ is an isomorphism on π_0 and π_1 (with any basepoint);

(c) the map $Y' \times_{X'} Y' \to Y \times_X Y$ is an isomorphism on π_0 and a surjection on π_1 (with any basepoint);

(d) the map $Y' \times_{X'} Y' \times_{X'} Y' \to Y \times_X Y \times_X Y$ is a surjection on π_0.

Then, the map $X' \to X$ induces isomorphisms on π_0 and π_1 (with any basepoint).

Proof. The assertion is equivalent to saying that the pull-back functor $\mathbf{Cov}_X \to \mathbf{Cov}_{X'}$ is an equivalence. This in turn is an exercise in descent. By assumption (a) and Proposition REF we need to show that the pull-back functor on descent data

$$\mathbf{DD}(Y/X) \longrightarrow \mathbf{DD}(Y'/X')$$

is an equivalence. An object of $\mathbf{DD}(Y/X)$ consists of a covering $Z \to Y$ together with an isomorphism $\phi\colon \mathrm{pr}_1^* Z \simeq \mathrm{pr}_2^* Z$ over $Y \times_X Y$ satisfying the cocycle condition, which is an equality of two maps over $Y \times_X Y \times_X Y$. By assumption (b) the pull-back functor

$$\mathbf{Cov}_Y \longrightarrow \mathbf{Cov}_{Y'}$$

is an equivalence, so each such $Z \to Y$ corresponds to a unique $Z' \to Y'$. By assumption (c), the pull-back functor

$$\mathbf{Cov}_{Y \times_X Y} \to \mathbf{Cov}_{Y' \times_{X'} Y'}$$

is fully faithful, so each ϕ corresponds to a unique $\phi'\colon \mathrm{pr}_1^* Z' \simeq \mathrm{pr}_2^* Z'$. Finally, assumption (d) implies that the pull-back functor

$$\mathbf{Cov}_{Y \times_X Y \times_X Y} \to \mathbf{Cov}_{Y' \times_{X'} Y' \times_{X'} Y'}$$

is faithful; thus ϕ satisfies the cocycle condition if and only if ϕ' does. This establishes that $\mathbf{DD}(Y/X) \to \mathbf{DD}(Y'/X')$ is an equivalence. □

We now proceed with the construction.

Definition 2.5.2. An **(abstract) simplicial complex** consists of a set V of vertices and a family of non-empty finite subsets $S \subseteq 2^V$ closed under taking non-empty subsets and containing all singletons.

Let (V,S) be an abstract simplicial complex. Its **geometric realization** is the subspace $\Delta(V,S)$ of the simplex

$$\Delta(V) = \{f\colon V \to [0,1] \,:\, f(v) = 0 \text{ for almost all } v \text{ and } \sum_{v \in V} f(v) = 1\}$$

(with the direct limit topology when we write V as a filtered colimit of its finite subsets), cut out by the condition

$$f \in \Delta(V,S) \quad \Leftrightarrow \quad \{v \in V \,:\, f(v) \neq 0\} \in S.$$

In other words, $\Delta(V,S)$ is the union of the simplices $\Delta(\sigma)$ for all $\sigma \in S$.

Lemma 2.5.3. *Let Γ be a finitely presented group. Then there exists a finite simplicial complex (V,S) with $\Delta(V,S)$ connected and*

$$\pi_1(\Delta(V,S)) \simeq \Gamma.$$

Proof. Take the usual construction of a two-dimensional CW-complex with $\pi_1 = \Gamma$ (the first steps of the construction of the classifying space $B\Gamma$), then subdivide every loop in three, and each disc into pizza slices. The details are tedious and omitted. □

We now define the projective realization of a finite simplicial complex similarly, by replacing simplices with complex projective spaces:

$$\mathbb{P}(V,S) := \{(x_v) \in \mathbb{P}(\mathbb{C}^V) \, : \, \{v : x_v \neq 0\} \in S\}.$$

There is an obvious inclusion map

$$\varphi \colon \Delta(V,S) \longrightarrow \mathbb{P}(V,S).$$

Theorem 2.5.4. *Let Δ be a finite simplicial complex. Then the map $\varphi \colon \Delta(V,S) \to \mathbb{P}(V,S)$ induces isomorphisms on π_0 and π_1 (with any basepoint).*

Proof. Let $\mathbb{P}'(V,S)$ be the disjoint union of the projective spaces $\mathbb{P}(\mathbb{C}^\sigma)$ for each $\sigma \in S$. Thus

$$\mathbb{P}'(V,S) \to \mathbb{P}(V,S)$$

is a proper surjection (a finite closed cover). We define $\Delta'(V,S)$ similarly as the disjoint union of simplices $\Delta(\sigma)$. We have a commutative square

$$\begin{array}{ccc} \mathbb{P}'(V,S) & \longrightarrow & \mathbb{P}(V,S) \\ \downarrow & & \downarrow \\ \Delta'(V,S) & \longrightarrow & \Delta(V,S) \end{array}$$

to which we wish to apply Lemma REF. The horizontal arrows are proper surjections and hence of effective descent (see REF). Assumptions (b), (c), and (d) of the lemma follow from the fact that for every $\sigma \in S$, the inclusion

$$\Delta(\sigma) \longrightarrow \mathbb{P}(\mathbb{C}^\sigma)$$

induces an isomorphism on π_0 (both spaces are connected) and on π_1 (both spaces are simply connected). □

Remark 2.5.5. Simpson [Simpson, 2011] showed that every finitely presented group is the fundamental group of an integral (irreducible and reduced) projective scheme over $\mathbb{C}$. See also [Kapovich and Kollár, 2014].

3. Lecture 3: GAGA

Summary. *We define complex analytic spaces and the analytification functors in §3.1 In §3.2, we sketch the proof of Serre's GAGA theorem: on a proper scheme, analytic and algebraic coherent sheaves coincide. We also derive numerous corollaries from this result, particularly those pertaining to topology of algebraic varieties. In §3.3 we discuss a beautiful example, again due to Serre, of a variety of a number field whose topological fundamental group depends on the embedding of the base field into* $\mathbb{C}$.

In the final §3.4 we prove Malcev's theorem that finitely generated linear groups are residually finite.

3.1. Complex analytic spaces and analytification

We briefly introduce complex analytic spaces. See the first sections of [Serre, 1956] or the first chapter of [Grauert and Remmert, 1984] for more.

We denote by $\mathcal{O}$ the sheaf of holomorphic functions on $\mathbb{C}^n$. It is a sheaf of $\mathbb{C}$-algebras whose stalks are noetherian (and henselian) local rings whose completions are isomorphic to the power series ring $\mathbb{C}[\![t_1,\dots,t_n]\!]$.

We describe the local models of complex analytic spaces. Let $U \subseteq \mathbb{C}^n$ be an open polydisc and let $f_1,\dots,f_r \in \mathcal{O}(U)$ be holomorphic functions on U. Let

$$Z = V(f_1,\dots,f_r) = \{x \in U \,:\, f_1(x) = \ldots = f_r(x) = 0\}.$$

be their vanishing locus. Denote the inclusion by $i\colon Z \hookrightarrow U$ and set

$$\mathcal{O}_Z = i^{-1}\mathcal{O}_U/(f_1,\dots,f_r).$$

The space $(Z,\mathcal{O}_Z)$ is a locally ringed space. The stalks of $\mathcal{O}_Z$ are local rings: we have $\mathcal{O}_{Z,x} = \mathcal{O}_{U,x}/(f_1,\dots,f_r)$.

Definition 3.1.1. A **complex analytic space** is a locally ringed space over $\mathbb{C}$ (that is, equipped with a morphism to $\operatorname{Spec}(\mathbb{C})$) which is locally isomorphic (over $\mathbb{C}$) to one of the spaces Z constructed above. A morphism of complex analytic spaces is a morphism of locally ringed spaces over $\mathbb{C}$. We denote the category of complex analytic spaces by $\mathbf{An}_{\mathbb{C}}$.

For example, every complex manifold is in particular a complex analytic space. One can also produce complex analytic spaces by analytifying schemes.

Definition 3.1.2. Let X be a scheme locally of finite type over $\mathbb{C}$. Its **analytification** is a complex analytic space X^{an} together with a map of locally ringed spaces over $\mathbb{C}$

$$\varepsilon\colon X^{\mathrm{an}} \longrightarrow X$$

such that every morphism $Y \to X$ of locally ringed spaces over $\mathbb{C}$ where Y is a complex analytic space factors uniquely through ε.

Usual definitions and constructions apply: open and closed immersions, smoothness, fiber products etc.

Proposition 3.1.3. *Let X be a scheme locally of finite type over $\mathbb{C}$. Then X admits an analytification $\varepsilon\colon X^{\mathrm{an}} \to X$. The map ε maps X^{an} bijectively onto $X(\mathbb{C})$.*

Proof (boring). Recall [SP Tag 01I1] that an affine scheme $\mathrm{Spec}(A)$ represents the functor

$$Y \mapsto \mathrm{Hom}(A, \Gamma(Y, \mathcal{O}_Y))$$

on all *locally ringed spaces* (not just schemes). Suppose that $X = \mathrm{Spec}(A)$ is affine, and choose a presentation $A = \mathbb{C}[T_1, \ldots, T_n]/(f_1, \ldots, f_r)$. Let X^{an} be the zero set $V(f_1, \ldots, f_r) \subseteq \mathbb{C}^n$ defined as before Definition 3.1.1. Then a map $Y \to \mathbb{C}^n$ is a tuple $(g_1, \ldots, g_n) \in \Gamma(Y, \mathcal{O}_Y)$. For this map to factor (uniquely) through X^{an} it is necessary and sufficient that $f_i(g_1, \ldots, g_n) = 0$ for $i = 1, \ldots, r$. This means that

$$\mathrm{Hom}_{\mathbf{An}_{\mathbb{C}}}(Y, X^{\mathrm{an}}) = \mathrm{Hom}_{\mathbb{C}}(A, \Gamma(Y, \mathcal{O}_Y)) = \mathrm{Hom}_{\mathbb{C}}(Y, X),$$

confirming that the obvious map $\varepsilon\colon X^{\mathrm{an}} \to X$ has the required universal property. The assertion that $X^{\mathrm{an}} \xrightarrow{\sim} X(\mathbb{C})$ is also clear.

We also easily check that if $U = D(f) \subseteq X = \mathrm{Spec}(A)$ is a distinguished affine open, then $U^{\mathrm{an}} \subseteq X^{\mathrm{an}}$ is defined by the condition $\{f \neq 0\}$. Consequently, affine open covers produce open covers of the analytification.

Suppose now that X is arbitrary, and admits an open cover $X = \bigcup X_i$ such that X_i^{an} and $(X_i \cap X_j)^{\mathrm{an}}$ exist. Then $(X_i \cap X_j)^{\mathrm{an}} \to X_i^{\mathrm{an}}$ are open immersions, along which we can stitch together a complex analytic space X'. One then checks easily that the induced $X' \to X$ produces an analytification of X. □

Lemma 3.1.4. *Let X be a scheme locally of finite type over $\mathbb{C}$. Then X is n-dimensional (reduced, normal, smooth, proper) if and only if X^{an} is n-dimensional (reduced, normal, smooth, compact).*

Proof. Conditions on singularities can be checked on complete local rings in both cases. For properness vs compactness see [■]. □

3.2. GAGA

We define analytic coherent sheaves (again, see [Serre, 1956] or the book [Grauert and Remmert, 1984] a more detailed treatment). Recall that for a locally ringed space $(X, \mathcal{O}_X)$, an $\mathcal{O}_X$-module $\mathcal{F}$ is **coherent**[7] if it satisfies the following two conditions

(a) $\mathcal{F}$ is of finite type; that is, locally on X there exists a surjection $\mathcal{O}_X^n \to \mathcal{F}$ for some $n \geq 0$;

(b) for every open $U \subseteq X$ and every map $\phi\colon \mathcal{O}_X^n \to \mathcal{F}$ for some $n \geq 0$, the kernel $\ker(\phi)$ is of finite type.

[7]For locally noetherian schemes, this is equivalent to the usual notion "locally on $\mathrm{Spec}(A)$ isomorphic to $\tilde{M}$ for some finitely generated A-module M" (e.g. [Hartshorne, 1977, III.5]).

We denote by $\mathbf{Coh}(\mathcal{O}_X)$ (or simply by $\mathbf{Coh}(X)$) the full subcategory of $\mathbf{Mod}(\mathcal{O}_X)$ consisting of coherent $\mathcal{O}_X$-modules (we often call them simply "coherent sheaves").

According to Oka's coherence theorem ([Grauert and Remmert, 1984, Chapter 2, §5]), the sheaf $\mathcal{O}_{\mathbb{C}^n}$ of holomorphic functions of $\mathbb{C}^n$ is coherent (as a module over itself). More generally, if $Z \subseteq U \subseteq \mathbb{C}^n$ are as above Definition 3.1.2, then $\mathcal{O}_Z$ is a coherent $\mathcal{O}_U$-module. It follows formally that for every complex analytic space X, the structure sheaf $\mathcal{O}_X$ is coherent. This is important because they imply that an $\mathcal{O}_X$-module is coherent if and only if locally on X it is isomorphic to the cokernel of a map $\mathcal{O}_X^m \to \mathcal{O}_X^n$ for some $n, m \geq 0$. Consequently, for any map $f\colon Y \to X$ of complex analytic spaces the pull-back functor $f^*\colon \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_Y)$ induces

$$f^*\colon \mathbf{Coh}(\mathcal{O}_X) \longrightarrow \mathbf{Coh}(\mathcal{O}_Y).$$

Moreover, one has the usual correspondence between closed immersions $i\colon Z \to X$ and coherent ideals $\mathcal{I} \subseteq \mathcal{O}_X$.

Similarly, if X is a scheme locally of finite type over $\mathbb{C}$, pull-back along the analytification map $\varepsilon^*\colon X^{\mathrm{an}} \to X$ induces a functor

$$\varepsilon^*\colon \mathbf{Coh}(X) \longrightarrow \mathbf{Coh}(X^{\mathrm{an}}). \tag{3.2.1}$$

We often denote $\varepsilon^*\mathcal{F}$ simply by $\mathcal{F}^{\mathrm{an}}$ and call it its **analytification**.

Example 3.2.1. We have $(\Omega^1_{X/\mathbb{C}})^{\mathrm{an}} \simeq \Omega^1_X$.

Lemma 3.2.2. *The functor* (3.2.1) *is faithful, conservative, and exact. For every coherent $\mathcal{O}_X$-module $\mathcal{F}$, the map $\Gamma(X,\mathcal{F}) \to \Gamma(X^{\mathrm{an}},\mathcal{F}^{\mathrm{an}})$ is injective.*

Proof. For every $x \in X^{\mathrm{an}} = X(\mathbb{C})$, the homomorphism of local rings $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathrm{an}},x}$ induces an isomorphism on completions and is (therefore) flat. Thus, if $f\colon \mathcal{G} \to \mathcal{F}$ is a map in $\mathbf{Coh}(X)$, then $f = 0$ if and only if for every $x \in X(\mathbb{C})$, the induced map of completed stalks $\widehat{\mathcal{G}}_x \to \widehat{\mathcal{F}}_x$ is zero. But this map can be recovered from $\mathcal{G}^{\mathrm{an}} \to \mathcal{F}^{\mathrm{an}}$, and hence is zero if the latter one is. This shows faithfulness and, using $\mathcal{G} = \mathcal{O}_X$, the injectivity on global sections. Similarly, exactness in both $\mathbf{Coh}(X)$ and $\mathbf{Coh}(X^{\mathrm{an}})$ can be checked on completed stalks, which shows that the functor is exact. Conservativity then follows since a map is an isomorphism if and only if $0 \to \mathcal{G} \to \mathcal{F} \to 0$ is exact. □

In addition to Oka's theorem, the second key result about analytic coherent sheaves is the finiteness theorem.

Theorem 3.2.3 (Cartan–Serre [Grauert and Remmert, 2004, VI])**.** *Let X be a compact complex analytic space, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\dim_{\mathbb{C}} H^n(X,\mathcal{F})$ is finite for all $n \geq 0$ (and zero for $n > \dim(X)$).*[8]

We are now ready to state and prove Serre's GAGA theorem.

[8] Grauert [Grauert, 1960], [Grauert and Remmert, 1984, Chapter 10] generalized this to the relative case: for a proper morphism $f\colon X \to S$, the higher direct images $R^n f_*\mathcal{F}$ are coherent $\mathcal{O}_S$-modules for $n \geq 0$.

Theorem 3.2.4 ([Serre, 1956], [Grothendieck, 1971, Exp. XII]). *Let X be a proper scheme over $\mathbb{C}$. Then* (3.2.1) *is an equivalence. Moreover, for every coherent $\mathcal{O}_X$-module $\mathcal{F}$, the map*

$$\varepsilon^* : H^p(X, \mathcal{F}) \longrightarrow H^p(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$$

is an isomorphism for $p \geq 0$.

Proof. The proof will proceed in several steps: $X = \mathbb{P}^n_{\mathbb{C}}$, X projective, X proper.

Step 1: The case $X = \mathbb{P}^n_{\mathbb{C}}$. The first deal with the crucial case $X = \mathbb{P}^n_{\mathbb{C}}$. As we shall see, the only non-trivial analytic input is the finiteness result Theorem 3.2.3.

Step 1.1: Cohomology for $\mathcal{F} = \mathcal{O}(m)$. We know how to compute $H^p(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(m))$ for all p, n, m. A similar computation on $(\mathbb{P}^n)^{\mathrm{an}}$ shows that they agree.

Step 1.2: Cohomology for a general $\mathcal{F}$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_{\mathbb{C}}$. By Hilbert's syzygy theorem, $\mathcal{F}$ admits a finite resolution of the form

$$0 \longrightarrow \mathcal{M}_q \longrightarrow \cdots \longrightarrow \mathcal{M}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where $q \leq n$ and where each $\mathcal{M}_i$ is a finite direct sum of $\mathcal{O}(m_{ij})$ for some integers m_{ij}. We proceed by induction on the length q of this resolution. If $q = 0$, then $\mathcal{F}$ is a finite direct sum of $\mathcal{O}(m_i)$ for some m_i and we use Step 1.1. For the induction step, suppose $q \geq 1$ and let $\mathcal{F}' = \ker(\mathcal{M}_0 \to \mathcal{F})$, so that $[\mathcal{M}_q \to \cdots \to \mathcal{M}_1]$ is a resolution of $\mathcal{F}'$ of length $q - 1$. Thus the statement holds for $\mathcal{F}'$, and the assertion about $\mathcal{F}$ follows by the five lemma applied to the following diagram whose rows are the cohomology exact sequences on $\mathbb{P}^n$ and its analytification:

$$\begin{array}{ccccccccccc}
\cdots \to & H^p(\mathbb{P}^n_{\mathbb{C}}, \mathcal{F}') & \to & H^p(\mathbb{P}^n_{\mathbb{C}}, \mathcal{M}_0) & \to & H^p(\mathbb{P}^n_{\mathbb{C}}, \mathcal{F}) & \to & H^{p+1}(\mathbb{P}^n_{\mathbb{C}}, \mathcal{F}') & \to & H^{p+1}(\mathbb{P}^n_{\mathbb{C}}, \mathcal{M}_0) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & H^p(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}, (\mathcal{F}')^{\mathrm{an}}) & \to & H^p(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}, \mathcal{M}_0^{\mathrm{an}}) & \to & H^p(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}, \mathcal{F}^{\mathrm{an}}) & \to & H^{p+1}(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}, (\mathcal{F}')^{\mathrm{an}}) & \to & H^{p+1}(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}, \mathcal{M}_0^{\mathrm{an}} & \to \cdots
\end{array}$$

Step 1.3: Fully faithfulness. We have $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = H^0(\mathbb{P}^n_{\mathbb{C}}, \mathcal{H}om(\mathcal{F}, \mathcal{G}))$ and the sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is coherent. Moreover, we have

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})^{\mathrm{an}} \simeq \mathcal{H}om(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}})$$

and thus the assertion follows from the cohomology comparison in Step 1.2.

Step 1.4: Global generation of analytic coherent sheaves. This is the key step. We prove: if $\mathcal{F}$ is an object of $\mathbf{Coh}(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}})$, then for $m \gg 0$ the sheaf $\mathcal{F}(m) = \mathcal{F} \otimes \mathcal{O}(1)^{\otimes m}$ is globally generated.

We proceed by induction on n, the case $n = 0$ being trivial. For $x \in \mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}$ we need to show that for $m \gg 0$ the map

$$\alpha_{m,x} : H^0(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}, \mathcal{F}(m)) \longrightarrow \mathcal{F}(m)_x$$

is surjective. Indeed, then $\alpha_{m,x}$ is surjective in an open neighborhood of x, and so is $\alpha_{m',x}$ for $m' \geq m$ (over the same neighborhood). Since $\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}$ is compact, it follows that we can choose $m \gg 0$ for which every $\alpha_{m,x}$ is surjective, and then

$$H^0(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}, \mathcal{F}) \otimes \mathcal{O} \longrightarrow \mathcal{F}$$

is surjective, i.e. $\mathcal{F}$ is globally generated.

Let $H \simeq \mathbb{P}^{n-1}_{\mathbb{C}}$ be a hyperplane passing through x, defined by a section $\ell \in H^0(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(1))$. By induction, it suffices to show that the restriction map

$$\alpha_m \colon H^0(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}, \mathcal{F}(m)) \longrightarrow H^0(H^{\mathrm{an}}, \mathcal{F}|_{H^{\mathrm{an}}}(m)) \tag{3.2.2}$$

is surjective for $m \gg 0$. The obstruction to this surjectivity lies the H^1 of the kernel. This leads us to the study of coherent cohomology. Note that we have an exact sequence in $\mathbf{Coh}(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}})$ (for every $m \in \mathbb{Z}$)

$$0 \longrightarrow \mathcal{K}(m) \longrightarrow \mathcal{F}(m-1) \xrightarrow{\ \ell\ } \mathcal{F}(m) \longrightarrow \mathcal{F}|_{H^{\mathrm{an}}}(m) \longrightarrow 0$$

which breaks into two short exact sequences

$$0 \longrightarrow \mathcal{K}(m) \longrightarrow \mathcal{F}(m-1) \longrightarrow \ell\mathcal{F}(m) \longrightarrow 0 \tag{3.2.3}$$

$$0 \longrightarrow \ell\mathcal{F}(m) \longrightarrow \mathcal{F}(m) \longrightarrow \mathcal{F}|_{H^{\mathrm{an}}}(m) \longrightarrow 0$$

Note that $\mathcal{K}$ and $\mathcal{F}|_{H^{\mathrm{an}}}$ are naturally objects of $\mathbf{Coh}(H^{\mathrm{an}}) = \mathbf{Coh}(\mathbb{P}^{n-1,\mathrm{an}}_{\mathbb{C}})$. In particular, by induction assumption there exists objects $\mathcal{K}_0$ and $\mathcal{F}_0$ of $\mathbf{Coh}(H^{\mathrm{an}})$ with $\mathcal{K} \simeq \mathcal{K}_0^{\mathrm{an}}$ and $\mathcal{F}|_H \simeq \mathcal{F}_0^{\mathrm{an}}$. Then, by the cohomology comparison of Step 1.2 and by [Hartshorne, 1977, III, Theorem 5.2], for $m \gg 0$ we have

$$H^p(H^{\mathrm{an}}, \mathcal{K}(m)) \simeq H^p(H, \mathcal{K}_0(m)) = 0 \quad \text{and} \quad H^p(H^{\mathrm{an}}, \mathcal{F}|_H(m)) \simeq H^p(H, \mathcal{F}_0(m)) = 0 \quad \text{for } p > 0.$$

Now let us look at the relevant pieces of the cohomology exact sequences associated to (3.2.3):

$$\begin{array}{ccccc} & & H^0(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}, \mathcal{F}(m)) & \xrightarrow{\alpha_m} & H^0(H^{\mathrm{an}}, \mathcal{F}|_{H^{\mathrm{an}}}(m)) \\ & \swarrow & & & \\ H^1(\mathbb{P}^{n,\mathrm{an}}, \ell\mathcal{F}(m)) & \xrightarrow[\beta_m]{} & H^1(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}, \mathcal{F}(m)) & \longrightarrow & \underbrace{H^1(H^{\mathrm{an}}, \mathcal{F}|_{H^{\mathrm{an}}}(m))}_{0} \end{array} \tag{3.2.4}$$

which shows that β_m is surjective for $m \gg 0$, and

$$H^1(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}, \mathcal{F}(m-1)) \xrightarrow{\gamma_m} H^1(\mathbb{P}^{n,\mathrm{an}}, \ell\mathcal{F}(m)) \longrightarrow \underbrace{H^2(H^{\mathrm{an}}, \mathcal{K}(m))}_{0}$$

which shows that γ_m is surjective for $m \gg 0$. We conclude that the map

$$\ell = \beta_m \circ \gamma_m \colon H^1(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}, \mathcal{F}(m-1)) \longrightarrow H^1(\mathbb{P}^{n,\mathrm{an}}, \mathcal{F}(m)) \tag{3.2.5}$$

is surjective for $m \gg 0$. But these cohomology groups are of finite dimension by Theorem 3.2.3. It follows that for $m \gg 0$ the dimension of $H^1(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}, \mathcal{F}(m))$ stabilizes, which forces the map (3.2.5) to be an isomorphism. Then β_m and γ_m are isomorphisms as well. Looking at the exact sequence (3.2.4) again, we obtain the desired surjectivity of α_m.

Step 1.5: Essential surjectivity. Let $\mathcal{F}$ be an object of $\mathbf{Coh}(\mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}})$. Pick m large enough so that $\mathcal{F}(m)$ is globally generated. We obtain a surjection $\mathcal{O}(-m)^r \to \mathcal{F}$; let $\mathcal{K}$ be its kernel. We pick m' large enough so that $\mathcal{K}(m')$ is globally generated. We obtain a surjection $\mathcal{O}(-m')^{r'} \to \mathcal{K}$, and together the two surjections combine into a presentation for $\mathcal{F}$ of the form

$$\mathcal{O}(-m')^{r'} \xrightarrow{\varphi} \mathcal{O}(-m)^r \longrightarrow \mathcal{F} \longrightarrow 0.$$

Now φ is an $m' \times m$ matrix of homogeneous polynomials of degree $r - r'$. We can thus define a coherent sheaf $\mathcal{F}_0$ on $\mathbb{P}^n_{\mathbb{C}}$ as the cokernel of

$$\varphi\colon \mathcal{O}(-m')^{r'} \longrightarrow \mathcal{O}(-m)^r.$$

Since the analytification functor is exact, we have $\mathcal{F}_0^{\mathrm{an}} \simeq \mathcal{F}$. This finishes the proof of Step 1.

Step 2: projective schemes. Let X be a projective scheme over $\mathbb{C}$, and let $i\colon X \to \mathbb{P}^n_{\mathbb{C}}$ be a closed immersion. Then coherent sheaves on X are identified with coherent sheaves on $\mathbb{P}^n_{\mathbb{C}}$ which are annihilated by the ideal of X in $\mathbb{P}^n_{\mathbb{C}}$. The result then follows simply from the case $X = \mathbb{P}^n_{\mathbb{C}}$.

Step 3: proper schemes. This part of the proof deduces the proper case from the projective case by means of Chow's lemma [Grothendieck, 1961, 5.6.1]. See [Grothendieck, 1971, Exp. XII, §4] for the details. □

Corollary 3.2.5 (Chow's theorem)**.** *Let X be a proper scheme over $\mathbb{C}$. Then closed subschemes of X correspond bijectively to closed analytic subspaces of X^{an}.*

Proof. A closed subscheme $Z \subseteq X$ corresponds to a surjective map $\mathcal{O}_X \to \mathcal{F}$ in $\mathbf{Coh}(X)$ (where $\mathcal{F} = \mathcal{O}_Z$ and $Z = \mathrm{supp}(\mathcal{F})$). The same assertion holds on the analytic side, and thus the statement follows from Theorem 3.2.4. □

Corollary 3.2.6. *The analytification functor, restricted to proper schemes over $\mathbb{C}$, is fully faithful.*

Proof. A map $Y \to X$ corresponds to a closed subscheme $Z \subseteq Y \times X$ such that the projection $Y \times X \to Y$ induces an isomorphism $Z \xrightarrow{\sim} Y$. The same holds on the analytic side, and the assertion follows from Corollary 3.2.5. □

Thanks to the above corollary, we can use without causing confusion the following terminology. We shall say that a compact complex analytic space is **algebraic** if it is of the form X^{an} for a proper $\mathbb{C}$-scheme X. Without the compactness assumptions, the "algebraic structure" might not be unique.

For the next corollary, we recall that for a locally noetherian scheme X, finite morphism of schemes $Y \to X$ correspond contravariantly to commutative coherent $\mathcal{O}_X$-algebras, i.e. coherent sheaves $\mathcal{A}$ endowed with an $\mathcal{O}_X$-linear multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ giving $\mathcal{A}$ the structure of a sheaf of commutative rings. This correspondence sends $f\colon Y \to X$ to $\mathcal{A} = f_*\mathcal{O}_Y$ and $\mathcal{A}$ to its "relative spectrum" $\mathrm{Spec}(\mathcal{A})$.

An analogous statement holds for complex analytic spaces. For simplicity, let us call a map of complex analytic spaces $f\colon Y \to X$ *finite* if it is proper and its fibers are finite. Then $f_*\mathcal{O}_Y$ is a coherent commutative $\mathcal{O}_X$-algebra. One can show that this construction again establishes a correspondence between finite maps to X and commutative algebras in $\mathbf{Coh}(X)$. The inverse functor, the analog of the relative spectrum, sends $\mathcal{A}$ to $Y \to X$ characterized uniquely by $\mathrm{Hom}_X(Z,Y) = \mathrm{Hom}(\mathcal{A}, g_*\mathcal{O}_Z)$ in the slice category of maps $g\colon Z \to X$ of complex analytic spaces over X.

Corollary 3.2.7. *Let X be a proper scheme over $\mathbb{C}$. The analytification functor establishes an equivalence between finite schemes over X and finite complex analytic spaces over X^{an}.*

Note that the case of closed immersions (Corollary 3.2.5) is a special case of this result. Another one is:

Corollary 3.2.8. *Let X be a proper scheme over $\mathbb{C}$. Finite étale morphisms $Y \to X$ corresponds, via analytification, to finite covering spaces of X^{an}.*

Proof. Let $Y' \to X^{\mathrm{an}}$ be a finite covering space. There exists a unique structure of a complex analytic space on Y' making this map into a local isomorphism. The resulting map of complex analytic spaces is finite, and hence is the analytification of a unique finite map $Y \to X$.

It remains to show that for a finite map of schemes $Y \to X$, its analytification $Y^{\mathrm{an}} \to X^{\mathrm{an}}$ is a local isomorphism if and only if $Y \to X$ is étale. We will learn more about étale morphisms in subsequent lectures, but now it is sufficient to know this: a map of $\mathbb{C}$-schemes locally of finite type $Y \to X$ is étale if and only if for every $y \in Y(\mathbb{C})$ mapping to $x \in X(\mathbb{C})$, the resulting map of completed local rings

$$\widehat{\mathcal{O}}_{X,x} \longrightarrow \widehat{\mathcal{O}}_{Y,y}$$

is an isomorphism. The assertion then follows from the (already used) fact that

$$\widehat{\mathcal{O}}_{X,x} \simeq \widehat{\mathcal{O}}_{X^{\mathrm{an}},x} \quad \text{and} \quad \widehat{\mathcal{O}}_{Y,y} \simeq \widehat{\mathcal{O}}_{Y^{\mathrm{an}},Y}$$

□

Example 3.2.9 (Non-unique algebraic structure)**.** We show that if we drop the properness assumption, there exist pairs of non-isomorphic complex algebraic varieties with isomorphic analytifications. This example is again due to Serre (I know, right?) and is explained beautifully in [Hartshorne, 1970, VI 3.2]. We explain it slightly differently. Let $\mathbb{Z}^2 \simeq \Lambda \subseteq \mathbb{C}$ be a lattice, which we let act on $\mathbb{C}^2$ by

$$\lambda \cdot (x_1, x_2) = (x_1 + \lambda, x_2 + \lambda).$$

This action is free and properly discontinuous. Denote the quotient complex surface $\mathbb{C}^2/\mathbb{Z}^2$ by X.

The first projection $\mathbb{C}^2 \to \mathbb{C}$ induces a map to the elliptic curve $E = \mathbb{C}/\Lambda$:

$$\begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & \mathbb{C} \\ \downarrow & & \downarrow \\ X & \longrightarrow & E \end{array}$$

and we check easily that $X \to E$ is an $\mathbb{C}$-bundle. Note that affine line bundles (as opposed to line bundles) on E are classified by $H^1(E, \mathcal{O}_E)$, which is one-dimensional. The variety

X corresponds to the unique non-split extension $\mathcal{F}$ of $\mathcal{O}_E$ by $\mathcal{O}_E$. This interpretation gives X an algebraic structure: $X \simeq Y^{\mathrm{an}}$ where $Y = \mathbb{P}(\mathcal{F}) \setminus E$. The variety Y is not affine because $H^0(Y, \mathcal{O}_Y) = \mathbb{C}$.

On the other hand, the action is isomorphic to the action

$$(n_1, n_2) \cdot (x_1, x_2) = (x_1 + n_1, x_2 + n_2)$$

of $\mathbb{Z}^2$ on $\mathbb{C}^2$. The quotient by this action is

$$(\mathbb{C}/\mathbb{Z}) \times (\mathbb{C}/\mathbb{Z}) \quad \simeq \quad \mathbb{C}^\times \times \mathbb{C}^\times$$

and hence X is also the analytification of the affine variety $Z = \mathbb{G}_m \times \mathbb{G}_m$.

Note that in this example Z is affine and Y is not. In particular, $X = Y^{\mathrm{an}}$ is Stein while Y is not affine.

Remark 3.2.10 (Warning)**.** Given the fully faithfulness statement Corollary 3.2.6, one could naively hope that if S is a projective variety and $f \colon X \to S^{\mathrm{an}}$ is a proper morphism of complex analytic spaces which locally on S is algebraizable, then f is algebraizable and in particular X is algebraic. The following example shatters this hope.

Let q be a complex number with $0 < |q| < 1$ and let $H_q = (\mathbb{C}^2 \setminus 0)/q^{\mathbb{Z}}$ be the Hopf surface. There is the obvious map $H_q \to \mathbb{P}^1 = (\mathbb{C}^2 \setminus 0)/\mathbb{C}^\times$ whose fibers are all isomorphic to the genus one curve $E_q = \mathbb{C}^\times / q^{\mathbb{Z}}$. Topologically, we have

$$H_q \sim S^1 \times S^3, \quad E_q \sim S^1 \times S^1, \quad \mathbb{P}^1 \sim S^2,$$

and the fibration $H^q \to \mathbb{P}^1$ is I think? the projection $S^1 \times S^3$ followed by the Hopf fibration $S^3 \to S^2$ with fiber S^1. The restriction of the family $H_q \to \mathbb{P}^1$ to every affine open $U \subseteq \mathbb{P}^1$ is isomorphic to the projection $E_q \times U \to U$. However, H_q is not algebraic (for example, because $H^1(H_q, \mathbb{Z}) \simeq \mathbb{Z}$).

The GAGA theorem also implies the following seemingly obvious fact.

Corollary 3.2.11. *Let X be a scheme locally of finite type over $\mathbb{C}$. Then the map ε induces a bijection $\pi_0(X^{\mathrm{an}}) \xrightarrow{\sim} \pi_0(X)$.*

Proof. Surjectivity is easy: if $Z \subseteq X$ is a connected component, then Z^{an} is nonempty since $Z(\mathbb{C}) \neq \emptyset$. We prove injectivity.

We first deal with the case X proper. We have

$$\Gamma(X, \mathcal{O}_X) = \mathrm{Hom}_{\mathbf{Coh}(X)}(\mathcal{O}_X, \mathcal{O}_X) = \mathrm{Hom}_{\mathbf{Coh}(X)}(\mathcal{O}_X, \mathcal{O}_X) = \Gamma(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}).$$

But connected components of X correspond to irreducible idempotent elements in $\Gamma(X, \mathcal{O}_X)$, and similarly for $\Gamma(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$.

The general case can be reduced to the case X affine normal connected. See [Grothendieck, 1971, ExpẊII] for the details.

Suppose now X is affine, normal, and connected. We need to show that X^{an} is connected. Let $X \subseteq \overline{X}$ be a normal projective compactification of X. One can construct it by embedding X

in $\mathbb{A}^n$, taking the closure in $\mathbb{P}^n$, then taking the normalization. Since $\overline{X}$ is connected and proper, we have already proved that $\overline{X}^{\mathrm{an}}$ is connected.

It remains to use the following fact: a non-empty Zariski open subset of a connected normal complex analytic space is connected. We omit the proof, see [Grothendieck, 1971, Exp. XII]. □

Note that the assertion of Corollary 3.2.11 is false over $\mathbb{R}$. For example, if X is the elliptic curve given by the equation $y^2 = x^3 - x$, then $X(\mathbb{R})$ has two connected components.

3.3. Serre's examples of conjugate varieties

Let X be a scheme locally of finite type over $\mathbb{C}$. We deal with the following general question:

$$\textit{Which topological properties of } X(\mathbb{C}) \textit{ can be defined algebraically?} \tag{3.3.1}$$

While we will not make this question completely precise, a property which is "defined algebraically" should, in particular, be preserved by field automorphisms of $\mathbb{C}$. The following paragraph explains why this question is nontrivial.

Let σ be an automorphism of $\mathbb{C}$. Then σ is continuous if and only if it is either the identity or conjugation. If X is a scheme locally of finite type over $\mathbb{C}$, we can form the "twist of X by σ" as the pullback

$$\begin{array}{ccc} X^\sigma & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow[\sigma]{} & \mathrm{Spec}(\mathbb{C}) \end{array}$$

(we treat X^σ as a scheme over $\mathbb{C}$ via the left vertical map). Note that $X^\sigma \to X$ is an isomorphism of schemes, but not of $\mathbb{C}$-schemes. In simple terms, X^σ is obtained from X by applying σ to the coefficients defining X. For example, the twist of the elliptic curve

$$E_\lambda = V(y^2z - x(x-z)(x-\lambda z)) \subseteq \mathbb{P}^2_{\mathbb{C}}, \qquad \lambda \in \mathbb{C} \setminus \{0,1\}$$

is the elliptic curve $E_{\sigma(\lambda)}$. Since σ might be discontinuous, the induced map $X^\sigma(\mathbb{C}) \to X(\mathbb{C})$ is not always continuous either. Thus there is no map

$$\pi_1(X^\sigma, x) \dashrightarrow\!\!\!\!\!\!\times\;\; \pi_1(X, x)$$

(one can of course substitute their favorite homotopy invariant in place of π_1.)

Anyway, we expect that $X^\sigma(\mathbb{C})$ should be topologically similar to $X(\mathbb{C})$. For example, we just saw that if X is an elliptic curve then so is X^σ, so the two spaces are homeomorphic in this case. Invariants for which the answer to Question (3.3.1) is positive include:

(a) The Betti numbers $b_n(X(\mathbb{C})) = \dim H^n(X(\mathbb{C}), \mathbb{Q})$. If X is smooth and projective, this can be seen using GAGA and the Hodge decomposition: we have

$$b_n(X(\mathbb{C})) = \sum_{p+q=n} \dim H^q(X^{\mathrm{an}}, \Omega^q_{X^{\mathrm{an}}}) = \sum_{p+q=n} \dim H^q(X, \Omega^q_{X/\mathbb{C}}),$$

so not only the Betti numbers but even the Hodge numbers are algebraically defined. If X is smooth (but possibly non-proper), we can use Grothendieck's comparison theorem [Grothendieck, 1966]:

$$b_n(X(\mathbb{C})) = \dim H^n(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}}) = \dim H^n(X, \Omega^\bullet_{X/\mathbb{C}}).$$

The result is true for X singular as well, and one can even deal with torsion. Perhaps this is easiest to show using ℓ-adic cohomology. Note that we do not claim here that we can define the cohomology groups $H^n(X(\mathbb{C}), \mathbb{Z})$ algebraically, but only their isomorphism type as a finitely generated abelian group.

(b) The profinite completion of the fundamental group $\pi_1^{\mathrm{top}}(X)$. For X proper, GAGA implies that finite étale coverings of $X(\mathbb{C})$ correspond to finite étale maps $Y \to X$ (see Corollary 3.2.8). As we shall later see, this statement is true even if X is not proper. The profinite completion of $\pi_1^{\mathrm{top}}(X)$ is identified with the étale fundamental group of X.

The goal of this section is to construct, following Serre [Serre, 1964], a smooth projective variety X over a number field F and two embeddings $\phi_0, \phi_1 : F \to \mathbb{C}$ such that the topological fundamental groups

$$\pi_1^{\mathrm{top}}(X_{\phi_0}) \quad \text{and} \quad \pi_1^{\mathrm{top}}(X_{\phi_1}$$

are not isomorphic, where for $\phi : F \to \mathbb{C}$ we denote by X_ϕ the base change of X along ϕ. In particular, $X_{\phi_0}(\mathbb{C})$ and $X_{\phi_1}(\mathbb{C})$ are not homotopy equivalent. (In order to turn this into a problem of conjugating a complex variety by an automorphism of $\mathbb{C}$, embed F into $\mathbb{C}$ by ϕ_0 and then extend $\phi_1 : F \to \mathbb{C}$ to an isomorphism $\sigma : \mathbb{C} \to \mathbb{C}$.)

Remark 3.3.1. The theory of the étale fundamental group developed later will imply that the groups $\pi_1^{\mathrm{top}}(X_{\phi_0})$ and $\pi_1^{\mathrm{top}}(X_{\phi_1})$ have isomorphic pro-finite completions. Moreover, Deligne's results [Deligne, 1970] show that the categories of *complex* representations of the two groups are equivalent. This makes the example even more surprising.

Remark 3.3.2. It is an open question whether there exists an example with $X_{\phi_0}(\mathbb{C})$ simply connected and $X_{\phi_1}(\mathbb{C})$ not. The corresponding group would have to have trivial profinite completion (an example of such a group, the Higman four group, is given in the homework problems).

Serre's example is based on the theory of complex multiplication for elliptic curves (reviewed below). It begins with the following observation: if E is an elliptic curve over $\mathbb{C}$ with complex multiplication by the ring of integers $\mathcal{O}_K$ in an imaginary quadratic number field $K = \mathbb{Q}(\sqrt{-p})$ (meaning that $\mathrm{End}(E) \simeq \mathcal{O}_K$), then

$$\pi_1^{\mathrm{top}}(E) = H_1(E(\mathbb{C}), \mathbb{Z}) \simeq \mathbb{Z}^2$$

carries the structure of a projective $\mathcal{O}_K$-module of rank one. This produces a class $c(E)$ in the (finite) class group $\mathrm{Cl}(\mathcal{O}_K) = \mathrm{Pic}(\mathrm{Spec}(\mathcal{O}_K))$. If we apply an automorphism of $\mathbb{C}$, the resulting scheme E_σ is still an elliptic curve with complex multiplication by $\mathcal{O}_K$. However, the resulting class $c(E_\sigma)$ might be different from c. In other words, the group $\mathrm{Aut}(\mathbb{C})$ operates nontrivially on the finite group $\mathrm{Cl}(\mathcal{O}_K)$. This means that the topological invariant $H_1(E(\mathbb{C}),\mathbb{Z})$ is actually sensitive to automorphisms of $\mathbb{C}$, but only if we treat it as an invariant of the variety E together with an endomorphism. Serre found a way to construct from E an abelian variety A such that the extra endomorphism of E gives rise to an order p automorphism of A, in such a way that $H_1(A(\mathbb{C}),\mathbb{Z}) \simeq \mathbb{Z}^{2\dim(A)}$ is still sensitive to the action of σ, but now if we treat it as a $\mathbb{Z}[\mathbb{Z}/p]$-module. In the final step, the abelian variety A is replaced with a smooth projective variety X whose fundamental group is an extension of $\mathbb{Z}/p$ by $H_1(A(\mathbb{C}),\mathbb{Z})$, which "encodes" the $\mathbb{Z}/p$-action on $H_1(A(\mathbb{C}),\mathbb{Z})$. Then $\pi_1^{\mathrm{top}}(X)$ will change when we replace X with X_σ for suitable σ.

Facts from algebraic number theory. A number field is a finite extension K of $\mathbb{Q}$. The integral closure of $\mathbb{Z}$ in K is the ring of integers $\mathcal{O}_K$, which is a Dedekind domain (its local rings are discrete valuation rings). The class group of K is by definition the class group $\mathrm{Cl}(\mathcal{O}_K) = \mathrm{Pic}(\mathrm{Spec}(\mathcal{O}_K))$. It is trivial if and only if $\mathcal{O}_K$ is a UFD; in general, it is a finite group whose order is called the class number of K.

- If K is an imaginary quadratic number field $K = \mathbb{Q}(\sqrt{-d})$, $d \geq 1$ squarefree, then $\mathcal{O}_K$ is spanned by 1 and $(1+\sqrt{-d})/2$ if $d \equiv -1 \bmod 4$, and by 1 and $\sqrt{-d}$ otherwise. The class number grows to infinity with d.

- If $K = \mathbb{Q}(\zeta_n)$ is the cyclotomic field (adjoining a primitive n-th root of unity), then $\mathcal{O}_K = \mathbb{Z}[\zeta_n]$. The class number of $\mathbb{Q}(\zeta_p)$ for p prime plays an important role in early attempts at proving Fermat's last theorem for some exponents p.

The Grothendieck group of $\mathcal{O}_K$-modules is equal to $\mathbb{Z} \times \mathrm{Cl}(\mathcal{O}_K)$; for a projective module M of rank r, the corresponding class is $(r, \det M)$ where $\det M = \bigwedge^r M$. In fact, $M \simeq \mathcal{O}_K^{r-1} \oplus \det(M)$ (see e.g. [Rosenberg, 1994, §1.4] or Theorem 7.2 in Peter May's notes[9]).

The Kronecker–Weber theorem asserts that every finite extension $K/\mathbb{Q}$ with abelian Galois group is contained in a cyclotomic field $\mathbb{Q}(\zeta_n)$. In particular, this implies that $\sqrt{-d}$ is a rational combination of roots of unity. Here is a concrete fact we will need:

Lemma 3.3.3. *Let p be a prime congruent to -1 mod 4. Then $\mathbb{Q}(\sqrt{-p}) \subseteq \mathbb{Q}(\zeta_p)$.*

Proof. Consider the quadratic Gauss sum $G = \sum_{i=0}^{p-1} \left(\frac{i}{p}\right) \zeta_p^i$. The assumption on p implies that -1 is not a square mod p, and therefore $\overline{G} = -G$, so G is purely imaginary. On the other hand, one can compute $|G|^2 = G\overline{G}$ to be equal to p. See [Ireland and Rosen, 1990, Proposition 6.3.2] for the details. □

If L/K is a finite extension of number fields of degree r, then $\mathcal{O}_L$ is a projective $\mathcal{O}_K$-module of rank r. It is endowed with the trace map $\mathrm{tr}\colon \mathcal{O}_L \to \mathcal{O}_K$. The map $\mathcal{O}_L \to \mathcal{O}_L^\vee = \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_K)$

[9] https://www.math.uchicago.edu/~may/MISC/Dedekind.pdf

sending x to the homomorphism $y \mapsto \mathrm{tr}(xy)$ induces by taking the r-th exterior powers a map between rank one projectives

$$\det(\mathcal{O}_L) \to \det(\mathcal{O}_L)^\vee,$$

or equivalently a map $\det(\mathcal{O}_L)^{\times 2} \to \mathcal{O}_K$. Its image is a nonzero ideal of $\mathcal{O}_K$ called the discriminant of L/K. We conclude that the class of this ideal in $\mathrm{Cl}(\mathcal{O}_K)$ is twice of the class $\det(\mathcal{O}_L)$ of $\mathcal{O}_L$. The discriminant can be computed in many cases:

- For $\mathbb{Q}(\sqrt{-d})/\mathbb{Q}$ it equals d if $d \equiv -1 \bmod 4$ and $4d$ otherwise.
- For $\mathbb{Q}(\zeta_p)$ it equals p^{p-1}.

We deduce the following fact:

- If p is a prime congruent to $-1 \bmod 4$, then $\mathbb{Q}(\zeta_p)/\mathbb{Q}(\sqrt{-p})$ is an extension of degree $(p-1)/2$ and discriminant a power of p. ■

Complex multiplication. Let E be an elliptic curve over $\mathbb{C}$. Then $E(\mathbb{C}) \simeq \mathbb{C}/\Lambda$ where $\Lambda \subseteq \mathbb{C}$ is a discretely embedded copy of $\mathbb{Z}^2$. Note that $\Lambda = H_1(E(\mathbb{C}), \mathbb{Z}) = \pi_1^{\mathrm{top}}(E)$. Conversely, every such quotient defines an elliptic curve, and two lattices Λ, Λ' give rise to isomorphic curves if and only if $\Lambda = z\Lambda'$ for some $z \in \mathbb{C}^\times$. The endomorphism ring $\mathrm{End}(E)$ can be described as the set of all $z \in \mathbb{C}$ such that $z\Lambda \subseteq \Lambda$. It is either isomorphic to $\mathbb{Z}$ or is a subring $\mathbb{Z} \neq R \subseteq \mathcal{O}_K$, $K = \mathbb{Q}(\sqrt{-d})$, and in the latter case Λ is a projective R-module of rank one, defining a class $c(\Lambda) = c(E) \in \mathrm{Pic}(R)$. If $J \subseteq R$ is an invertible ideal of class c, then $\mathbb{C}/J$ (for the natural embedding of J into $\mathbb{C}$) defines an elliptic curve $E(J)$ with $c(E(J)) = c$. This construction establishes a bijection between isomorphism classes of complex elliptic curves with $\mathrm{End}(E) = R$ and $\mathrm{Pic}(R)$. We say that E has **complex multiplication** by R if $\mathrm{End}(E) \simeq R$ (and $R \neq \mathbb{Z}$). If E has complex multiplication (by some $R \neq \mathbb{Z}$), then $j(E)$ is an algebraic number, as its conjugates under $\mathrm{Aut}(\mathbb{C})$ are j-invariants of elliptic curves with endomorphism ring R and hence are finite in number. Therefore every elliptic curve with complex multiplication is defined over a number field.

An important fact we shall use is the Weber–Fueter theorem [Serre, 1967]: let E and E' be two complex elliptic curves with complex multiplication by $\mathcal{O}_K$, $K = \mathbb{Q}(\sqrt{-d})$. Then $j(E)$ and $j(E')$ are conjugate (under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The full picture is the following:

Theorem 3.3.4. *Let $K = \mathbb{Q}(\sqrt{-d})$. There exists a Galois extension F/K with an isomorphism $\mathrm{Gal}(F/K) \simeq \mathrm{Cl}(\mathcal{O}_K)$* [10]*, an elliptic curve E over F with $\mathrm{End}(E) \simeq \mathcal{O}_K$, and an embedding $\phi_0 \colon F \to \mathbb{C}$ such that for every $c \in \mathrm{Cl}(\mathcal{O}_K)$, if we denote by $\phi_c \colon F \to \mathbb{C}$ the composition of ϕ_0 with the automorphism of F/K corresponding to c, and by E_c the base change of E along ϕ_c, then $H^1(E_c(\mathbb{C}), \mathbb{Z})$ is an invertible $\mathcal{O}_K$-module of class c.*

[10] In fact F is the Hilbert class field of K, the maximal abelian unramified extension.

Construction of E. Fix a prime p congruent to -1 mod r and let $K=\mathbb{Q}(\sqrt{-p})$. We denote the class number of K by h. For reasons soon to become apparent, we assume that $h>1$ and that h is prime to $p-1$. For example, this happens for $p=23$ with $h=3$. Let $c\in \mathrm{Cl}(\mathcal{O}_K)$ be a nonzero class.

We set F/K to be the Hilbert class field of K as in Theorem 3.3.4, endowed with the complex embeddings $\phi_c\colon F\to\mathbb{C}$ ($c\in\mathrm{Cl}(\mathcal{O}_K)=\mathrm{Gal}(F/K)$) and the elliptic curve E over F. We set $\phi_0\colon F\to\mathbb{C}$ to be the embedding corresponding to $0\in\mathrm{Cl}(\mathcal{O}_K)$ and $\phi_1=\phi_c\colon F\to\mathbb{C}$ to be the embedding corresponding to the nonzero class c.

By construction, we we have that $H_1(E_{\phi_0}(\mathbb{C}),\mathbb{Z})$ is a free $\mathcal{O}_K$-module and $H_1(E_{\phi_1}(\mathbb{C}),\mathbb{Z})$ is not.

Construction of A. By our assumption on p, we have $K\subseteq\mathbb{Q}(\zeta_p)$. Set $G=\mathbb{Z}/p$ with generator σ and note that $\mathbb{Z}[\zeta_p]=\mathbb{Z}[G]/(\tau)$ where $\tau=1+\sigma+\cdots+\sigma^{p-1}$ (as $\sigma^p-1=\tau(\sigma-1)$). Thus $\mathbb{Z}[\zeta_p]$ is both an $\mathcal{O}_K$-module and a $\mathbb{Z}[G]$-module.

We claim that $\mathbb{Z}[\zeta_p]$ is a free $\mathcal{O}_K$-module (of rank $(p-1)/2$). Indeed, it is free if and only if its determinant $\bigwedge^{(p-1)/2}\mathbb{Z}[\zeta_p]$ is free, but by the assertion in REF its square is the class of the discriminant of F/K, which is a power of p and hence principal. Since $h=\#\mathrm{Cl}(\mathcal{O}_K)$ is odd, we conclude that $\bigwedge^{(p-1)/2}\mathbb{Z}[\zeta_p]$ is free as well, and hence so is $\mathbb{Z}[\zeta_p]$.

We define the abelian variety $A=E^{(p-1)/2)}$. The ring of endomorphisms $\mathrm{End}(A)$ is the ring of $(p-1)/2\times(p-1)/2$ matrices over $\mathrm{End}(E)=\mathcal{O}_K$, and hence picking a basis of $\mathbb{Z}[\zeta_p]$ it can be identified with the module of $\mathcal{O}_K$-module endomorphisms of $\mathbb{Z}[\zeta_p]$. This choice gives us an action of $\mathbb{Z}[\zeta_p]$, and hence of $G=\mathbb{Z}/p$, on A.

We claim that $H_1(A_{\phi_0}(\mathbb{C}),\mathbb{Z})$ is a free $\mathbb{Z}[\zeta_p]$-module and $H_1(A_{\phi_1}(\mathbb{C}),\mathbb{Z})$ is not. We have

$$H_1(A_{\phi_i}(\mathbb{C}),\mathbb{Z})\simeq H_1(E_{\phi_i}(\mathbb{C}),\mathbb{Z})\otimes_{\mathcal{O}_K}\mathbb{Z}[\zeta_p],$$

so the first assertion is clear. For the second assertion, we check the stronger claim that $H_1(A_{\phi_1}(\mathbb{C}),\mathbb{Z})$ is not free as an $\mathcal{O}_K$-module. To this end, we check that the class of its determinant is $(p-1)/2$ times the class c of $H_1(E_{\phi_1}(\mathbb{C}),\mathbb{Z})$. Since $c\neq 1$ and $h=\#\mathrm{Cl}(\mathcal{O}_K)$ is prime to $p-1$, the assertion follows.

Construction of X. We have constructed an abelian variety A over F with an action of $G=\mathbb{Z}/p$. By Theorem 2.4.1 there exists a simply connected smooth projective variety Y with a free G-action (which we may construct already over F). (As Serre remarks, in this situation it is sufficient to take for Y the Fermat hypersurface of degree p in $\mathbb{P}^{p-1}$.) Thus G acts freely on $Y\times A$, and we can take the quotient

$$X=(Y\times A)/G.$$

Thus X is a smooth projective variety admitting $Y\times A$ as a G-covering space, and hence for any embedding $\phi\colon F\to\mathbb{C}$ we have a short exact sequence

$$1\longrightarrow H_1(A_\phi(\mathbb{C}),\mathbb{Z})\longrightarrow\pi_1^{\mathrm{top}}(X_\phi)\longrightarrow G\longrightarrow 1,$$

a nonabelian extension of $\mathbb{Z}/p$ by $\mathbb{Z}^{p-1}$. We finally arrive at:

Theorem 3.3.5. *The groups $\pi_1^{\mathrm{top}}(X_{\phi_0})$ and $\pi_1^{\mathrm{top}}(X_{\phi_1})$ are not isomorphic.*

Proof. We first claim[11] that for every ϕ, the subgroup

$$M = H_1(A_\phi(\mathbb{C}),\mathbb{Z}) \subseteq \pi_1^{\mathrm{top}}(X_\phi) = \Pi$$

is the unique abelian normal subgroup of index p. Indeed, suppose $M' \subseteq \Pi$ is another such, with $M \neq M'$. Then M' contains an element x mapping to the generator σ of G. This forces $M \cap M'$ to be a trivial G-submodule of M of index p. Indeed, the action of the generator $\sigma \in G$ on M is given by $m \mapsto xmx^{-1}$. Thus if $m \in M \cap M'$, then $\sigma m = xmx^{-1} = m$ since M' is abelian. However, in our situation M is a nonzero projective $\mathbb{Z}[G]/(\tau) = \mathbb{Z}[\zeta_p]$-module (here $\tau = 1+\sigma+\cdots+\sigma^{p-1}$). If it has a trivial (i.e. annihilated by $\sigma - 1$) submodule of index p, then the whole module would be annihilated by $(\sigma-1)^2$. But $\mathbb{Z}[\zeta_p]$ is a Dedekind domain, and M is torsion free.

Now, suppose that there exists an isomorphism $\pi_1^{\mathrm{top}}(X_{\phi_0}) \xrightarrow{\sim} \pi_1^{\mathrm{top}}(X_{\phi_1})$. By the result of the previous paragraph, it extends to an isomorphism of extensions

$$\begin{array}{ccccccccc}
1 & \longrightarrow & H_1(A_{\phi_0}(\mathbb{C}),\mathbb{Z}) & \longrightarrow & \pi_1^{\mathrm{top}}(X_{\varphi_0}) & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & H_1(A_{\phi_1}(\mathbb{C}),\mathbb{Z}) & \longrightarrow & \pi_1^{\mathrm{top}}(X_{\varphi_1}) & \longrightarrow & G & \longrightarrow & 1.
\end{array}$$

This means that the G-modules $H_1(A_{\phi_0}(\mathbb{C}),\mathbb{Z})$ and $H_1(A_{\phi_1}(\mathbb{C}),\mathbb{Z})$ differ by an automorphism of G. But the element $\tau = 1+\sigma+\cdots+\sigma^{p-1} \in \mathbb{Z}[G]$ is preserved by all automorphisms of G, and so is the property "M is annihilated by τ and is free as a $\mathbb{Z}[G]/(\tau)$-module." But $H_1(A_{\phi_0}(\mathbb{C}),\mathbb{Z})$ has this property while $H_1(A_{\phi_1}(\mathbb{C}),\mathbb{Z})$ does not, a contradiction. □

Remark 3.3.6. Since Serre's paper, numerous examples of pairs of conjugate varieties with different topological properties have been constructed. See in particular [Milne and Suh, 2010] for examples coming from Shimura varieties and Charles [Charles, 2009] who used elliptic curves with complex multiplication to construct a pair with non-isomorphic real cohomology rings $H^*(X_\sigma(\mathbb{C}),\mathbb{R}) \not\simeq H^*(X(\mathbb{C}),\mathbb{R})$.

3.4. Malcev's theorem (A)

Definition 3.4.1. A group Γ is **residually finite** if the intersection of its finite index subgroups is the trivial group, or equivalently if the profinite completion map $\Gamma \to \widehat{\Gamma}$ is injective.

Theorem 3.4.2. *Let Γ be a finitely generated group. Suppose that Γ admits a faithful representation*

$$\rho : \Gamma \longrightarrow \mathrm{GL}_n(K)$$

for some field K. Then Γ is residually finite.

[11] I hope that's what he meant by *On vérifie sans difficultés que...* [Serre, 1964, bottom of p. 4195]

Proof. Let $\gamma \in \Gamma$. We shall find a representation valued in a finite field k,

$$\rho' \colon \Gamma \longrightarrow \mathrm{GL}_n(k)$$

with $\rho'(\gamma) \neq 1$. Then $\ker(\rho') \subseteq \Gamma$ is a subgroup of finite index which does not contain γ. Since we can find such a subgroup for every $\gamma \in \Gamma$, it follows that Γ is residually finite.

Consider the following functor $\mathrm{Rep}_n(\Gamma)$ on the category of $\mathbb{Z}$-algebras:

$$\mathrm{Rep}_n(\Gamma)(R) = \mathrm{Hom}(\Gamma, \mathrm{GL}_n(R)).$$

It is represented by a scheme of finite type over $\mathbb{Z}$ (which we denote by X). Indeed, let $\gamma_1, \ldots, \gamma_r \in \Gamma$ be a finite set of generators; then giving a representation $\Gamma \to \mathrm{GL}_\mathrm{n}(R)$ is equivalent to giving a system of r invertible matrices $A_1, \ldots, A_r$ of size $n \times n$ with coefficients in r such that for every relation

$$w \in \ker(F_r \longrightarrow \Gamma)$$

between $\gamma_1, \ldots, \gamma_r$, we have

$$w(A_1, \ldots, A_r) = 1,$$

which translates into a system of polynomial relations between the entries of the matrices A_i. Note that since $\mathbb{Z}$ is noetherian, finitely many of these equations suffice to describe $\mathrm{Rep}_n(\Gamma)$, even though Γ might not be finitely presented. Thus

$$X \simeq \mathrm{Spec}\left(\mathbb{Z}[a_{ijk}\,(i \leq r, 1 \leq j,k \leq n), d_i\,(i \leq r)]/(d_i \det([a_{ijk}]_{j,k}) - 1, \text{(relations coming from } \Gamma)\right).$$

We now define the subfunctor $Y_\gamma \subseteq X$ by the condition that the element γ is sent to $1 \in \mathrm{GL}_n(R)$. It is clearly a closed subscheme of X. We define $U_\gamma \subseteq X$ to be its open complement, which is again a scheme of finite type over $\mathbb{Z}$, though it might no longer be affine.

Note that the faithful representation ρ provides a point

$$[\rho] \in U_\gamma(K).$$

In particular, U_γ is non-empty, and hence it contains a closed point $u \in U_\gamma$. The residue field $k = k(u)$ is a finite field. By the modular interpretation of U_γ we obtain a representation

$$\rho' \colon \Gamma \longrightarrow \mathrm{GL}_n(k)$$

with $\rho'(\gamma) \neq 1$, as desired. □

A. Complements to Part I

(todo: Summary (no proofs) of what is known in 2025 about Kähler groups, groups of class $\mathcal{P}$, fundamental groups of smooth/normal quasiprojective varieties, fundamental group of curve complements and complements of hyperplane arrangements. Fanos/rc varieties are simply connected (Debarre). Maybe examples of quotients of abelian varieties by finite group actions. Corollaries of Yau/BB decomposition: CY have virtually abelian pi1. Something about strange surfaces of general type? Double covers of P2?)

Part II
The étale fundamental group

4. Lecture 4: Étale morphisms and Galois categories

Summary. *Having discussed the topological fundamental group of complex varieties, we turn our attention towards étale fundamental groups of schemes. We review étale morphisms in §4.1–§4.2, and in §4.3 we reinterpret Galois theory of fields in terms of étale algebras. This serves as a motivation for the definition of Galois categories in §4.4. In the final §4.5 we introduce the key example of a Galois category: the category of finite étale coverings of a connected scheme (part of the proof is postponed to Lecture 5). This allows us to define the étale fundamental group as the Galois group of this Galois category.*

In addition to [Grothendieck, 1971], a good reference for this part of the lecture are the relevant chapters of the Stacks Project [SP Chapter 024J] and [SP Chapter 0BQ6]. For a great review of étale morphisms, with detailed proofs, see [Raynaud, 1970].

4.1. Étale morphisms

The notion of an étale morphism of schemes is motivated by the implicit function theorem, which says that a map of smooth manifolds $f\colon Y \to X$ is a diffeomorphism onto its image in an open neighborhood of a point $y \in Y$ if and only if the mapping on tangent spaces $T_yY \to T_{f(y)}X$ is an isomorphism. In local coordinates $t_1, \ldots, t_n$ at Y and $f_1, \ldots, f_n$ at $f(y)$, the latter condition can be expressed in terms of the Jacobian determinant:

$$\det \left[\frac{df_i}{dx_j}\right](y) \neq 0.$$

In algebraic geometry, being a local isomorphism is much stronger than the above differential condition, which for maps of smooth varieties is equivalent to being étale. For example, the map $f(z) = z^n\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ is not a local isomorphism, but it is étale away from zero if the characteristic of k does not divide n.

Recall that if X is a variety (or a scheme) over a field k, and $x \in X(k)$ is a rational point, then tangent vectors $v \in T_xX = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ correspond to maps

$$v\colon \operatorname{Spec}(k[\varepsilon]/(\varepsilon^2)) \longrightarrow X$$

which reduce to $x\colon T_0 = \operatorname{Spec}(k) \to X$ modulo ε. (If you haven't seen this before, see [Hartshorne, 1977, Exercise II 2.8] and Lemma 4.1.2 below.) Diagrammatically, one can picture the tangent space T_xX as the set of dotted arrows making the diagram below commute

$$\begin{array}{ccc} T_0 = \operatorname{Spec}(k) & \xrightarrow{\quad x \quad} & X \\ \big\downarrow & \overset{v}{\nearrow} & \big\downarrow \\ T = \operatorname{Spec}(k[\varepsilon]/(\varepsilon^2)) & \longrightarrow & \operatorname{Spec}(k). \end{array}$$

Let now $f\colon Y \to X$ be a map of k-schemes and let $y \in Y(k)$ with $f(y) = x$. Then lifts $w \in T_yY$ of a tangent vector $v \in T_xX$ correspond to dotted arrows making a similar diagram commute

$$\begin{array}{ccc} T_0 = \operatorname{Spec}(k) & \xrightarrow{\quad y \quad} & Y \\ \big\downarrow & \overset{w}{\nearrow} & \big\downarrow f \\ T = \operatorname{Spec}(k[\varepsilon]/(\varepsilon^2)) & \xrightarrow[\quad v \quad]{} & X. \end{array}$$

In particular, the map on tangent spaces $df\colon T_yY \to T_xX$ is an isomorphism for every $y \in Y(k)$ over x if and only if the map

$$i^*\colon \operatorname{Hom}_X(T,Y) \longrightarrow \operatorname{Hom}_X(T_0,Y)$$

is bijective.

It is convenient to formulate the definition of an étale morphism, as well as the closely related notions of smooth and unramified morphisms, in terms of "infinitesimal lifting problems." A **nilpotent thickening** is a closed immersion of schemes $T_0 \to T$ such that the corresponding ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_T$ is locally nilpotent. A **square zero thickening** is one for which $\mathcal{I}^2 = 0$. Note that in this case we have

$$\mathcal{I} = \mathcal{I}/\mathcal{I}^2 = \mathcal{I} \otimes_{\mathcal{O}_T} \mathcal{O}_{T_0}$$

and hence $\mathcal{I}$ is naturally an $\mathcal{O}_{T_0}$-module. Moreover, every nilpotent thickening with $\mathcal{I}^{n+1} = 0$ (globally) is a composition of square zero thickenings:

$$T_0 \hookrightarrow T_1 \hookrightarrow \cdots \hookrightarrow T_n = T, \qquad T_i = V(\mathcal{I}^{n+1}).$$

Definition 4.1.1. An **infinitesimal lifting problem** is a commutative square in the category of schemes of the shape

$$\begin{array}{ccc} T_0 & \longrightarrow & Y \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ T & \longrightarrow & X \end{array}$$

where $T_0 \to T$ is a square zero thickening of affine schemes. Dotted arrows making the resulting diagram commute are called its **solutions**.

We note that solutions of the lifting problem are precisely the preimages of the top arrow $(T_0 \to Y)$ under the map

$$i^*\colon \operatorname{Hom}_X(T,Y) \longrightarrow \operatorname{Hom}_X(T_0,Y) \tag{4.1.1}$$

The following lemma establishes a close relationship between infinitesimal lifting problems and derivations.

Lemma 4.1.2. *Let $A \to B$ and $f\colon B \to R$ be a map of rings, $I \subseteq R$ an ideal with $I^2 = 0$, and $\delta\colon B \to I$ an A-linear map. The following are equivalent:*

(a) the map $f + \delta\colon B \to R$ is an A-algebra homomorphism;

(b) δ is an A-linear derivation of B into the B-module I.

Proof. For $b, b' \in R$ we write

$$(f(b) + \delta(b))(f(b') + \delta(b')) = f(b)f(b') + (f(b)\delta(b') + f(b')\delta(b)) + \underbrace{\delta(b)\delta(b')}_{0},$$

which is equal to $f(bb') + \delta(bb')$ if and only if δ satisfies the Leibniz rule. □

Corollary 4.1.3. *Given an infinitesimal lifting problem*

$$\begin{CD} T_0 @>{u_0}>> Y \\ @V{i}VV @VV{f}V \\ T @>>> X, \end{CD}$$

there is a natural action of $\mathrm{Hom}(u_0^*\Omega^1_{Y/X}, \mathcal{I})$ *on its set of solutions. If this set is non-empty, this action is free and transitive.*

Hopefully we now have enough motivation to digest the following definition.

Definition 4.1.4. Let $f\colon Y \to X$ be a morphism of schemes.

(a) We say that f is **formally étale** (resp. **formally smooth**, **formally unramified**) if for every ring R, every ideal $I \subseteq R$ with $I^2 = 0$, every lifting problem

$$\begin{CD} T_0 = \mathrm{Spec}(R/I) @>>> Y \\ @VVV @VV{f}V \\ T = \mathrm{Spec}(R) @>>> X, \end{CD}$$

has a unique solution (resp. at least one solution, at most one solution). In other words, for every map $T = \mathrm{Spec}(R) \to X$, the map

$$\mathrm{Hom}_X(T, Y) \longrightarrow \mathrm{Hom}_X(T_0, Y)$$

is bijective (resp. surjective, resp. injective).

(b) We say that f is **étale** (resp. **smooth**, **unramified**) if f is formally étale (resp. smooth, unramified) and locally of finite presentation (resp. locally of finite presentation, locally of finite type).

Example 4.1.5. (a) Every closed immersion is unramified.[12]

[12] This justifies why we did not impose local finite presentation on unramified morphisms in the above definition: not every closed immersion is finitely presented!

(b) Every open immersion is étale.

(c) The map $\mathbb{A}^n_X \to X$ is smooth.

(d) Let $A = k[x]$ and let $n \geq 1$ be an integer invertible in k. Set

$$B = k[x^{1/n}] = (A[T]/(T^n - x))_x.$$

Then $A \to B$ is étale.

Generalizing the above example, we have the following notion.

Definition 4.1.6. Let A be a ring. An A-algebra B is called **standard étale** if there exist a monic polynomial $f \in A[T]$ and an element $g \in A[X]/(f)$ such that the image of $f' = df/dT$ in $(A[T]/(f))_g$ is invertible, and an isomorphism $B \simeq (A[T]/(f))_g$ over A.

Lemma 4.1.7. *Let A be a ring and let B be a standard étale A-algebra. Then* $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *is étale.*

Proof. Note that we can replace g with f'. Suppose we are given a lifting problem

$$\begin{array}{ccc} R/I & \longleftarrow & (A[T]/(f))_g \\ \uparrow & \swarrow & \uparrow \\ R & \longleftarrow & A \end{array}$$

This translates to: an A-algebra R with $I \subseteq R$ of square zero and an element $\bar{t} \in R/I$ such that $f(\bar{t}) = 0$ and $f'(\bar{t}) \in (R/I)^\times$, which we wish to uniquely lift to $t \in R$ with $f(t) = 0$ and $f'(t) \in R^\times$. Note that the latter condition is automatic: since I is nilpotent, an element of R is invertible if and only if its image in R/I is. Let t_0 be any lifting of $\bar{t}$, so that $f(t_0) \in I$, and we need to show that there exists a unique $\delta \in I$ such that $f(t_0 + \delta) = 0$. But, since $\delta^2 = 0$, we have the "Taylor expansion"

$$f(t_0 + \delta) = f(t_0) + \delta f'(t_0),$$

and $\delta = -f(t_0)/f'(t_0)$ is the unique solution. □

Lemma 4.1.8. *Each of the classes of morphisms: étale, smooth, and unramified, is*

(a) closed under composition,

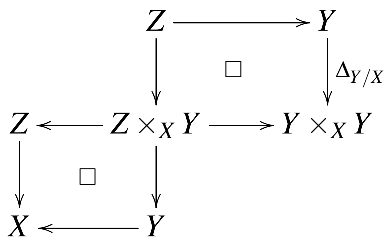
(b) closed under pull-back,

(c) local on source and target.

Moreover, unramified and étale satisfy cancellation:

(d) If $Y \to X$ and $Z \to X$ are unramified (resp. étale), then every map $Z \to Y$ over X is unramified (resp. étale).

Proof. Parts (a) and (b) are formal and left as an exercise, and so is (c) for étale and unramified morphisms. Assertion (c) for smooth morphisms will be shown later (see ■). For (d) we use the well-known trick expressing $Z \to Y$ as the vertical composition in the diagram with cartesian squares



This diagram shows that if $\mathcal{P}$ is any property of morphisms stable under composition and pullback, and $Z \to Y \to X$ are such that $Z \to X$ and $\Delta_{Y/X} \colon Y \to Y \times_X Y$ have $\mathcal{P}$, then $Z \to Y$ has $\mathcal{P}$.

We are now left with showing that the relative $\Delta_{Y/X}$ is unramified (resp. étale) if $Y \to X$ is unramified (resp. étale). This follows from the stronger assertion below. □

Lemma 4.1.9. *If $f \colon Y \to X$ is unramified then $\Delta_{Y/X} \colon Y \to Y \times_X Y$ is an open immersion.*

Proof. These assertions are local, so we may assume that $Y \to X$ is separated, so that $\Delta_{Y/X}$ is a closed immersion defined by an ideal sheaf $\mathcal{I}$. Then $\Omega^1_{Y/X} = \Delta^*_{Y/X}\mathcal{I} = \mathcal{I}/\mathcal{I}^2$. Since f is unramified, we have $\Omega^1_{Y/X} = 0$, and hence $\mathcal{I} = \mathcal{I}^2$. The ideal $\mathcal{I}$ is of finite type: indeed, this can be checked locally, and if $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ with $x_1, \dots, x_n \in B$ generating B as an A-algebra, then $I = \ker(B \otimes_A B \to B)$ is generated as an ideal by the elements $x_i \otimes 1 - 1 \otimes x_i$. By Nakayama, we have $\mathcal{I} = 0$ in a neighborhood of Y, so that $\Delta_{Y/X}$ is an open immersion. □

The following three theorems are cornerstones of the theory. We will prove them in subsequent sections.

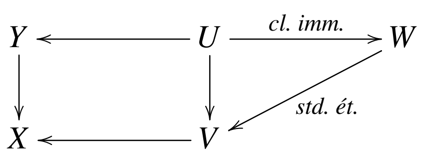
Theorem 4.1.10 (Characterizations of unramified morphisms)**.** *Let $f \colon Y \to X$ be a map locally of finite type. The following are equivalent:*

(U1) f is unramified;

(U2) $\Omega^1_{Y/X} = 0$;

(U3) the diagonal $\Delta_{Y/X} \colon Y \to Y \times_X Y$ is an open immersion;

(U4) for every $y \in Y$ there exists an open affine neighborhood $U = \operatorname{Spec}(A)$ of $f(y)$, an open neighborhood $V = \operatorname{Spec}(B)$ of y mapping to U, a standard étale A-algebra C, and a surjection of A-algebras $C \to B$. Diagrammatically:



Theorem 4.1.11 (Characterizations of étale morphisms)**.** *Let $f\colon Y \to X$ be a map locally of finite presentation. The following are equivalent:*

(E1) *f is étale;*

(E2) *f is flat and unramified;*

(E3) *for every $y \in Y$ there exists an open affine neighborhood $U = \operatorname{Spec}(A)$ of $f(y)$, an open neighborhood $V = \operatorname{Spec}(B)$ of y mapping to U, a monic polynomial $f \in A[T]$, an element $g \in A[X]$ such that the image of $f' = df/dT$ in $(A[T]/(f))_g$ is invertible, and an isomorphism of A-algebras*

$$(A[T]/(f))_g \xrightarrow{\sim} B;$$

(E4) *for every $y \in Y$ there exists an open affine neighborhood $U = \operatorname{Spec}(A)$ of $f(y)$, an open neighborhood $V = \operatorname{Spec}(B)$ of y mapping to U, and a presentation*

$$B \simeq A[T_1, \ldots, T_n]/(f_1, \ldots, f_n)$$

such that $\det(df_i/dT_j) \in B^\times$.

Theorem 4.1.12 (Characterizations of smooth morphisms)**.** *Let $f\colon Y \to X$ be a map locally of finite presentation. The following are equivalent:*

(S1) *f is smooth;*

(S2) *f is flat and $\Omega^1_{Y/X}$ is locally free of rank equal to $\dim(Y/X)$ (i.e. at a point $y \in Y$ we have $\Omega^1_{Y/X,y} \simeq \mathcal{O}^r_{Y,y}$ where $r = \dim(\mathcal{O}_{Y,y}/\mathfrak{m}_{X,f(y)}\mathcal{O}_{Y,y})$);*

(S3) *locally on Y there exists an integer $n \geq 0$ and a factorization $Y \to \mathbb{A}^n_X \to X$ where $Y \to \mathbb{A}^n_X$ is étale;*

(S4) *for every $y \in Y$ there exists an open affine neighborhood $U = \operatorname{Spec}(A)$ of $f(y)$, an open neighborhood $V = \operatorname{Spec}(B)$ of y mapping to U, and a presentation*

$$B \simeq A[T_1, \ldots, T_n]/(f_1, \ldots, f_c)$$

with $n \geq c$ such that $\det(df_i/dT_j)^c_{i,j=1} \in B^\times$.

[Grothendieck, 1971, Exp. I] [Bost et al., 2000]

4.2. Proofs

We will prove Theorems 4.1.10 and 4.1.12, leaving Theorem 4.1.12 for another occasion. We first deal with the easy implications.

(U1)⇔(U2): The ⇒ part follows directly from Corollary 4.1.3. For the converse, in order to show $\Omega^1_{Y/X}$ is zero, it is enough to show that for every map $u_0\colon T_0 = \operatorname{Spec}(R_0) \to Y$, we have $u_0^*\Omega^1_{Y/X} = 0$. Call this module I. Set R to be the trivial square zero extension of R_0 by I, that is

$R = R_0 \oplus I$ with $(r,\omega)(r',\omega') = (rr', r\omega' + r'\omega)$. We thus have $R_0 = R/I$, and since $Y \to X$ is unramified there is at most one extension $u\colon \operatorname{Spec}(R) \to Y$. But these extensions are permuted by the group $\operatorname{Hom}(I,I)$, which therefore must be zero. So $\mathrm{id}_I = 0$ and $I = 0$.

(U2)⇔(U3): This is Lemma 4.1.9.

(U4)⇔(U1): The given condition expresses f locally as the composition of a closed immersion (which is unramified) and an étale morphism (Lemma 4.1.7).

(E3)⇔(E2): Unramifiedness has just been established. To show flatness, note that the map is the composition of $\operatorname{Spec}(A[T]/(f)) \to \operatorname{Spec}(A)$ and the open immersion $\operatorname{Spec}(B) \to \operatorname{Spec}(A[T]/(f))$. But since f is monic, $A[T]/(f) \simeq \bigoplus_{i=0}^{\deg(f)-1} AT^i$ is a free A-module and hence flat.

(E3)⇔(E4): This follows from the presentation

$$(A[T]/(f))_g = A[T,S]/(f, Sg-1)$$

(E4)⇔(E1): This is a multi-variable variant of the argument of Lemma 4.1.7. Given an infinitesimal lifting problem

$$\begin{array}{ccc} R/I & \xleftarrow{(\bar t_1,\dots,\bar t_n)} & A[T_1,\dots,T_n]/(f_1,\dots,f_n) = B \\ \uparrow & \overset{(t_1,\dots,t_n)}{\swarrow} & \uparrow \\ R & \longleftarrow & A \end{array}$$

we pick any lifts $t_i' \in R$ of the images $\bar t_i$ of T_i in R/I. Setting $t_i = t_i' + \delta_i$ for the sought for solutions, we thus seek $\delta_1, \dots, \delta_n \in I$ such that

$$f_j(t_1' + \delta_1, \dots, t_n' + \delta_n) = 0 \quad j = 1, \dots, n.$$

But

$$f_j(t_1' + \delta_1, \dots, t_n' + \delta_n) = f(t_1', \dots, t_n') + \sum_{i=1}^{n} \delta_i \frac{\partial f_j}{\partial x_i}$$

and we arrive at a system of linear equations which has a unique solutions since the determinant is invertible by assumption.

With what we know so far, we are left with showing (U3)⇒(U1), (E1)⇒(E3), and (E2)⇒(E3). We note here that characterization (U2) in particular means (by Nakayama) that being unramified is a fiberwise notion. That is, a map locally of finite type $f\colon Y \to X$ is unramified if and only if for every $x \in X$, the fiber $Y_{k(x)} \to \operatorname{Spec}(k(x))$ is unramified. We are thus led to study unramified schemes over spectra of fields.

Lemma 4.2.1. *Let Y be a scheme locally of finite type over a field k. Then $Y \to \operatorname{Spec}(k)$ is unramified if and only if it is étale if and only if Y is the disjoint union of $\operatorname{Spec}(k_i)$ for a family of finite separable extensions k_i of k.*

Proof. By the primitive element theorem, a finite separable field extension is standard étale. Thus the last condition implies étale, which implies unramified. Suppose that $Y \to \mathrm{Spec}(k)$ is unramified and let $\overline{k}$ be an algebraic closure of k. At every closed point y of $Y_{\overline{k}}$ we have $\mathfrak{m}_y = \mathfrak{m}_y^2$. This implies that $Y_{\overline{k}}$ is the disjoint union of copies of $\mathrm{Spec}(\overline{k})$, and that Y is the disjoint union of $\mathrm{Spec}(k_i)$ for some finite field extensions k_i of k. Then $k_i \otimes_k \overline{k}$ is a product of copies of $\overline{k}$ and hence reduced. Suppose that $\alpha \in k_i$ with $\alpha^p \in k$, then the element

$$\beta = \alpha \otimes 1 - 1 \otimes \alpha \quad \in k_i \otimes_k \overline{k}$$

satisfies β^p, and hence $\beta = 0$, so that $\alpha \in k$, and k_i is separable over k. □

Corollary 4.2.2. *An unramified morphism is quasi-finite.*

This allows us to apply Zariski's main theorem in the following form:

Theorem 4.2.3 ([SP Lemma 00QB], [Raynaud, 1970, IV])**.** *Let $f\colon Y \to X$ be a quasi-finite morphism of schemes. Then locally on X and Y the map f factors as an open immersion $Y \to Y'$ and a finite morphism $Y' \to X$.*

We are now ready to prove the remaining assertions in Theorems 4.1.10 and 4.1.11.

(U3)⇔(U4): We try to lift the presentation from a fiber. Let $y \in Y$, $x = f(y)$, $k = k(x)$. Since $k(y)$ is finite separable over $k(x) = k$, by the primitive element theorem, we can write $k(y) = k[T]/(f)$ for f separable and irreducible. By Zariski's main theorem Theorem 4.2.3, we may assume that $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $B = B'_h$ for a finite A-algebra B' and some element $h \in B'$. We may find $t \in B'$ whose image in $k(y)$ is the generator T and which vanishes on the rest of the fiber of $\mathrm{Spec}(B') \to \mathrm{Spec}(A) = X$ over x. Let $C = A[t] \subseteq B'$ be the A-subalgebra generated by t. We claim that $g\colon Y = \mathrm{Spec}(B) \to \mathrm{Spec}(C)$ is an isomorphism in a neighborhood of y. First, this map is quasi-finite and y is the unique point in its fiber by construction. This implies that $\mathcal{O}_{\mathrm{Spec}(C),g(y)} \to \mathcal{O}_{Y,y}$ is finite. Since this map is surjective on the fiber over x, it is surjective by Nakayama. But it is injective by construction. Thus we may assume that $B = C_g$ for some $g \in C$. But $C \simeq A[T]/(f)$ with f monic, and we win.

(E1)⇔(E3): By (U4) we may assume that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ where $B = C/J$ for some standard étale A-algebra C and a finitely generated ideal $J \subseteq C$. We want to show that $J = 0$ in a neighborhood of $y \in Y$. By Nakayama it is again enough to show $J = J^2$. Now we look at the infinitesimal lifting problem

$$\begin{array}{ccc} C/J & = & B \\ \uparrow & \swarrow & \uparrow \\ C/J^2 & \longleftarrow & A \end{array}$$

A solution will split the exact sequence

$$0 \longrightarrow J/J^2 \longrightarrow C/J^2 \longrightarrow B \longrightarrow 0$$

and hence it will stay exact upon restricting to the fiber over $f(y)$. But C and B have the same fiber, so J/J^2 is zero on the fiber, and hence zero by Nakayama.

(E2)⇔(E3): We argue as above using (U4) and note that the short exact sequence

$$0 \longrightarrow J \longrightarrow C \longrightarrow B \longrightarrow 0$$

will stay exact on the fiber since B is flat over A.

4.3. Étale algebras over a field

Let K/k be a (possibly infinite) Galois extension of fields. Let $\{k_\alpha\}_{\alpha \in I}$ be the family of all intermediate extensions $k \subseteq k_\alpha \subseteq K$ such that k_α/k is finite and Galois. We treat the index set I as a partially ordered set: $\alpha \le \beta$ if $k_\alpha \subseteq k_\beta$. It is then a filtering poset: for every $\alpha, \alpha' \in I$ there Galois closure k_β of the compositum $k_\alpha \cdot k_\beta$ contains both k_α and $k_{\alpha'}$. For $\beta \ge \alpha$, every automorphism of k_β/k preserves k_α and induces an automorphism of k_α/k. We thus obtain a (surjective) homomorphism of finite groups $\mathrm{Gal}(k_\beta/k) \to \mathrm{Gal}(k_\alpha/k)$. Similarly, we obtain compatible surjections $\mathrm{Gal}(K/k) \to \mathrm{Gal}(k_\alpha/k)$. Together, these assemble into an isomorphism of groups

$$\mathrm{Gal}(K/k) \xrightarrow{\sim} \varprojlim_{\alpha \in I} \mathrm{Gal}(k_\alpha/k),$$

endowing $\mathrm{Gal}(K/k)$ with the structure of a profinite group. Explicitly, a subset U of $\mathrm{Gal}(K/k)$ is an open neighborhood of 1 if and only if there exists a k_α/k such that every element of U acts trivially on k_α.

For a profinite group Γ, we denote by Γ-**sets** the category of *finite* sets endowed with a continuous action of Γ. Explicitly, continuity of the action means that it factors through an action of Γ/H for some open normal subgroup $H \subseteq \Gamma$.

(todo: eventually the notes should contain a proper review of profinite groups and profinite completion)

Theorem 4.3.1 (Galois theory)**.** *Let k be a field and let k^{sep} be a separable closure of k. Let $\Gamma = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. Then the functor*

$$F(A) = \mathrm{Hom}_k(A, k^{\mathrm{sep}}) \colon \{\textit{finite étale } k\textit{-algebras}\} \longrightarrow \Gamma\textbf{-sets}$$

is an equivalence.

Proof. By Lemma 4.2.1, a finite dimensional k-algebra A is étale if and only if $A \otimes_k k^{\mathrm{sep}}$ is isomorphic to $\prod_{i=1}^r k^{\mathrm{sep}}$. Thus, by Galois descent (include review of Galois descent?), finite étale k-algebras correspond to k^{sep}-algebras of the form $\prod_{i=1}^r k^{\mathrm{sep}}$ endowed with a compatible Γ-action which factors through Γ/H for an open normal subgroup $H \subseteq \Gamma$. It is easy to check that such an action permutes the factors and is uniquely determined by the induced permutation. In other words, finite étale k-algebras correspond to finite sets with a continuous Γ-action. (too sketchy?) □

4.4. Galois categories

In view of Theorem 4.3.1, it makes sense to try and axiomatize categories of the form Γ-sets for a profinite group Γ.

Definition 4.4.1. Let $\mathcal{C}$ be a category.

(a) We say that $\mathcal{C}$ is a **Galois category** if there exists a profinite group Γ and an equivalence

$$\mathcal{C} \simeq \Gamma\text{-sets}.$$

(b) A functor $F\colon \mathcal{C} \to \textbf{sets}$ to the category of finite sets is a **fiber functor** if it is isomorphic to the composition of such an equivalence $\mathcal{C} \simeq \Gamma$-sets and the forgetful functor Γ-sets $\to$ **sets**.

(c) A **pointed Galois category** is a pair $(\mathcal{C}, F)$ consisting of a Galois category $\mathcal{C}$ equipped with a fiber functor F.

Proposition 4.4.2. *Let $\mathcal{C}$ be an essentially small category. The following conditions are equivalent:*

(a) $\mathcal{C}$ is a Galois category;

*(b) $\mathcal{C}$ admits a functor $F\colon \mathcal{C} \to$ **sets** to the category of finite sets such that the pair $(\mathcal{C}, F)$ satisfies the following axioms (see [SP Tag 0BMY]):*

- ***(GC1)*** *$\mathcal{C}$ has finite limits and finite colimits;*
- ***(GC2)*** *F preserves finite limits and finite colimits;*
- ***(GC3)*** *F is conservative ($F(f)$ is an isomorphism implies that f is an isomorphism);*
- ***(GC4)*** *every object of $\mathcal{C}$ is isomorphic to a finite coproduct of connected objects. (Here, we say that an object X of $\mathcal{C}$ is* **connected** *if for every monomorphism $Y \to X$, either $Y \xrightarrow{\sim} X$ or $Y \simeq \emptyset$ is the initial object.)*

Moreover, a functor $F\colon \mathcal{C} \to$ **sets** *is a fiber functor if and only if it satisfies* **(GC2)** *and* **(GC3)**.

Proof of (a)⇒(b). Without loss of generality, $\mathcal{C} = \Gamma$-sets, and we take F to be the forgetful functor. Assertions on finite limits in (GC1) and (GC2) are clear, and so is (GC3). For (GC4) we easily check that a finite Γ-set X is a connected object if and only if the action is transitive. Moreover, every finite Γ-set is the disjoint union of its orbits.

It remains to check the assertions about finite colimits, i.e. initial object (check), binary coproducts (check), and coequalizers. □

In order to prove (b) implies (a), we need to construct a profinite group Γ. The natural candidate is the automorphism group of the functor F, appropriately topologized.

Definition 4.4.3 (Topology on $\mathrm{Aut}(F)$)**.** Let $\mathcal{C}$ be an essentially small category, so that there exists a set $S \subseteq \mathrm{Ob}(\mathcal{C})$ such that every object of $\mathcal{C}$ is isomorphic to an object from S, and let $F\colon \mathcal{C} \to \mathbf{sets}$ be a functor. We have an injective map (in particular showing that $\mathrm{Aut}(F)$ is a set)

$$\mathrm{Aut}(F) \longrightarrow \prod_{X \in S} \mathrm{Aut}(F(S)).$$

We give each finite permutation group $\mathrm{Aut}(F(S))$ the discrete topology, the product the product topology, and $\mathrm{Aut}(F)$ the subspace topology.

(todo: finish this proof)

Remark 4.4.4. As we shall see when we discuss the pro-étale fundamental group later, the above definition has an important extension to functors valued in the category **Set** of all sets, with one important difference that the natural topology on the possibly infinite permutation group $\mathrm{Aut}(F(S))$ is the compact-open topology.

Definition 4.4.5. Let $(\mathcal{C}, F))$ be a pointed Galois category. We denote the group $\mathrm{Aut}(F)$, endowed with the structure of a profinite group as in REF, by $\pi_1(\mathcal{C}, F)$ and call it the **Galois group** of $(\mathcal{C}, F)$.

(add remark about fiber functors being noncanonically isomorphic)

Corollary 4.4.6. *Let $(\mathcal{C}, F))$ be a pointed Galois category. Then F induces an equivalence*

$$F\colon \mathcal{C} \xrightarrow{\sim} \pi_1(\mathcal{C}, F)\text{-}\mathbf{sets}.$$

[Grothendieck, 1971, Exp. V]

4.5. The étale fundamental group

[Grothendieck, 1971, Exp. V]

Definition 4.5.1. Let X be a scheme. We denote by $\mathbf{FÉt}_X$ the category of finite étale maps $Y \to X$.

Definition 4.5.2. A **geometric point** of a scheme X is a morphism $\bar{x} \to X$ where $\bar{x}$ is the spectrum of a separably closed field.

(convention: empty scheme is not connected)

Theorem 4.5.3. *Let X be a connected scheme. Then $\mathbf{FÉt}_X$ is a Galois category, and for every geometric point $\bar{x} \to X$, the functor $F_{\bar{x}}$ is a fiber functor.*

We shall prove this theorem in the next lecture.

Definition 4.5.4. Let X be a connected scheme and let $\bar{x} \to X$ be a geometric point. We define the **étale fundamental group** of $(X, \bar{x})$ to be the profinite group[13]

$$\pi_1(X, \bar{x}) = \pi_1(\mathbf{FÉt}_X, F_{\bar{x}}),$$

the Galois group of the pointed Galois category $(\mathbf{FÉt}_X, F_{\bar{x}})$.

(add examples, it's too dry)

[13] Sometimes denoted by $\pi_1^{\text{ét}}(X, \bar{x})$ or $\pi_1^{\text{alg}}(X, \bar{x})$.

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