LIFTABILITY OF FROBENIUS
(JOINT WITH J. WITASZEK AND M. ZDANOWICZ)

Throughout the talk we fix an algebraically closed field k of characteristic $p > 0$.

1. Liftability to(wards) characteristic zero

It often very useful to reduce a given scheme over a field K of characteristic 0 modulo p and use characteristic p techniques to study the reduction. One important example is the bend and break technique of Mori, used to show that every Fano variety contains a rational curve. Similarly, given a scheme X over our k of positive characteristic, it is a good idea to lift X to characteristic 0 and use e.g. complex analysis. So the first question is:

Q 1. Does a given X/k lift to characteristic zero?

More precisely, the question asks whether there exists a local domain R whose residue field is of characteristic zero and residue field k and a flat $\widetilde{X}/R$ with $\widetilde{X} \otimes_R k \cong X$. Unfortunately the answer is no in general (the first such example is due to Serre). We will see another example later.

As first observed by Deligne and Illusie, for many purposes the following weaker version of the question suffices:

Q 2. Does X admit a lift modulo p^2?

The precise statement is whether there exists a flat $\widetilde{X}$ over $\mathbf{Z}/p^2\mathbf{Z}$ with $\widetilde{X} \otimes_{\mathbf{Z}/p^2\mathbf{Z}} \mathbf{F}_p \cong X$. This is equivalent to the existence of a flat $\widetilde{X}$ over the ring $W_2(k)$ of Witt vectors of length two with $\widetilde{X} \otimes_{W_2(k)} k$. The reason why this question is interesting is the following:

Theorem (Deligne–Illusie [DI87])**.** Suppose that X is smooth projective, liftable modulo p^2 and that $p > \dim X$. Then

(1) *Kodaira vanishing* holds, that is
$$H^j(X, \Omega^i_{X/k} \otimes L) = 0, \quad (L \text{ ample}, \ i + j > \dim X).$$
(2) The Hodge to de Rham spectral sequence
$$E_1^{ij} = H^j(X, \Omega^i_{X/k}) \quad \Rightarrow \quad H^{i+j}(X, \Omega^\bullet_{X/k})$$
degenerates.

There are known counterexamples to (1) (Mumford, Raynaud) and hence to Q2.

2. Liftability of Frobenius

For a k-scheme X, let us denote by $F_X : X \to X$ or simply by F the Frobenius morphism of X. It is not k-linear, which is remedied by considering the relative Frobenius $F_{X/k} : X \to X'$ instead; we choose to be a bit sloppy with this distinction here. The main motivating question of this talk is:

Q 3. When does X admit a lifting to characteristic zero or modulo p^2 together with its Frobenius morphism F_X?

Thus we are asking for the existence of an $\tilde{X}$ as before together with $\tilde{F} : \tilde{X} \to \tilde{X}$ such that the square

$$\begin{array}{ccc} X & \hookrightarrow & \tilde{X} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \tilde{F}} \\ X & \hookrightarrow & \tilde{X} \end{array}$$

commutes. Again, in the modulo p^2 case such an $\tilde{F}$ automatically commutes with the Witt vector Frobenius on $W_2(k)$. We will focus on the mod p^2 case and if the answer to the above question is yes we will say that X is *F-liftable*.

Moral of the talk: F-liftability is *extremely rare*.

In the case when X is *smooth and affine*, the answer is always positive. Troubles start if X is either non-affine or singular, so we actually mean two things: that F-liftable smooth projective varieties are rare, and that F-liftable singularities are rare. In this talk I will focus on the former case. The known examples of F-liftable smooth projective varieties are:

Example 1 (Toric varieties)**.** Every toric variety is F-liftable. A toric variety over a base ring R (one can take $R = \mathbf{Z}$) is glued from spectra of monoid algebras $R[P]$ for some monoids P, and the multiplication by p maps $P \to P$ glue to give a global lift of Frobenius.

Example 2 (Ordinary abelian varieties)**.** An abelian variety A/k is F-liftable if and only if it is ordinary. In this situation *Serre–Tate theory* provides a canonical lift $\tilde{X}/W(k)$ satisfying certain functoriality properties, which in particular imply the existence of a unique Frobenius lifting $\tilde{F} : \tilde{X} \to \tilde{X}$.

Example 3 (Quotients of ordinary abelian varieties)**.** More generally, suppose that a finite group G acts freely on an ordinary abelian variety A. Then $X = A/G$ is F-liftable [MS87].

Example 4 (Toric fibrations over ordinary abelian varieties)**.** To mix the first two examples: let T be a torus over k, Y a toric variety on which T acts, A an ordinary abelian variety, $P \to A$ a principal bundle under T. Take $X = P(Y) := P \times^T Y$. This comes with a fibration $a : X \to A$, Zariski-locally trivial with toric fiber Y. One can show that X is Frobenius liftable.

The analog of Deligne–Illusie in the presence of a Frobenius lift is the following much stronger

Theorem ([BTLM97, MS87])**.** Suppose that X is smooth projective and F-liftable (modulo p^2). Then

(1) *Bott vanishing* holds:

$$H^j(X, \Omega^i_{X/k} \otimes L) = 0 \quad (L \text{ ample}, \ j > 0).$$

(2) The Hodge to de Rham spectral sequence degenerates.
(3) X is *Frobenius split* (that is, the morphism $F^* : \mathscr{O}_X \to F_*\mathscr{O}_X$ is a split injection).

3. Nonliftability of Frobenius

The above theorem makes it very easy to find non-F-liftable varieties, as it imposes the following restrictions:

3.1. **Kodaira dimension.** Since X is smooth, then F is flat, and hence by duality for the finite flat morphism we have a Frobenius-linear isomorphism

$$\mathrm{Hom}(F_*\mathscr{O}_X, \mathscr{O}_X) \cong \mathrm{Hom}(\mathscr{O}_X, F^!\mathscr{O}_X) \cong \mathrm{Hom}(\mathscr{O}_X, \omega_{X/k} \otimes F^*\omega^{-1}_{X/k}) \cong H^0(X, \omega_X^{1-p}).$$

So if X is Frobenius split, then its Kodaira dimension is ≤ 0.

3.2. **Fano case.** Another restriction comes from Bott vanishing, which is seen most easily if X is Fano (i.e., ω_X^{-1} is ample). In this case, we have

$$0 = H^1(X, \Omega^{n-1}_{X/k} \otimes \omega^{-1}_{X/k}) \cong H^1(X, T_X),$$

so X must be *rigid*.

3.3. **The Calabi–Yau case.** (i.e., $\omega_{X/k}$ numerically trivial) is also heavily restricted. It follows from the main result of [MS87] that a F-liftable smooth projective variety with $\omega_{X/k}$ numerically trivial is of the type mentioned in Example 3. In particular, $\pi_1(X)$ has to be nontrivial.

3.4. **Homogeneous spaces.** A rich family of examples which do not fall into these types are homogeneous spaces. Their F-liftability was studied in [BTLM97], where the authors showed using Bott non-vanishing that most homogeneous spaces are not F-liftable. For example, a smooth quadric hypersurface $Q \subseteq \mathbf{P}^n_k$ $(n > 3)$ is not F-liftable because

$$H^1(Q, \Omega^{n-2}_{Q/k} \otimes \mathscr{O}_Q(n-3)) \neq 0.$$

They conjectured that the only F-liftable homogeneous spaces are products of projective spaces.

3.5. **Hypersurfaces.** The above examples together allow one to show easily that a degree $d > 1$ hypersurface in $\mathbf{P}^n_k$ $(n > 1)$ is not F-liftable unless $n + d \leq 5$. Indeed, for $d = 2$ we have a smooth quadric, for $2 < d < n + 1$ we have a non-rigid Fano, for $d = n + 1$ we have a simply-connected Calabi-Yau, and for $d > n + 1$ we have a variety of positive Kodaira dimension.

3.6. **A nonliftable example.** Using the above considerations it is relatively easy to provide an example of a smooth projective variety which does not lift modulo p^2. Let $Q \subseteq \mathbf{P}^n_k$ $(n > 3)$ be a smooth quadric, and let $\pi : X \to Q \times Q$ be the blowup of $Q \times Q$ along the graph Γ_F of the Frobenius morphism. Using deformation theory, one can show that if X was liftable to $W_2(k)$, so would be the pair $(Q \times Q, \Gamma_F)$, and hence Q would be F-liftable, a contradiction.

4. The main conjecture

Given our lists of examples and restrictions, we pose the following conjecture (almost) characterizing F-liftability.

Conjecture 1 (Main Conjecture – answer to Q3)**.** Let X be a smooth projective variety over k. Suppose that X lifts modulo p^2 together with the Frobenius. Then there exists a finite étale Galois cover $f : Y \to X$ such that the Albanese morphism of Y

$$a_Y : Y \longrightarrow A = \mathrm{Alb}(Y)$$

is a Zariski-locally trivial fibration over an ordinary abelian variety A whose fibers are smooth toric varieties.

In particular, a Frobenius liftable rationally connected variety is toric.

This is an *almost* 'if an only if' statement. The subtlety comes from the fact that one cannot combine Examples 3 and 4: if a finite group G acts freely on X is in Example 4, then X/G may or may not be F-liftable.

In addition to the aforementioned list of non-examples, we are able to verify the conjecture in the following special cases.

Theorem 1. Conjecture 1 is true in the following cases:

(1) If X has a normal crossings divisor $D \sim -K_X$ which admits a lifting $\widetilde{D} \subseteq \widetilde{X}$ compatible with $\widetilde{F}$ in the sense that

$$\widetilde{F}^* \widetilde{D} = p\widetilde{D}.$$

(2) If $\dim X \leq 2$.

(3) If X is a Fano threefold.
(4) If X is homogeneous space.

Case (1) is a logarithmic version of the main result of [MS87], to which paper our project owes a great deal. Case (2) is quickly reduced to rational surfaces, in which case the question is still not completely trivial. Case (3) is tedious as it relies on the Mori–Mukai classification with about 100 families; the example which caused us the most trouble is the blow-up of $\mathbf{P}^3_k$ along the twisted cubic. The most subtle case is (4) which in particular settles the question phrased in [BTLM97].

5. Images of toric varieties

Our second motivation comes from characteristic zero. The template question is:

Q 4. Let $f : X \to Y$ be a surjective morphism of smooth projective varieties. If X belongs to a certain class of varieties, what can one say about Y?

Here are a few famous examples of such statements:

Theorem. Let $f : X \to Y$ be as in Q4, over a base field of characteristic zero.

(1) (Lazarsfeld [Laz84]) If $X \cong \mathbf{P}^n$ then $Y \cong \mathbf{P}^n$.
(2) (Demailly–Hwang–Mok–Peternell) If X is an abelian variety then Y admits as a finite étale cover the product of an abelian variety and projective spaces.
(3) (Occhetta–Wiśniewski [OW02]) If X is a toric variety *and* $\operatorname{Pic} Y \cong \mathbf{Z}$ then Y is a toric variety (i.e., $Y \cong \mathbf{P}^n$).

Occhetta and Wiśniewski conjectured that:

Conjecture 2 (Occhetta–Wiśniewski)**.** Let $f : X \to Y$ be a surjective morphism of smooth projective varieties in characteristic zero. If X is toric then so is Y.

Smoothness of Y is an important assumption: consider $Y = X/G$ for a finite group G acting on $X = \mathbf{P}^n$. Then usually Y will be singular. It is not difficult to reduce the question to the case when f is a finite morphism by considering the Stein factorization.

Our second main result links the two conjectures:

Theorem 2. If Conjecture 1 is true then Conjecture 2 is true.

Sketch of proof. As mentioned before, we can assume that f is finite; let $m = \deg f$.

We spread out $f : X \to Y$ over a finitely generated $\mathbf{Z}$-algebra R. Shrinking the base $\operatorname{Spec} R$, we can assume that R is smooth over $\mathbf{Z}$ and that for every maximal $\mathfrak{p} \subseteq R$, $X_\mathfrak{p}$ is a smooth toric variety and $Y_\mathfrak{p}$ is smooth, and that $p = \operatorname{char} R/\mathfrak{p}$ is prime to m.

The main point is to show that each $Y_\mathfrak{p}$ is Frobenius liftable. Since R is smooth over $\mathbf{Z}$, there is a $\tilde{\mathfrak{p}} \subseteq \mathfrak{p}$ such that $R/\tilde{\mathfrak{p}} = W_2(R/\mathfrak{p})$. We set $\tilde{X}_\mathfrak{p} = X_{\tilde{\mathfrak{p}}}$ and $\tilde{Y}_\mathfrak{p} = Y_{\tilde{\mathfrak{p}}}$. By deformation theory, the obstruction $\sigma_{\tilde{X}_\mathfrak{p}}$ of lifting $F_{X_\mathfrak{p}}$ to $\tilde{X}_\mathfrak{p}$ lies in $\operatorname{Ext}^1(F^*_{X_\mathfrak{p}}\Omega^1_{X_\mathfrak{p}}, \mathscr{O}_{X_\mathfrak{p}})$. These obstructions are functorial in the sense that the diagram

$$\begin{array}{ccc} F^*_{Y_\mathfrak{p}}\Omega^1_{Y_\mathfrak{p}} & \xrightarrow{\sigma_{\tilde{Y}_\mathfrak{p}}} & \mathscr{O}_{Y_\mathfrak{p}}[1] \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*[1]} \\ f_{\mathfrak{p}*}F^*_{X_\mathfrak{p}}\Omega^1_{X_\mathfrak{p}} & \xrightarrow[\sigma_{f_{\mathfrak{p}*}\tilde{X}_\mathfrak{p}}]{} & f_{\mathfrak{p}*}\mathscr{O}_{X_\mathfrak{p}}[1] \end{array}$$

commutes. Since m is prime to p, the right arrow is a split injection. Thus if the bottom arrow is zero, so is the top one. But $X_\mathfrak{p}$ is a toric variety, and hence is Frobenius liftable.

Now what is left is showing that if $Y_\mathfrak{p}$ is a toric variety for all maximal $\mathfrak{p} \subseteq R$ then Y is toric. We omit the proof. ∎

6. Homogeneous spaces

Finally, we would like to sketch the proof of Theorem 1(4). Let X be a homogeneous space, that is, a smooth projective variety over k whose automorphism group acts transitively. Suppose that X is F-liftable.

Step 1. The *Borel–Remmert decomposition*:

$$X \cong A \times \prod_{i=1}^{r} G_i/P_i$$

where A is an abelian variety, G_i are simple linear algebraic groups, and the $P_i \subseteq G_i$ are parabolic subgroup *schemes* (possibly nonreduced). One can show easily that X is F-liftable if and only if the factors in the above decomposition are. This reduces the question to the case $X = G/P$ with G simple linear algebraic and $P \subseteq G$ a possibly non-reduced parabolic.

Step 2. A theorem of Lauritzen and Mehta states that if $X = G/P$ is Frobenius split (which is weaker than F-liftable) then $X \cong G/P_{\rm red}$, so we can assume that P is reduced, so X is a 'flag variety.'

Step 3. We reduce the question to the case where $P \subseteq G$ is *maximal*, i.e. $\operatorname{Pic} X \cong \mathbf{Z}$. In this case the statement is that $X \cong \mathbf{P}^n_k$. Let $P \subseteq Q$ be a maximal (reduced) parabolic. Then one can descend the Frobenius lifting along the map

$$\pi : X = G/P \to G/Q.$$

If the assertion is true in the Picard rank one case, then $G/Q \cong \mathbf{P}^n$ for all maximal Q containing P. Using the classification of homogeneous spaces, one can show that this can hold only if P itself is maximal, or if X is the incidence variety

$$F_{1,n}(k^{n+1}) = \{(\ell, H) \in \mathbf{P}^n \times (\mathbf{P}^n)^* \,:\, \ell \subseteq H\},$$

which is easily shown to be non-F-liftable (already in [BTLM97]).

Step 4. From now on, the only assumptions we need are as follows: X is an F-liftable smooth projective Fano variety of Picard rank one whose tangent bundle T_X is nef. It is easily shown that X must by simply connected and $H^0(X, \Omega^1_{X/k}) = 0$.

Interlude: Cartier operation. The Frobenius lift $\widetilde{F}$ induces the following 'Cartier operation' on the level of $\Omega^1_{X/k}$: take a local section $\omega \in \Omega^1_{X/k}$, lift it to a section $\widetilde{\omega} \in \Omega^1_{\widetilde{X}/W_2(k)}$ and pull back by $\widetilde{F}$. The resulting form $\widetilde{F}^*\widetilde{\omega}$ reduces to 0 mod p (since F^* acts as zero on $\Omega^1_{X/k}$), and hence $\widetilde{F}^*\widetilde{\omega} = p \cdot \varphi(\omega)$ for some $\varphi(\omega) \in \Omega^1_{X/k}$ which depends only on ω and not on the choice of $\widetilde{\omega}$. If ω is multiplied by $f \in \mathscr{O}_X$, then $\varphi(\omega)$ is multiplied by f^p, and this defines a map

$$\varphi : F^*\Omega^1_{X/k} \to \Omega^1_{X/k}$$

whose adjoint $\Omega^1_{X/k} \to F_*\Omega^1_{X/k}$ has image in $F_*Z^1_{X/k}$ (closed forms) and is a splitting of the Cartier operator $F_*Z^1_{X/k} \to \Omega^1_{X/k}$. The map φ is an isomorphism over a dense open $U \subseteq X$, and its determinant

$$\det \varphi \in \operatorname{Hom}(\det F^*\Omega^1_{X/k} \to \det \Omega^1_{X/k}) \cong \operatorname{Hom}(F^*\omega_{X/k}, \omega_{X/k}) \cong H^0(X, \omega^{1-p}_{X/k})$$

is the map corresponding to the Frobenius splitting induced by $\widetilde{F}$.

We regard φ as a p-linear endomorphism of the cotangent bundle T^*X. Let $Y = (T^*X)^\varphi \subseteq T^*X$ be its fixed point locus. Then the projection $\pi : Y \to X$ is étale and its restriction to U is finite of degree $p^{\dim X}$.

Thus if $U = X$ then we are done: either X is not simply connected or it admits a global one-form. Similarly if some component of Y (except for the zero section $T^*_X X \subseteq Y$) is finite over X.

Step 5. We will try to produce such a component of Y by looking at its restriction to rational curves $f : \mathbf{P}^1 \to X$. Since T_X is nef, we have

$$f^*\Omega^1_{X/k} \cong \mathcal{O}^{\oplus r} \oplus \bigoplus \mathcal{O}(a_i)$$

with $r \geq 0$ and $a_i < 0$ (the curve f is *free*). It is easy to see that $f^*\varphi$ preserves the $\mathcal{O}^{\oplus r}$ factor, and hence that $Y \times_X \mathbf{P}^1 \to \mathbf{P}^1$ has p^r sections. If we find enough curves with $r > 0$ covering X then we might be able to 'glue' these sections to a finite étale cover of X.

Step 6. Let us call a rational curve $f : \mathbf{P}^1 \to X$ *very free* if f^*T_X is ample (i.e., $r = 0$). We would like to show that if X is not a projective space then we can find a covering family of rational curves which are not very free. The following celebrated result is due to Mori:

Theorem 6.1. *Let X be a smooth projective Fano variety. Pick a general point $x \in X$ and assume that every rational curve through x is very free. Then $X \simeq \mathbf{P}^n$. (In particular, if T_X is ample then $X \cong \mathbf{P}^n$).*

Step 7. The final step is a bit involved, but the key idea is to construct (under the assumption that X is not a projective space) a family of non-very free rational curves in X parametrized by a proper scheme M,

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ {\scriptstyle\pi}\downarrow & & \\ M & & \end{array}$$

such that $f : C \to X$ is proper and smooth, and such that over a big open subset $M^\circ \subseteq M$, the corresponding rational curves intersect the open subset $U \subseteq X$.

A version of the argument in Step 5 can be used to construct a component of $Y \times_X C$ which is finite over $C^\circ = \pi^{-1}(M^\circ)$ and then using Zariski–Nagata purity one can show the existence of a suitable component of $Y \times_X C$ finite over C. One can then 'push down' this component to Y.

References

[BTLM97] Anders Buch, Jesper F. Thomsen, Niels Lauritzen, and Vikram Mehta, *The Frobenius morphism on a toric variety*, Tohoku Math. J. (2) **49** (1997), no. 3, 355–366. MR 1464183

[DI87] Pierre Deligne and Luc Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. **89** (1987), no. 2, 247–270. MR 894379

[Laz84] Robert Lazarsfeld, *Some applications of the theory of positive vector bundles*, Complete intersections (Acireale, 1983), Lecture Notes in Math., vol. 1092, Springer, Berlin, 1984, pp. 29–61. MR 775876

[MS87] V. B. Mehta and V. Srinivas, *Varieties in positive characteristic with trivial tangent bundle*, Compositio Math. **64** (1987), no. 2, 191–212, With an appendix by Srinivas and M. V. Nori. MR 916481

[OW02] Gianluca Occhetta and Jarosław A. Wiśniewski, *On Euler-Jaczewski sequence and Remmert-van de Ven problem for toric varieties*, Math. Z. **241** (2002), no. 1, 35–44. MR 1930984