Regular logarithmic connections

Motivic Geometry
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I

Regular connections

(after Deligne)

Regularity in dimension one

$$K = \mathbf{C}((t)) \quad \supseteq \quad \mathbf{C}[[t]] = \mathcal{O} \qquad \circlearrowleft \quad \tau = t\frac{d}{dt}$$

$$\mathrm{MIC}(K/\mathbf{C}) = \left\{ \begin{array}{c} M \text{ fin. dim. over } K \text{ with a } \mathbf{C}\text{-linear } \nabla_\tau : M \to M \\ \text{satisfying } \nabla_\tau(fm) = \tau(f)m + f\nabla_\tau(m) \end{array} \right\}$$

Definition

$M \in \mathrm{MIC}(K/\mathbf{C})$ is *regular* if it admits a ∇_τ-stable $\mathcal{O}$-lattice $\overline{M} \subseteq M$.

Examples.

1. "$t^{-\lambda}$" $= (K, 1 \mapsto \lambda)$, $\lambda \in \mathbf{C}$ is regular, "$e^{1/t}$" $= (K, 1 \mapsto \frac{1}{t})$ is not.
2. (K^n, ∇_τ) cyclic corresponding to a DE

$$\left(\tau^n + a_{n-1}(t)\tau^{n-1} + \cdots + a_0(t)\right)u = 0, \quad a_i(t) \in K$$

is regular iff all $a_i(t) \in \mathcal{O}$. *(N.B. Every $M \in \mathrm{MIC}(K/\mathbf{C})$ is cyclic.)*

Residue and monodromy

$M \in \mathrm{MIC}(K/\mathbf{C})$ regular, $\overline{M} \subseteq M$ a ∇_τ-stable $\mathcal{O}$-lattice

$$\nabla_\tau : \overline{M} \to \overline{M} \quad \rightsquigarrow \quad \overset{\textit{residue map}}{\rho : \overline{M}_0 \to \overline{M}_0, \quad \overline{M}_0 := \overline{M}/t\overline{M}}$$

Its eigenvalues* are the *exponents* of $\overline{M}$.

Theorem (Canonical extension)

For $M \in \mathrm{MIC}(K/\mathbf{C})$ regular, there is a unique $\overline{M} = \overline{M}_{\mathrm{can}}$ with exponents in

$$\{0 \le \mathrm{Re}(z) < 1\}.$$

If M is obtained by base change from a meromorphic $\mathcal{M} \in \mathrm{MIC}_{\mathrm{mero}}(\Delta^*)$ on the punctured disc Δ^*, then the monodromy of $\mathcal{M}^\nabla$ is conjugate to

$$\exp(-2\pi i \rho_{\mathrm{can}}).$$

Regularity in higher dimension

$X/\mathbf{C}$ smooth scheme

$$\mathrm{MIC}(X/\mathbf{C}) = \left\{E \in \mathrm{Coh}(X),\ \nabla : E \to E \otimes \Omega^1_X \text{ integrable conn.}\right\}$$

Definition

$E \in \mathrm{MIC}(X/\mathbf{C})$ is *regular* (at infinity) if for every formal punctured disc

$$s\colon \operatorname{Spec} \mathbf{C}((t)) \to X,$$

the induced connection $s^*E \in \mathrm{MIC}(\mathbf{C}((t))/\mathbf{C})$ is regular.

If $\overline{X}$ is a smooth compactification of X with $D = \overline{X} \setminus X$ sncd, then

$$E \in \mathrm{MIC}(X/\mathbf{C}) \text{ is regular} \iff \begin{array}{c}\text{it extends to a log connection}\\ \overline{E} \to \overline{E} \otimes \Omega^1_{\overline{X}}(\log D),\end{array}$$

and $\exists!\, \overline{E} = \overline{E}_{\mathrm{can}}$ ("canonical extension") with exponents in $\{0 \leq \mathrm{Re}(z) < 1\}$.

Regularity in higher dimension

Existence Theorem

For a smooth scheme $X/\mathbf{C}$, the analytification functor

$$E \mapsto E_{\mathrm{an}} \quad : \quad \mathrm{MIC}_{\mathrm{reg}}(X/\mathbf{C}) \longrightarrow \mathrm{MIC}(X_{\mathrm{an}}/\mathbf{C}) \quad \simeq \quad \mathrm{LocSys}_{\mathbf{C}}(X_{\mathrm{an}})$$

is an equivalence.

Comparison Theorem

For $E \in \mathrm{MIC}_{\mathrm{reg}}(X/\mathbf{C})$ we have

$$H^*_{\mathrm{dR}}(X, E) \quad \simeq \quad H^*_{\mathrm{dR}}(X_{\mathrm{an}}, E_{\mathrm{an}}) \quad \simeq \quad H^*(X_{\mathrm{an}}, E^{\nabla}_{\mathrm{an}}).$$

II

Logarithmic connections

(Kato, Nakayama, Illusie, Ogus)

Log schemes

$X/\mathbf{C}$ idealized log smooth log scheme ... that is, X étale locally looks like

$$Y = \operatorname{Spec} \mathbf{C}[P]/\Sigma, \qquad \mathcal{M}_Y \text{ induced by } P \to \mathbf{C}[P]$$

monoid   monomial ideal

Note: $\Omega^1_Y \simeq P^{\mathrm{gp}} \otimes \mathbf{C}[P]/\Sigma$ is free and spanned by $d\log p$'s.

Log strata are locally described as torus orbits ($T = \operatorname{Hom}(P, \mathbf{G}_m) \circlearrowright Y$)

Examples.

1. $\underline{X}$ smooth, $D \subseteq \underline{X}$ sncd $\rightsquigarrow$ log scheme X with $\Omega^1_X \simeq \Omega^1_{\underline{X}}(\log D)$.
2. $Z \subseteq D$ stratum $\rightsquigarrow$ induced log str. on Z with $\Omega^1_Z \simeq \Omega^1_{\underline{X}}(\log D)|_Z$.

In general, log strata of a log smooth scheme will be idealized log smooth, which allows for inductive arguments.

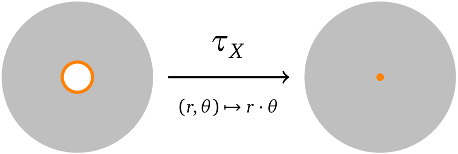
Betti realization (Kato–Nakayama)

$X/\mathbf{C}$ idealized log smooth log scheme $\rightsquigarrow$ a manifold with corners $X_{\log}$ + a proper map $\tau_X : X_{\log} \to X_{\mathrm{an}}$

Examples.

1. $X = (\mathbf{A}^1, 0)$

$\rightsquigarrow$ $X_{\log} = \overline{\mathbf{C}} = \mathbf{R}_{\geq 0} \times \mathbf{S}^1$



$\mathbf{C} = X_{\mathrm{an}}$

2. $X = \operatorname{Spec} \mathbf{C}[P]$ $\rightsquigarrow$ $X_{\log} = \operatorname{Hom}(P, \overline{\mathbf{C}}) \xrightarrow{\tau_X} \operatorname{Hom}(P, \mathbf{C}) = X_{\mathrm{an}}$
3. (X, D) snc pair $\rightsquigarrow$ $X_{\log} \to X_{\mathrm{an}}$ "oriented real blow-up"

$X_{\log}$ is the Betti realization of X in the sense that $H^*(X_{\log}, \mathbf{C}) \simeq H^*_{\mathrm{dR}}(X/\mathbf{C})$.

Logarithmic connections, Ogus' theorem

$X/\mathbf{C}$ idealized log smooth log scheme or complex analytic space

$$\mathrm{MIC}(X/\mathbf{C}) = \left\{ \begin{array}{c} E \in \mathrm{Coh}(\underline{X}) \text{ endowed with an} \\ \text{integrable connection } \nabla : E \to E \otimes \Omega^1_X \end{array} \right\}$$

Warning: E might not be locally free! E.g. $\mathcal{O}_{\overline{Z}}$ for a log stratum $Z \subseteq X$.

Theorem (Ogus' logarithmic Riemann–Hilbert correspondence)

For an idealized log smooth log analytic space X there is an equivalence

$$\mathrm{MIC}(X/\mathbf{C}) \simeq L_{\mathrm{coh}}(\mathbf{C}^{\log}_X)$$

between $\mathrm{MIC}(X/\mathbf{C})$ and certain $\overline{\mathcal{M}}^{\mathrm{gp}}_X \otimes \mathbf{C}$-graded $\mathbf{C}[\overline{\mathcal{M}}^{\mathrm{gp}}_X]$-modules on $X_{\log}$.

III

Regular logarithmic connections

Splittings of log structures

X log scheme with $\overline{\mathcal{M}}_X$ locally constant

Definition

A *splitting* of the log structure on X is a homomorphism $\varepsilon\colon \overline{\mathcal{M}}_X \to \mathcal{M}_X$ inducing a splitting of

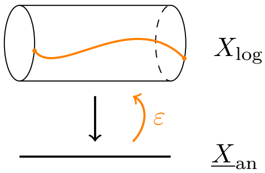
$$(\star) \qquad 1 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathcal{M}_X^{\mathrm{gp}} \longrightarrow \overline{\mathcal{M}}_X^{\mathrm{gp}} \longrightarrow 1.$$

Splittings of $(\star)$ form a torsor $\pi\colon V_X \to \underline{X}$ under torus $T_X = \mathrm{Hom}(\overline{\mathcal{M}}_X, \mathbf{G}_m)$.

$$\varepsilon_{\mathrm{univ}} := \text{universal splitting of } \mathcal{M}_X \text{ on } V_X$$

Splittings of log structures

Intuition: for $\overline{\mathcal{M}}_X$ locally constant, the log structure is a torus bundle



and a splitting ε is a section "ε": $\underline{X} \to X$.

For example, ε induces

$$“\varepsilon^*”\colon \Omega^1_X \to \Omega^1_{\underline{X}} \qquad \text{and} \qquad “\varepsilon^*”\colon \mathrm{MIC}(X/\mathbf{C}) \to \mathrm{MIC}(\underline{X}/\mathbf{C}).$$

Regular logarithmic connections. Definition

$X/\mathbf{C}$ idealized log smooth

Definition

1. Suppose that $\underline{X}$ is smooth and $\overline{\mathcal{M}}_X$ locally constant. Have $\pi : V_X \to X$ and $\varepsilon_{\text{univ}}$ on V_X. Then $E \in \mathrm{MIC}(X/\mathbf{C})$ is *regular* (at infinity) if

$$\text{“}\varepsilon^*_{\text{univ}}\text{”}(\pi^* E) \in \mathrm{MIC}(\underline{V}_X/\mathbf{C})$$

   is regular at infinity in the classical sense.

2. In general, let $\sigma : X_{\text{strat}} \to X$ be the (reduced) log stratification. Then $E \in \mathrm{MIC}(X/\mathbf{C})$ is *regular* if $\sigma^* E \in \mathrm{MIC}(X_{\text{strat}}/\mathbf{C})$ is regular in sense (1).

Write $\mathrm{MIC}_{\text{reg}}(X/\mathbf{C}) \subseteq \mathrm{MIC}(X/\mathbf{C})$ for the full subcategory.

Regular logarithmic connections. Properties

1. $\underline{X}$ proper $\Rightarrow$ $\mathrm{MIC}_{\mathrm{reg}}(X/\mathbf{C}) = \mathrm{MIC}(X/\mathbf{C})$
(not obvious since $V_{X_{\mathrm{strat}}}$ is usually not proper)

2. If $\overline{X} \supseteq X$ "good" compactification, then $\mathrm{MIC}_{\mathrm{reg}}(X/\mathbf{C})$ is the essential image of $\mathrm{MIC}(\overline{X}/\mathbf{C})$, and there is a "canonical extension."

3. Regularity is étale local, and "birational": if $U \subseteq X$ contains associated primes of E, then $E|_U$ regular $\Rightarrow$ E regular.

4. "Cut-by-curves" criterion: E is regular iff its restriction to every formal log punctured disc is regular.

Main theorems

Theorem 1 (Existence theorem)

The analytification functor

$$E \mapsto E_{\mathrm{an}} \quad : \quad \mathrm{MIC}_{\mathrm{reg}}(X/\mathbf{C}) \longrightarrow \mathrm{MIC}(X_{\mathrm{an}}/\mathbf{C}) \quad \underset{\text{Ogus}}{\overset{\tau_X^*}{\simeq}} \quad L_{\mathrm{coh}}(\mathbf{C}_X^{\log})$$

is an equivalence.

Theorem 2 (Comparison theorem)

For $E \in \mathrm{MIC}_{\mathrm{reg}}(X/\mathbf{C})$, we have

$$H^*_{\mathrm{dR}}(X, E) \quad \simeq \quad H^*_{\mathrm{dR}}(X_{\mathrm{an}}, E_{\mathrm{an}}) \quad \underset{\text{Ogus}}{\simeq} \quad H^*(X_{\log}, \tau_X^*(E_{\mathrm{an}})_0).$$

About the proofs. Good compactifications

Theorem (Toroidal compactification, version of Włodarczyk 2020)

$X/\mathbf{C}$ log smooth, i.e. a toroidal embedding $(\underline{X}, D)$.

Then étale locally X admits a *good compactification* $j : X \hookrightarrow \overline{X}$, i.e.

1. $\overline{X}$ is log smooth, i.e. a toroidal embedding $(\underline{\overline{X}}, \overline{D})$.
2. $\overline{D} = (\text{closure of } D) + (\overline{X} \setminus X)$. In particular, $\mathcal{M}_{\overline{X}} = j_* \mathcal{M}_X$.
3. Locally, $(\overline{X}, X)$ looks like

$$(\operatorname{Spec} \mathbf{C}[P][x_1, \ldots, x_r],\ \operatorname{Spec} \mathbf{C}[P][x_1^{\pm 1}, \ldots, x_r^{\pm 1}]).$$

A good compactification (especially the form (3)) allows us to perform canonical extensions from X to $\overline{X}$ and invoke GAGA on $\overline{X}$.

IV

Hodge theory on rigid spaces

(work in progress)

Rigid-analytic spaces

Setup.

- $K = \mathbf{C}(\!(t)\!) \quad \supseteq \quad \mathbf{C}[\![t]\!] = \mathcal{O}$
- X/K smooth qcqs rigid-analytic space
- $\mathcal{X}/\mathcal{O}$ a (generalized) semistable formal model of X
- $Y = \mathcal{X}_0/\mathbf{C}$ its log special fiber (is idealized log smooth)
- $Y_{\log} \to (\operatorname{Spf} \mathcal{O})_{0,\log} \simeq \mathbf{S}^1$ its Kato–Nakayama space

$$\begin{array}{cccccccc}
\text{generic fiber} & & \text{formal model} & & \text{special fiber} & & \text{Kato–Nakayama space} \\
X & \longrightarrow & \mathcal{X} & \longleftarrow & Y & \longleftarrow & Y_{\log} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Sp} \mathbf{C}(\!(t)\!) & \longrightarrow & \operatorname{Spf} \mathbf{C}[\![t]\!] & \longleftarrow & \operatorname{Spec} \mathbf{C} & \longleftarrow & \mathbf{S}^1
\end{array}$$

Slogan: the topology $Y_{\log}$ reflects the topology of $\mathcal{X}$ with its monodromy

Homotopy types of rigid-analytic spaces

Theorem (A.–Talpo)

The homotopy type of $Y_{\log}/\mathbf{S}^1$ does not depend on the choice of $\mathcal{X}$.

This gives rise to a functor

$$\Psi\colon \{\text{smooth rigid-analytic spaces over } K\} \longrightarrow (\infty\text{-category of spaces}).$$

Theorem (Stewart–Vologodsky, Berkovich)

The cohomology groups

$$H^*(\widetilde{\Psi}(X), \mathbf{Z}) := H^*(\widetilde{Y}_{\log}, \mathbf{Z}), \quad \widetilde{Y}_{\log} = Y_{\log} \times_{\mathbf{S}^1} \mathbf{R}(1)$$

carry a natural MHS.

Riemann–Hilbert on rigid-analytic spaces

$\mathrm{MIC}(X/\mathbf{C}) = \{\mathbf{C}\text{-linear int. conn. on } X\}$ (so $\tau = t\frac{d}{dt}$ acts)

$\mathrm{MIC}_{\mathrm{reg}}(X/\mathbf{C}) \subseteq \mathrm{MIC}(X/\mathbf{C})$ *regular* connections

$\mathrm{LocSys}_{\mathbf{C}}(\Psi(X)) = \mathbf{C}\text{-local systems on } Y_{\log}$ (indep. of model $\mathcal{X}$)

"Theorem" 3 (Riemann–Hilbert for rigid-analytic spaces)

Let X be a smooth qcqs rigid-analytic space over $K = \mathbf{C}((t))$. There is an equivalence of categories

$$\mathrm{RH}\colon \mathrm{MIC}_{\mathrm{reg}}(X/\mathbf{C}) \simeq \mathrm{LocSys}_{\mathbf{C}}(\Psi(X)).$$

VMHS on rigid-analytic spaces

Definition (tentative)

A *variation of mixed Hodge structure* (VMHS) on X consists of

- $V \in \mathrm{MIC}_{\mathrm{reg}}(X/\mathbf{C})$ with a Griffiths-transverse Hodge filtration $F^\bullet V$,
- $\mathcal{V} \in \mathrm{LocSys}_{\mathbf{Q}}(\Psi(X))$ with a weight filtration $W_\bullet \mathcal{V}$,
- an isomorphism $\iota : \mathrm{RH}(V) \simeq \mathcal{V}_{\mathbf{C}}$,

such that for every classical point $s\colon \operatorname{Sp} \mathbf{C}(\!(t^{1/N})\!) \to X$, the pull-back

$$s^*(V, F^\bullet, \mathcal{V}, W_\bullet, \iota)$$

is an "admissible limit VMHS."