Topics in Hodge theory

What is Hodge theory?

Hodge theory studies structures of analytic/transcendental nature that exist on the cohomology of complex algebraic varieties (*Hodge structures*). They are easiest to explain in the case of a smooth and projective complex variety X, in which case the singular cohomology groups $H^*(X, \mathbf{C})$ admit a natural and functorial *Hodge decomposition*

$$H^n(X, \mathbf{C}) \simeq \bigoplus_{p+q=n} H^q(X, \Omega_X^p).$$

Here $H^q(X, \Omega_X^p)$ is the q-th sheaf cohomology group of the sheaf Ω_X^p of holomorphic p-differential forms. This decomposition has the extra property that the action of id⊗conjugation on $H^n(X, \mathbf{C}) = H^n(X, \mathbf{Z}) \otimes \mathbf{C}$ exchanges p and q. In particular, one has the *Hodge symmetry*: $H^q(X, \Omega_X^p)$ and $H^p(X, \Omega_X^q)$ are complex vector spaces of the same dimension. A posteriori, $H^n(X, \mathbf{C})$ is of even dimension for n odd, which is a non-trivial constraint on the homotopy type of X.

We can rephrase the existence of the Hodge decomposition by saying that $H := H^n(X, \mathbf{Z})$ carries a natural *Hodge structure of weight n*, which is simply a decomposition $H \otimes \mathbf{C} = \bigoplus_{p+q=n} H^{p,q}$ with the property that id ⊗ conjugation maps $H^{p,q}$ into $H^{q,p}$. This shift of perspective allows one to separate the very interesting linear algebra of such objects from the geometry of X, consider spaces parametrizing Hodge structures (*period domains*), families (*variations of Hodge structures*) and so on.

If X and Y are "similar" algebraic varieties (for example, two elliptic curves), then their underlying manifolds are often diffeomorphic, and hence $H^*(X, \mathbf{Z}) \simeq H^*(Y, \mathbf{Z})$. However, in many cases the consideration of the Hodge structures allows one to distinguish X and Y. Statements of the kind "X is uniquely determined by $H^*(X, \mathbf{Z})$ together with its Hodge structure," often dubbed *Torelli theorems*, hold for abelian varieties, K3 surfaces, hyperkähler varieties etc.

Why learn Hodge theory?

If one is interested in complex algebraic varieties, the possibility of using Hodge theory to turn a geometric problem into linear algebra can obviously be very useful. More surprisingly, if one is interested in algebraic varieties and other geometric objects defined over arbitrary fields, like number fields, p-adic fields, or even fields of positive characteristic, one often finds structures quite similar in nature to those coming from Hodge theory, like actions of Galois groups, Frobenius operators on crystalline cohomology, the Hodge–Tate decomposition in p-adic Hodge theory, Hodge decompositions arising from liftings modulo p^2 and so on. A good working knowledge of "classical" Hodge theory can be of tremendous help in those faraway contexts.

Scope of the seminar / possible topics

1. the Hodge decomposition and its corollaries; pure Hodge structures; the Hodge conjecture
2. variations of Hodge structures; period domains
3. mixed Hodge structures
4. limits of Hodge structures, the weight-monodromy theorem
5. mixed Hodge modules

Prerequisites

A good working knowledge of algebraic geometry over $\mathbf{C}$ is necessary. We will need some basics of algebraic topology, homological algebra (including spectral sequences) and sheaf theory and cohomology, all of these topics are reviewed in Appendices A and B in [1].

Bibliography

[1] Peters, Steenbrink *Mixed Hodge structures*, Ergebnisse der Mathematik 52, Springer.

[2] Claire Voisin *Théorie de Hodge et géometrie algébrique complexe*, Cours Spécialisés 10, SMF.

Basics of étale and ℓ-adic cohomology

What is étale cohomology?

If X is a complex algebraic variety, then one can endow (the set of closed points of) X with the classical metric topology, making it a well-behaved topological space. In particular, one can apply the tools of algebraic topology (cohomology, fundamental group, higher homotopy groups) directly to that space to obtain interesting invariants. In turn, these invariants tend to carry interesting additional structure (e.g. mixed Hodge structures on cohomology) which encodes deep geometric information.

If X is a variety over some field k, one would like to similarly make use of algebraic topology. The *étale homotopy theory*, developed mostly by Grothendieck and Artin, fulfills this need by providing algebraically defined topological invariants (the *étale fundamental group*, *étale cohomology* etc.). The coefficient groups for cohomology are typically taken to be ℓ-adic numbers, and the extra structures, vaguely analogous to Hodge structures, are provided by the natural action of the Galois group of k. In particular, algebraic varieties are a good source of interesting Galois representations, which explains the relevance of ℓ-adic cohomology to number theory.

During the seminar, we will aim to give a good outline of the key ideas behind the construction of étale cohomology and its most fundamental properties.

Scope of the seminar (tentative)

1. étale and smooth morphisms, the étale site
2. sheaves on the étale site; locally constant sheaves and the fundamental group
3. construction of étale and ℓ-adic cohomology
4. comparison with singular cohomology over $\mathbf{C}$
5. finite fields: the Grothendieck–Lefschetz trace formula, Weil conjectures
6. cohomology with compact supports, Poincaré duality
7. weights, pure and mixed sheaves

Prerequisites

Schemes and sheaf cohomology, basics of homological algebra.

Bibliography

[1] Deligne *Cohomologie étale: les points de départ*, Cohomologie étale, 4–75, Lecture Notes in Math., 569

D-modules

What are D-modules?

Let X be a smooth complex algebraic variety (or a complex manifold), and denote by T_X the sheaf of derivations of $\mathscr{O}_X$ into itself (the tangent sheaf). The *sheaf of differential operators* D_X is simply the subring of the sheaf of all $\mathbf{C}$-linear endomorphisms of $\mathscr{O}_X$ generated by T_X and $\mathscr{O}_X$. A *D-module* on X is simply a sheaf of D_X-modules which is quasi-coherent as an $\mathscr{O}_X$-module.

One may think of a D-module as a "differential equation with singularities" on X. For example, if we are given an ordinary linear differential equation

$$f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1 f' + a_0 f = 0$$

with $a_i \in \mathbf{C}(x)$ being rational functions, then we can construct the corresponding differential operator

$$\delta = \partial_x^n + a_{n-1}\partial_x^{n-1} + \ldots + a_0 \in \Gamma(X, D_X)$$

on $X = \mathbf{A}^1 \setminus \{\text{poles of the } a_i\}$. We can then set $M = D_X / D_X \cdot \delta$, the quotient of D_X by the left ideal generated by δ.

Another source of interesting D-modules comes from topology. If X is a complex manifold and V is a local system (locally constant sheaf) of $\mathbf{C}$-vector spaces on X, then the vector bundle $\mathscr{V} = V \otimes \mathscr{O}_X$ carries a natural action of holomorphic differential operators D_X, making it into an $\mathscr{O}_X$-coherent D_X-module (equivalently: endowing it with a holomorphic flat connection). The *Riemann–Hilbert correspondence* is the basic fact that the association $V \mapsto \mathscr{V}$ is an equivalence of categories. If X is a smooth complex algebraic variety, then there is a much more subtle equivalence proved by Deligne:

$$(\text{complex local systems on } X_{\text{an}}) \simeq (\mathscr{O}_X\text{-coherent } D_X\text{-modules regular at infinity}).$$

If a D-module on the right is a differential equation, the corresponding local system is the sheaf of local holomorphic solutions of that equation.

To have a good theory (e.g. to allow push-forwards along closed immersions $X \hookrightarrow Y$), one needs to allow D-modules with singularities (i.e. drop the $\mathscr{O}_X$-coherence condition). In that case, local systems have to be replaced with *perverse sheaves*. As proved by Kashiwara and Mebkhout, one then has a Riemann–Hilbert correspondence

$$(\text{perverse sheaves on } X_{\text{an}}) \simeq (\text{regular holonomic } D\text{-modules on } X).$$

One of the main goals of the seminar could be to understand the above statement.

Prerequisites

Some basic algebraic and differential geometry and algebraic topology.

Scope of the seminar (tentative)

1. D_X, basic properties of D-modules, the Weyl algebra and the Fourier transform
2. vector bundles with connection and local systems (complex-analytic Riemann–Hilbert)
3. meromorphic vector bundles on a punctured disc, Fuchs' theory of regular singularities
4. Deligne's Riemann–Hilbert correspondence and canonical extensions
5. singular support and holonomic D_X-modules
6. perverse sheaves and the Riemann–Hilbert correspondence for regular holonomic D-modules
7. applications to representation theory

Bibliography

[1] Borel et al. *Algebraic D-modules*, Perspectives in Mathematics, 2.

[2] Deligne *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics 163.

[3] Hotta, Takeuchi, Tanisaki *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, 236.

Rigid geometry

What is rigid geometry?

An idea which pervades much of characteristic 0 algebraic geometry (and also geometry in positive characteristic) is the Lefschetz principle:

Lefschetz principle: *Let F be a field of characteristic* 0*. Then, 'geometric properties' of varieties over F hold true if and only if they hold true for varieties over* $\mathbf{C}$*.*

While this can be made precise, one should treat this more as a philosophical idea. In particular, if one can reduce a question Q about varieties over any field F of characteristic 0 (e.g. $\mathbf{Q}$) to $\mathbf{C}$ one can then employ the extra structure that varieties over $\mathbf{C}$ possess to try and answer Q. Namely, varieties X over $\mathbf{C}$ possess an *analytification* $X^{\text{an}} = X(\mathbf{C})$ which is (at least in the case when X is smooth) a complex manifold. Assuming that one can study X effectively by studying X^{an} (which is the purvue of Serre's *GAGA theorems*) this opens one up to using many analytic/topological techniques not available over your original field F. For example, one can apply Hodge theory. Even more simply one can cover your space X^{an} by **disks** which allows one to compute the cohomology of any local system using Cech cohomology.

Of course, this principle, while kind to questions of a purely geometric nature, is uncaring about arithmetic. For example, if $F = \mathbf{Q}$ and X is a variety over F then one might be interested in studying the space $X_{\overline{\mathbf{Q}}}$ with its action by $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ (for example this action allows one to, in good situations, compute the number of points that X might have over a finite field $\mathbf{F}_q$). The passage from $F = \mathbf{Q}$ to $\mathbf{C}$ via the Lefschetz principle does not at all care about this arithmetic—it treats varieties X and Y over $\mathbf{Q}$ identically if they are isomorphic over $\overline{\mathbf{Q}}$.

It would be nice to be able to have a theory for varieties over certain fields with the benefits that the Lefschetz principle affords us (topological and analytic techniques) in a way that does not entirely destroy the interesting. Namely, if F is $\mathbf{Q}_p$ (or a finite extension thereof) then $\overline{F}$ itself has an interesting notion of topology and analysis and therefore one might imagine that $X(\overline{F})$ may be studyable by analytic or topological techniques similar to what happens over $\mathbf{C}$ (e.g. maybe there are p-adic disks or a p-adic version of Hodge theory). But, this time, there is a natural action of $\text{Gal}(\overline{F}/F)$ on $X(\overline{F})$ and so the arithmetic of the situation is not destroyed. The theory of rigid geometry seeks to make such a desire reality.

Why learn rigid geometry?

Rigid geometry has taken a central role in arithmetic geometry and number theory in the last few decades. For example, the majority of Peter Scholze's ouvre has used the theory of rigid geometry. Namely, it's used to

- p-adic Hodge theory, an arithmetic version of Hodge theory for varieties over $\mathbf{Q}_p$, using much more intuitive analytic techniques (notably the existence of a p-adic de Rham resolution of the constant sheaf).
- Understand explicit relationships between geometry over highly ramified extensions of $\mathbf{Q}_p$ (which are thus characteristic 0!) and geometry over characteristic p fields—this is his notion of perfectoid spaces and tilting. This theory has been used to spectacular effect (e.g. to solve the so-called *weight-monodromy conjecture* in certain cases).
- Propose a program, toether with L. Fragues, about how to understand the local Langlands correspondence for p-adic fields in a way similar to the work of the Lafforgues on Langlands over function fields.

While these are very visible examples, all three of these bullets have general analogues.

- Rigid geometry has been a key component to studying the theory of p-adic Galois representations of $\text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ (e.g. the theory of étale (φ, Γ)-modules and p-adic differential equations used to prove the so-called *p-adic monodromy theorem* by work of Berger and Kedlaya).

- There is a deep interplay between rigid geometry in characteristic 0 and (classical) algebraic geometry in characteristic p that benefits sides (e.g. the solution to Abhyankar's conjecture on the fundamental groups of curves in positive characteristic uses rigid geometry).
- Rigid geometry has been pivotal in many of the advances in number theory in the last decade: from Kisin's work on Galois deformation rings to Bellaïche and Chenevier's work on the Bloch-Kato conjecture.

Most importantly:

It's fun!

If you like thinking topologically/analytic but also enjoy thinking arithmetically (e.g. in terms of number fields or finite fields) then you will enjoy thinking about why the disk is not simply connected (it has Artin-Schreier covers!) or how the analytic description of elliptic curves $\mathbf{C}^\times/q^{\mathbf{Z}}$ makes sense even in a p-adic setting.

Scope of the seminar (tentative)

(1) Study the basic theory of rigid spaces over a non-archimedean field.
(2) Study formal schemes and their relationship to rigid spaces.
(3) Study the p-adic uniformization of elliptic curves.
(4) Study modular curves and the interplay between their rigid geometry in characteristic 0 and their algebraic geometry in characteristic p.

Prerequisites

A good working knowledge of algebraic geometry over general fields (and over DVRs) is necessary. A good understanding of the p-adic numbers, their extensions, and other basic properties will be necessary as well. No advanced number theory (e.g. Galois representations or class field theory) is a prerequisite.

Bibliography

[1] Y. Tian *Introduction to rigid geometry* (See JAGS webpage)

[2] B. Conrad *Several approaches to non-archimedean geometry*

[3] R. Huber *A generalization of formal schemes and rigid analytic varieties*