Étale cohomology : an outline

Motivating questions:

① $X/\mathbb{C}$ algebraic variety
Can we study the topology of $X(\mathbb{C})$ algebraically?
(e.g. define the Betti numbers $b_i(X) = \dim_{\mathbb{Q}} H^i(X(\mathbb{C}), \mathbb{Q})$)

② $X/\mathbb{F}_q$ variety over a finite field
~~How~~ What is the cardinality of $X(\mathbb{F}_q)$ $(X(\mathbb{F}_{q^m}))$?

<u>Miracle</u>: ① & ② are closely related!

<u>Example</u>. $X = \mathbb{P}^N$

① $b_i(X(\mathbb{C})) = \begin{cases} 1 & i \text{ even } \leq 2N \\ 0 & \text{otherwise} \end{cases}$

② $\# X(\mathbb{F}_q) = 1 + q + q^2 + \dots + q^N$

(Similarly for some other spaces, e.g. Grassmannians, or TV.
but for general varieties not clear what to say)

To formulate the precise relationship (part of the Weil conjectures), we need the <u>Hasse–Weil zeta function</u>

$$Z(X,t) = \exp\Big(\sum_{n\geq 1} \frac{\# X(\mathbb{F}_{q^n})}{n} t^n\Big) \in \mathbb{Q}[\![t]\!]$$

<u>Example</u> $Z(\mathbb{P}^N_{\mathbb{F}_q}, t) = \prod_{i=0}^{N} \exp\Big(\sum \underbrace{\frac{(q^i t)^n}{n}}_{-\log(1-q^i t)}\Big)$

$$= \prod_{i=0}^{N} \frac{1}{1-q^i t}$$

<u>Theorem</u> (Weil conjectures; Dwork, Artin–Grothendieck, Deligne).
Let $X/\mathbb{F}_q$ smooth + projective. Then

$$Z(X,t) = \frac{P_1(t) P_3(t) \dots P_{2\dim X-1}(t)}{P_0(t) P_2(t) \dots P_{2\dim X}(t)},$$

where $P_i(t) = \prod_{j=1}^{\beta_i} (1 - \alpha_{ij} t) \in \mathbb{Z}[\![t]\!]$

2)

Moreover (1) $|\alpha_{ij}| = q^{i/2}$ ("Riemann hypothesis")

(2) if $\mathcal{X}/R$ is a smooth lifting of X over a dvr R with residue field $\mathbb{F}_q$ and fraction field $K \subseteq \mathbb{C}$, then

$$\beta_i := \deg P_i = b_i := \dim_{\mathbb{Q}} H^i_{sing}(\mathcal{X}(\mathbb{C}), \mathbb{Q})$$

(3) functional equation (~ Poincaré duality)

$$Z\left(X, \frac{1}{q^{\dim X} t}\right) = \pm\, q^{\dim X \cdot \chi(X)/2}\, t^{\chi(X)} Z(X, t),$$

where $\chi(X) = \sum (-1)^i \beta_i = \sum (-1)^i b_i = (\Delta . \Delta)$.

(Why "Riemann hypothesis"?

X scheme of finite type over $\mathbb{Z}$

$$\rightsquigarrow \quad \zeta_X(s) := \prod_{\substack{x \in |X| \\ (\text{closed pt})}} \frac{1}{1 - (\# k(x))^{-s}} \quad \zeta\text{-fn}$$

Then (1) $X = \operatorname{Spec} \mathbb{Z} \Rightarrow \zeta_X =$ Riemann ζ-fn

(2) X over $\mathbb{F}_q \Rightarrow \zeta_X(s) = Z(X, q^{-s})$

(3) $|\alpha_{ij}| = q^{i/2}$ means the poles & zeros of $\zeta_X(s)$ lie on some vertical lines)

<u>How are the conjectures proved</u>?

- construct well-behaved cohomology theory (<u>ℓ-adic cohomology</u> $H^i(\bar{X}, \mathbb{Q}_\ell)$, $\ell \in k^\times$ aux. prime)

.. a "base change theorem" will show for $X/\mathbb{F}_q$, $\bar{X} = X \otimes \bar{\mathbb{F}}_q$

$$H^i(\bar{X}, \mathbb{Q}_\ell) \simeq H^i(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}_\ell) \quad \text{in situation (2)}$$

... a "comparison theorem" will show for $X/\mathbb{C}$

$$H^i(X, \mathbb{Q}_\ell) \simeq H^i_{sing}(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_\ell$$

.... a "Lefschetz fixed point / trace formula" will show (for $X/\mathbb{F}_q$ again):

$$(\Delta) \quad \# X(\mathbb{F}_{q^n}) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(F^n \mid H^i(\bar{X}, \mathbb{Q}_\ell))$$

where $F=$ Frobenius ($(x_0 : \dots : x_N) \mapsto (x_0^q : \dots : x_N^q)$, $\operatorname{Fix}(F^n) = X(\mathbb{F}_{q^n})$)

10

3) ..... the "Riemann hypothesis" is equivalent to: fix $\iota: \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$

(*) the eigenvalues α_{ij} of F on $H^i(\overline{X}, \mathbb{Q}_\ell)$ satisfy $\alpha_{ij} \in \overline{\mathbb{Z}}$ (algebraic integers)
$$|\iota \alpha_{ij}| = q^{i/2}.$$
+?

(this is due to Deligne & uses lots of other properties of ℓ-adic cohomology).

<u>Note.</u> Plug in the trace formula (Δ) into the def of $Z(X,t)$:

$$Z(X,t) = \exp\Big(\sum_{n\geq 1} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\Big)$$
$$= \exp\Big(\sum_{n\geq 1}\Big(\sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(F^n \mid H^i(\overline{X}, \mathbb{Q}_\ell)\Big)\frac{t^n}{n}$$
$$= \prod_{i=0}^{2\dim X} \exp\Big(\sum_{n\geq 1} \operatorname{Tr}(F^n \mid H^i(\overline{X}, \mathbb{Q}_\ell))\frac{t^n}{n}\Big)^{(-1)^i}$$
$$= \prod_{i=0}^{2\dim X}\Big(\exp\Big(\sum_{n\geq 1}\sum_{j=1}^{b_i} \alpha_{ij}^n \frac{t^n}{n}\Big)\Big)^{(-1)^i}$$
$$= \prod_{i=0}^{2\dim X}\prod_{j=1}^{b_i}\Big(\exp\Big(\sum_{n\geq 1}\frac{(\alpha_{ij}t)^n}{n}\Big)\Big)^{(-1)^i}$$
$$= \prod_{i=0}^{2\dim X}\Big(\underbrace{\prod_{j=1}^{b_i}(1-\alpha_{ij}t)}_{\text{this is } P_i(t)}\Big)^{(-1)^{i+1}}$$

Applications.

① know $\#X(\mathbb{F}_{q^n})$ for fin. many n dep. on b_i $\Rightarrow$ know for all

② Hasse inequality
$$|\#C(\mathbb{F}_q) - (1+q)| < 2g\sqrt{q}$$

<u>How is $H^i(X, \mathbb{Q}_\ell)$ defined?</u> Say $X/k = \overline{k}$ smooth.

- a map $Y \to X$ is "étale" if it is a "(formal) local iso" i.e. $\forall_{y \in Y(k)}$ $\hat{\mathcal{O}}_{X, f(y)} \xrightarrow{\sim} \hat{\mathcal{O}}_{Y,y}$

(ex: $X = \mathbb{G}_m := \mathbb{A}^1 \setminus \{0\}$, $Y = X$, $f(x) = x^n$, $n \in k^\times$)

•• an "étale sheaf" is a functor
$$\mathcal{F}: \acute{E}t_{/X}^{op} = \{\text{étale maps } Y \to X\}^{op} \longrightarrow \text{Sets (or Ab)}$$
which satisfies a sheaf condition for all surjective $Z \to Y$.

With Ab, we get an abelian category with enough inj. obj.

$$\rightsquigarrow \quad H^i(X_{\acute{e}t}, \mathcal{F}) = \left(\begin{array}{c} i\text{-th derived functor of} \\ \mathcal{F} \longmapsto \mathcal{F}(X) \end{array} \right)$$

$$\ldots \quad H^i(X, \mathbb{Q}_\ell) = \Big(\varprojlim_n H^i(X_{\bar{e}t}, \underbrace{\mathbb{Z}/\ell^n}_{\text{constant ét. sheaf}}) \Big) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Outline of the seminar.

(1. Intro)

2. Étale maps

3–4. Étale fundamental group $\pi_1^{\acute{e}t}(X)$

5–6. Étale topology, cohomology of étale sheaves

7. Examples: curves & abelian varieties (...)

8. Artin comparison ("comparison thm" above)
 + Artin vanishing (coh. of affines)

9. (Sm) Proper base change ("base change thm" above)
 + nearby cycles

10. Properties of ℓ-adic cohomology (Poincaré duality &c.)

11. ζ-Lefschetz trace formula (point counting / $\mathbb{F}_q$)
 — Weil conjectures except for RH

12. Riemann hypothesis ?

Literature:
- Deligne's intro in SGA $4\frac{1}{2}$ & Weil I
- Milne's book
- SGA 1, 4_{III}, and 5
- Kleiman's article
