Junior Algebraic Geometry Seminar, Fall 2019

Basics of étale and ℓ-adic cohomology

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What is étale cohomology?

If X is a complex algebraic variety, then one can endow (the set of closed points of) X with the classical metric topology, making it a well-behaved topological space. In particular, one can apply the tools of algebraic topology (cohomology, fundamental group, higher homotopy groups) directly to that space to obtain interesting invariants. In turn, these invariants tend to carry interesting additional structure (e.g. mixed Hodge structures on cohomology) which encodes deep geometric information.

If X is a variety over some field k, one would like to similarly make use of algebraic topology. The *étale homotopy theory*, developed mostly by Grothendieck and Artin, fulfills this need by providing algebraically defined topological invariants (the *étale fundamental group*, *étale cohomology* etc.). The coefficient groups for cohomology are typically taken to be ℓ-adic numbers, and the extra structures, vaguely analogous to Hodge structures, are provided by the natural action of the Galois group of k. In particular, algebraic varieties are a good source of interesting Galois representations, which explains the relevance of ℓ-adic cohomology to number theory.

During the seminar, we will aim to give a good outline of the key ideas behind the construction of étale cohomology and its most fundamental properties.

Talks

The outlines below are only guidelines. The general philosophy we would like to follow is to try to first explain the corresponding notion in the context of algebraic topology or geometry over the complex numbers before giving the étale analog (e.g. the relationship between the fundamental group and coverings, the Lefschetz fixed point formula, Poincaré duality etc.).

1. Introduction. Overview of the topic, with the Weil conjectures as one of the main motivations.

2. Morphisms. Start with a motivation for étale morphisms via local homeomorphisms over $\mathbf{C}$. Define flat, unramified, étale, and smooth morphisms and give equivalent conditions, including infinitesimal lifting criteria. Draw many pictures illustrating the concepts (e.g. unramifiedness being equivalent to open diagonal, the infinitesimal lifting criterion, etc.) Discuss henselian rings and étale algebras over a field (or more generally over henselian local rings). (Chapter I of [2] or Exposés I–IV of [3]).

3. Étale fundamental group (I). For motivation, discuss the relationship between π_1 and coverings spaces in algebraic topology. Define the étale fundamental group $\pi_1^{\mathrm{et}}(X)$ in terms of the fiber functor and compute it for $X = \operatorname{Spec} k$. Explain how every étale covering is dominated by a Galois covering and give the more down-to-earth definition of $\pi_1^{\mathrm{et}}(X)$ as $\varprojlim \mathrm{Gal}(Y/X)$. Discuss basic properties of $\pi_1^{\mathrm{et}}(X)$ like functoriality, when the induced map is injective (surjective, bijective) and dependence on base point. Discuss pro-finite groups and pro-finite completion in general. (Chapter I §5 of [2], Exposé V of [3]).

4. Étale fundamental group (II). Give examples: compute $\pi_1^{\mathrm{et}}(X)$ of X when X is $\mathbf{P}^1_k$ (for k an arbitrary field) and when X is $\mathbf{A}^1_k$ $\mathbf{G}_{m,k}$ (where k is a characteristic 0 field), an elliptic curve, an abelian variety, the nodal cubic, and cuspidal cubic. Discuss $\pi_1(\mathbf{A}^1_k)$ when k has positive characteristic. State and sketch the proof of the comparison theorem over $\mathbf{C}$ (Grauert–Remmert/Riemann's existence theorem). Discuss Grothendieck's specialization theorem. (Chapter I §5 of [2], Exposés X–XII of [3]).

5. Étale topology (I). Discuss sites (Grothendieck topologies) and sheaves on them. Define the étale site and cohomology of sheaves. Relate $H^1(X_{\mathrm{et}}, \mathscr{F})$ to torsors under $\mathscr{F}$ and locally constant sheaves to $\pi_1^{\mathrm{et}}(X)$. Discuss faithfully flat descent and Hilbert's theorem 90. Define the Kummer sequence, which will be one of our basic tools. (§I–II of [1], Chapter II of [2], Exposés VI and VIII of [3]).

6. Étale topology (II). Constructible sheaves. define Čech cohomology and discuss how they are related (e.g. state spectral sequence and agreement for H^1). Define and discuss torsors and their relationship with Čech cohomology. In particular, explain the explicitly meaning of $\check{H}^1(\mathcal{U}, \mathcal{F})$ where $\mathcal{U}$ is a cover of X. Give lots of examples (e.g $\mathbf{Z}/l^n, \mu_n, \mathbf{G}_m$), discuss the relationship between forms of an object S and Aut(S)-torsors, define and discuss principal homogenous spaces. (§2 and §4 of Chapter III of [2]).

7. Cohomology of curves and abelian varieties. Compute the étale cohomology of a smooth curve or an abelian variety with $\mathbf{Z}/\ell^n$ coefficients over an algebraically closed field. Compare with singular cohomology over $\mathbf{C}$. (E.g. §III of [1])

8. Artin comparison and Artin vanishing. Sketch one of the proofs of Artin's comparison theorem over $\mathbf{C}$: via $K(\pi, 1)$ neighborhoods (Exposé XI of [4]) or via devissage (Exposé XVI of [4]). Sketch the proof of Artin's vanishing theorem for affine varieties and deduce the hyperplane Lefschetz theorem.

9. Proper base change and nearby cycles. For motivation, discuss proper base change in topology, Ehresmann's theorem (a proper submersion of manifolds is a fiber bundle), and (optionally) the Milnor fibration. Translate these to the étale setting, stating the proper base change and smooth proper base change theorems. Discuss applications e.g. the unramifiedness of the cohomology of a smooth proper variety over $\mathbf{Q}_p$ with good reduction. Mention the Néron–Ogg–Shafarevich theorem for abelian varieties). Define nearby cycles and algebraic Milnor fibers. (§IV–V of [1], [7])

10. Properties of ℓ-adic cohomology. Define the ℓ-adic cohomology groups $H^*(X, \mathbf{Q}_\ell)$ for a scheme of finite type over a field k. Discuss their fundamental properties: finite dimensionality, Galois action, Künneth formula, Poincaré duality. Application of Poincaré duality to reduction tricks (surjective morphisms induce injective maps on cohomology). Motivate and discuss compactly supported cohomology. Compute some examples. ([6])

11. Cohomology over finite fields. Lefschetz trace formula and the Weil conjectures. Discuss the Grothendieck–Lefschetz trace formula expressing the number of points of a variety over a finite field using ℓ-adic cohomology. Deduce the Weil conjectures except for the Riemann hypothesis. Discuss some concrete examples ζ-functions and explicitly show the equality between the 'cohomological ζ-function' and the 'point counting ζ-function' (e.g. $\mathbb{P}^n$, the nodal/cuspidal cubic, etc.). Give some applications of rationality. (Chapters 2 and 3 of [1], and Chapters 1,2, and 4 of [7])

12. Purity and the Riemann hypothesis. Discuss the notion of purity for sheaves and explain Deligne's theorem on the purity of the higher pushforwards for smooth smooth proper maps. Explain why this gives a solution to the Riemann hypothesis part of the Weil conjectures. Explain, in general, the Lang-Weil estimates. Explain what needs to be modified for non-smooth or non-proper varieties. (Chapter 2 and 3 of [1], and Chapters 1,2, and 4 of [7], and [8])

Prerequisites

Schemes and sheaf cohomology, basics of homological algebra.

Bibliography

[1] P. Deligne *Cohomologie étale: les points de départ*, Cohomologie étale (SGA $4\frac{1}{2}$), Springer LNM 569

[2] P. Milne *Étale Cohomology*, Princeton Univ. Press

[3] A. Grothendieck et al. *Revetements Etales et Groupe Fondamental* (SGA1)

[4] A. Grothendieck et al. *Theorie des Topos et Cohomologie Etale des Schemas* (SGA4 part III)

[5] P. Deligne and N. Katz *Groupes de Monodromie en Geometrie Algebrique* (SGA7)

[6] S. Kleiman *Algebraic cycles and Weil conjectures*, in Dix exposés sur la cohomologie des schémas

[7] M. Mustata, *Zeta functions in algebraic geometry*, http://www-personal.umich.edu/~mmustata/zeta_book.pdf

[8] http://virtualmath1.stanford.edu/~conrad/Weil2seminar/Notes/L18.pdf