Flatness, blowups, and valuations

Study seminar, Spring 2023

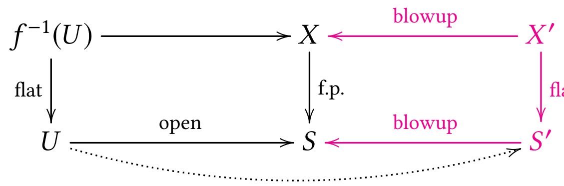
https://achinger.impan.pl/flatness.html

The flattening theorem

The basic example of a non-flat morphism of schemes is that of a blowup. The deep theorem of Gruson and Raynaud, dubbed **flattening by blowup**, implies that in a certain sense, blowing up is the only source of non-flatness. Below is the precise statement:

Theorem (Gruson–Raynaud)**.** *Let S be a quasi-compact and quasi-separated scheme (e.g. noetherian or affine), let $X \longrightarrow S$ be a morphism of finite presentation, and let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. Let $U \subseteq S$ be a quasi-compact open subset such that $\mathscr{F}|_{f^{-1}(U)}$ is flat over U. Then there exists a blowup $S' \longrightarrow S$ at a finitely presented closed subscheme $Z \subseteq S$ disjoint from U such that the strict transform of $\mathscr{F}$ is finitely presented over the structure sheaf $\mathscr{O}_{X'}$ of the strict transform X' of X in S' and flat over S'.*

As a special case, taking $\mathscr{F} = \mathscr{O}_X$, we obtain flatness of the strict transform $X' \longrightarrow S'$:



This fundamental result has many consequences. We say that $X \longrightarrow S$ is a *U-modification* if it is proper, $f^{-1}(U) \simeq U$, and $f^{-1}(U)$ is schematically dense in X. The above theorem implies that for every U-modification $X \longrightarrow S$ there exists a blowup $S' \longrightarrow S$ away from U which dominates $X \longrightarrow S$ (and $S' \longrightarrow X$ is a blowup too). In particular, while not every proper birational map $X \longrightarrow S$ is a blowup, there always exists a blowup $X' \longrightarrow X$ such that $X' \longrightarrow X \longrightarrow S$ is a blowup. Since blowups are projective morphisms, we obtain **Chow's lemma** as a corollary. A more advanced application is a modern proof of **Nagata's compactification theorem**.

In a different direction, the results of Raynaud and Gruson surrounding the flattening theorem can be used to prove the following striking result: let A be an integral domain and let B be a finite type and flat A-algebra. Then B is **finitely presented** over A.

Many foundational results in **non-Archimedean geometry** rely on the flattening theorem. It can be used to prove that a morphism of rigid spaces is flat if and only if it admits a flat formal model, which implies that flat maps of rigid spaces are open. Conversely, ideas from non-Archimedean geometry led to a **new proof of the flattening theorem** (due independently to Temkin, Fujiwara–Kato, and Guignard). The key idea is that it is easy to check flatness of a module over a valuation ring (it is equivalent to torsion-freeness), and the inverse limit $\mathrm{RZ}(U/S)$ of all U-modifications of S, called the **relative Riemann–Zariski** space by Temkin, can be identified with a certain space of valuations on U. In fact, the above theorem about a flat map $A \longrightarrow B$ in the case when A is a valuation ring has been first proved by Nagata.

Goals of the seminar

The first goal of the seminar is to understand the original proof of the flattening theorem (using the notion of "devissage") and its basic corollaries. The original paper [1] is complicated, so we will take our time. Fortunately, there is a survey by Raynaud [2], and the whole content has been included in the Stacks Project [8] in the mysterious chapter titled *More on Flatness*.

After some applications (Nagata's compactification theorem), we will study modifications and their link with spaces of valuations, based on Temkin's beautiful paper [4]. This will enable us to understand the second proof of the flattening theorem based e.g. on [5].

The final goal is to study the applications of the flattening theorem and related ideas in non-Archimedean geometry, particularly [6], [7], [9].

The theory is not set in stone, and the seminar may result in some new research ideas, particularly the potential simplification of the proofs of some foundational results in non-Archimedean geometry.

Talks

The outlines below are only guidelines. The content is likely to change as we get a better understanding of our goals.

1. Overview. Give a quick presentation of the flattening theorem, its corollaries and applications. Sketch the original method of proof.

2. Proof of Flattening Theorem (I): warm-up. Prove the theorem in the projective case [2, §4.2] using the Quot scheme. Define Fitting ideals and explain how they enter the proof.

3. Proof of Flattening Theorem (II): devissage. Explain the notion of a relative devissage of a module and the content of the first three sections of [1]. See also [10].

4. Proof of Flattening Theorem (II): the proof. Sections 4–5 of [1] except 5.7.

5. Applications of the flattening theorem. See [1, §5.7] and [8, Tag 081Q].

6. Spaces of valuations and relative Riemann–Zariski spaces. [4]

7. More on relative Riemann–Zariski spaces. [3], [4]

8. Nagata's compactification theorem. [3], [4], [8, Tag 0F3T]

9. Second proof of the Flattening Theorem. [5]

10. Flattening in non-Archimedean geometry (I). [6,10]

11. Flattening in non-Archimedean geometry (II). [6,7,10]

12. The reduced fiber theorem. [9]

Prerequisites

The first half of the seminar will require basic knowledge of scheme theory and commutative algebra. In the second half, some knowledge of non-Archimedean geometry will be helpful but not required. In fact, the whole seminar can serve as a good, if slightly unorthodox, introduction to non-Archimedean geometry for algebraic geometers.

Coordinates

Thursdays 10:15–11:45, room 1780 at MIMUW

Dates: March 2, 9, 16, 23, 30, April 13 (optional – right after Easter), April 20 (Piotr will be absent), April 27, May 11, 18, 25, June 1, 15.

Bibliography

[1] M. Raynaud, L. Gruson *Critères de platitude et de projectivité. Techniques de "platification" d'un module*, Invent. Math. 13 (1971), 1–89.

[2] M. Raynaud *Flat modules in algebraic geometry*, Compositio Math. 24 (1972), 11–31.

[3] K. Fujiwara, F. Kato *Rigid geometry and applications*, Moduli spaces and arithmetic geometry, 327–386, Adv. Stud. Pure Math., 45, Math. Soc. Japan, Tokyo, 2006.

[4] M. Temkin *Relative Riemann-Zariski spaces*, Israel J. Math. 185 (2011), 1–42.

[5] Q. Guignard *A new proof of Raynaud-Gruson's flattening theorem*, Int. Math. Res. Not. IMRN 2021, no. 9, 6932–6966.

[6] S. Bosch, W. Lütkebohmert *Formal and rigid geometry. II. Flattening techniques*, Math. Ann. 296 (1993), no. 3, 403–429.

[7] A. Ducros *Dévisser, découper, éclater et aplatir les espaces de Berkovich*, Compos. Math. 157 (2021), no. 2, 236–302.

[8] The Stacks Project Authors *More on Flatness*, Stacks Project chapter 057M.

[9] S. Bosch, W. Lütkebohmert, M. Raynaud *Formal and rigid geometry. IV. The reduced fibre theorem*, Invent. Math. 119 (1995), no. 2, 361–398.

[10] A. Abbes *Éléments de géométrie rigide. Volume I.* Progress in Mathematics, 286. Birkhäuser / Springer Basel AG, Basel, 2010.