Raynaud-Gruson seminar 16/03/2023

Dévissage and criterion of flatness

1. Motivations
2. Dévissages
3. Criterion of flatness

1. Motivations:

A ring ; B A-alg of f.p ; M B-mod of f.p

Goal: Study the A-flatness of M.

Idea: Reduce the study of "A-flat modules" to "free modules over some smooth A-alg."

Reason: M is A-flat $\iff$ locally for étale topology on $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$, the A-mod M has a finite composition sequence $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ with quotient M_i/M_{i-1} free over a smooth A-alg B_i.

How it works:
- $(X,x) \xrightarrow{f} (S,s)$ morphism of schemes loc. of finite presentation.
- $\mathcal{M}$ $\mathcal{O}_X$-mod of finite presentation
  $\dim_x(\mathcal{M}/S) = n$

We can find (étale locally):
- a decomposition "dévissage" of $\mathcal{M}$:

$$D_1 : (X,x) \xrightarrow[\text{finite}]{g} (T_1,t_1) \xrightarrow{\text{smooth}} (S,s)$$

with morph of T_1-modules:

$$\mathcal{L}_1 \xrightarrow{\alpha_1} \mathcal{N}_1 = g_* \mathcal{M} \longrightarrow \mathrm{coker}(\alpha_1) = \mathcal{P}_1$$

$\mathcal{L}_1$ free of f.t.

satisfying $\dim(\mathcal{P}_1 \otimes k(s)) < n$

and: $\mathcal{M}_x$ is S-flat $\iff$ α_1 injective and $\mathcal{P}_{1,t_1}$ S-flat

- find a similar dévissage D_2 over $(T_1,t_1) \to (S,s)$ of $\mathcal{P}_1$, then $D_3, \dots, D_r$ such that $\mathcal{P}_r = 0$.

In that case: $\mathcal{M}_x$ is S-flat $\iff$ $\alpha_i : \mathcal{L}_i \to \mathcal{N}_i$ is injective, $\forall i$

Here M is only assume of f.t.

② Dévissage: **Remark**: if $X \xrightarrow{g} \overset{\tau}{\overset{\cap}{T}}$ if finite,

$\mathcal{N} = g_* \mathcal{M}$ is a $\mathcal{O}_T$-mod of f.t

so $\mathcal{N} \otimes k(\tau)$ is a finite dim. $k(\tau)$ v.s.

Any sequence of $\Gamma(\mathcal{N}, T)$ giving a basis of $\mathcal{N} \otimes k(\tau)$ induces:

$\alpha: \mathcal{O}_T^r \to \mathcal{N}$ with $\alpha \otimes k(\tau)$ bijection.

définition:

A S-dévissage of M in dimension n at the point x is a factorization: $X \supseteq X' \xrightarrow{g} T \longrightarrow S$

with $X' \subseteq X$ closed imm. and $\mathrm{Supp}(M) \subseteq X'$

and a morph. of T-modules: $\alpha: \mathcal{L} \longrightarrow \mathcal{N} = g_* M'$ with:

(i) $\dim(X' \otimes k(s)) \leq n$

(ii) $X' \xrightarrow{g} T$ finite and $T \to S$ smooth affine with geometrically int. fibres of dim n. τ = gen point of $T \otimes k(s)$

(iii) $\mathcal{L}$ free of f.t and $\alpha \otimes k(\tau)$ bijective.

(iv) x is the only point of $g^{-1}(k(t))$

Remark 1: $\mathcal{P} = \mathrm{coker}(\alpha)$, $\mathcal{P}$ is of f.t / T

$\mathcal{P}_\tau \otimes k(\tau) = 0 \overset{\text{(Nak)}}{\Longrightarrow} \mathcal{P}_\tau = 0$ (α_τ surj.)

τ gen point of $T \otimes k(s) \Longrightarrow \dim(\mathcal{P} \otimes k(s)) < n$

Remark 2: $(X' \to T \text{ finite} + (iv)) \Longrightarrow M'_{x'}$ and $\mathcal{N}_t$ represent the same $\mathcal{O}_{S,s}$-module

définition: • A S-dévissage of M at x in dim $n_1 > n_2 > \dots > n_r \geq 0$ is the data, for $i = 1, \dots, r$, of S-dévissages:

$D_i = (X_i \to T_i,\ \mathcal{L}_i \xrightarrow{\alpha_i} \mathcal{N}_i \longrightarrow \mathcal{P}_i)$ of dim n_i

of the T_{i-1}-mod $\mathcal{P}_{i-1}$ at t_{i-1}

$(\mathcal{P}_0 = M,\ T_0 = X,\ t_0 = x)$

• $(D_i)_{i=1\dots r}$ is total if $\mathcal{P}_r = 0$

Terminology: Is it in dimensions between n and n' if:

$n \geq n_1 > \dots > n_r \geq n' > \dim(\mathcal{P}_r \otimes k(s))$

Thm: (Existence, locally for the étale topology, of a total dévissage of dim between n and $n-r$)

Assume $M_x \neq 0$, $\begin{cases} n = \dim_x(M \otimes k(s)) \\ r = \mathrm{codepth}_{\mathcal{O}_{X,x}\otimes k(s)}(M_x \otimes k(s)) \end{cases}$

There exists a commut. diag.
$$\begin{tikzcd} (X',x') \arrow[r] \arrow[d] & (S',s') \arrow[d] \\ (X,x) \arrow[r] & (S,s) \end{tikzcd}$$
with columns affine étale neigh.
and total S'-dévissage of M' at x' of dim. between n and $n-r$.

Remark: if A local ring, M A-mod of finite type:
$\mathrm{depth}_A(M)$ = length of maximal reg. sequences for M in m_A ($\leq \dim_A(M)$)
$= \min\{i \geq 0,\ \mathrm{Ext}^i(k_A, M) \neq 0\}$
$\mathrm{codepth}_A(M) = \dim_A(M) - \mathrm{depth}_A(M) \geq 0$
- If A local regular: $\mathrm{codepth}_A(M) = 0 \iff M$ proj $\iff M$ free (Auslander-Buchsbaum formula)

proof: 1) Can assume X and S affine and $\dim_x(X/S) = n$ (easy)
2) Can assume a factorizat°:
$$(X,x) \xrightarrow[x \text{ only point above } t]{\text{finite}} (T,t) \xrightarrow[\text{geo. int. fibres of dim } n]{\text{smooth}} (S,s)$$
(difficult, consequence of Zariski Main Theorem)

3) $r = \mathrm{codepth}_{\mathcal{O}_{T,t}\otimes k(s)}(N_t \otimes k(s))$

Lemma: EGA 0_{IV} 16.1.9, 16.4.8. A, B local rings, $A \to B$ local
M B-mod with M_A of f.t: $\begin{cases} \mathrm{depth}_A(M_A) = \mathrm{depth}_B(M_B) \\ \dim_A(M_A) = \dim_B(M_B) \end{cases}$

Induction on r: • If $r = 0$: $N_t \otimes k(s)$ free
$\exists\ \alpha: L = \mathcal{O}_T^k \longrightarrow N \longrightarrow P$
with $P_t \otimes k(s) = 0 \Rightarrow P_t = 0$ (Nakayama)
- $\exists\ U$ open neigh of t in T, $P_U = 0$
- $T \to S$ is open because smooth $\exists\ S' \underset{\text{open}}{\subseteq} S$ containing image of U

Replacing X and T by their inverse image over S':
we get a total dévissage of length 1: $(X \to T, \mathcal{L} \xrightarrow{\alpha} \mathcal{N} \to 0)$

- If $r > 0$: $\exists$ a S-dévissage $(X \to T, \mathcal{L} \xrightarrow{\alpha} \mathcal{N} \to \mathcal{P})$ of M at x of dim n.

Also: $\alpha_r \otimes k(s)$ inj $\Rightarrow$ $\alpha_t \otimes k(s)$ inj

But $\alpha_t \otimes k(s)$ non bijective, otherwise $\mathcal{N}_t \otimes k(s)$ free and $r = 0$.

$$0 \to \mathcal{L}_t \otimes k(s) \to \mathcal{N}_t \otimes k(s) \to \mathcal{P}_t \otimes k(s) \to 0$$

So $\forall i \geq 0$, $\mathrm{Ext}^i(k(t), \mathcal{N}_t \otimes k(s)) = \mathrm{Ext}^i(k(t), \mathcal{P}_t \otimes k(s))$

So $\mathrm{depth}_{\mathcal{O}_{T,t} \otimes k(s)}(\mathcal{P}_t \otimes k(s)) = \mathrm{depth}_{\mathcal{O}_{T,t} \otimes k(s)}(\mathcal{N}_t \otimes k(s))$

But $\dim(\mathcal{P} \otimes k(s)) < n \Rightarrow \mathrm{codepth}(\mathcal{P}_t \otimes k(s)) < r$

By induction, $\exists$ a total dévissage of $\mathcal{P}$ of dim. between $(n-1)$ and $(n-r)$.

③ _ Criterion of flatness: M is assumed to be of f.p

There exists commu. diagram:

$$\begin{tikzcd} (X', x') \arrow[r] \arrow[d] & (S', s') \arrow[d] \\ (X, x) \arrow[r] & (S, s) \end{tikzcd}$$

with columns affine elementary étale neigh.

and a dévissage $(Y \to T, \mathcal{L} \xrightarrow{\alpha} \mathcal{N} \to \mathcal{P})$ of M' at x' of dim $n = \dim_x(M/S)$. Let t = image of x' and τ gen. point of $T \otimes k(s)$.

Thm 2: TFAE:

(i) M_x is S-flat
(i') $\mathcal{N}_t$ is S'-flat
(ii) α_r bijective and $\mathcal{P}_t$ $\mathcal{O}_{T,t}$-flat
(ii') α_t injective and $\mathcal{P}_t$ $\mathcal{O}_{T,t}$-flat
(ii'') $\exists$ open neigh $s' \in U' \subseteq S'$ with α S'-univ. injective above U' and $\mathcal{P}_t$ $\mathcal{O}_{T,t}$-flat

Remark: M_1 and M_2 $\mathcal{O}_X$-modules, $u: M_1 \to M_2$

u is S univ. injective $\Leftrightarrow$ $\forall S' \to S$, $u \otimes S'$, $M_1 \otimes S' \to M_2 \otimes S'$ is injective

If M_1 S-flat, u S-univ. injective $\Leftrightarrow$ M_2/M_1 S-flat

Corollary: Let $D_i = (Y_i \to T_i, \mathcal{L}_i \xrightarrow{\alpha_i} \mathcal{N}_i \to \mathcal{P}_i)_{1 \le i \le r}$ a S'-dévissage of length r :

(i) M_x is S-flat
(ii) α_i bij at τ_i and $\mathcal{P}_r$ is S'-flat at t_r
(iii) α_i inj at t_i and $\mathcal{P}_r$ is S'-flat at t_r
(iv) $\exists$ open neighb. U' of s' in S', α_i is (S-univ. inj over U' and $\mathcal{P}_r$ is S'-flat at t_r.

Remark:
- If the dévissage is total ($\mathcal{P}_r = 0$, then $\mathcal{P}_r$ is S'-flat at t_r).
- If M_x is S'-flat, up to shrinking S' one can obtain all α_i to be injective.
- Let $A = \Gamma(S, \mathcal{O}_S)$; $M' = \Gamma(X', \mathcal{M}')$
$M'_i = \mathrm{Ker}(M' \to \Gamma(T_i, \mathcal{P}_i))$

Then : $0 = M'_0 \subsetneq M'_1 \subsetneq \dots \subsetneq M'_r = M'$
$M'_i / M'_{i-1} = \Gamma(T_i, \mathcal{L}_i)$

For the proof of Thm 2, we will need this lemma.

Lemma 2: If $(T,t) \to (S,s)$ is a smooth morphism, with $T \otimes k(s)$ integral of gen. point τ, and $\alpha : \mathcal{L} \to \mathcal{N}$ morph. of T-modules ($\mathcal{L}$ free of f.t) with $\alpha_\tau \otimes k(s)$ injective.

Then (i) α_τ injective
$\Updownarrow$
(ii) α_t is S-universally injective

proof: One can assume S noetherian.

(ii) $\Rightarrow$ (i) : easy because α_τ surjective by Nakayama.

(i) $\Rightarrow$ (ii) : The following diagram is commutative : (of $\mathcal{O}_{T,t}$-modules)

$$\begin{array}{ccc} \mathcal{L}_t & \longrightarrow & \mathcal{M}_t \\ \downarrow & & \downarrow \\ \mathcal{L}_\tau & \underset{\hookrightarrow}{\xrightarrow{\alpha_\tau}} & \mathcal{M}_\tau \end{array}$$

so it is enough to prove $\mathcal{L}_t \to \mathcal{L}_\tau$ is S-univ. injective, and $\mathcal{L}$ is free, so it is enough to prove $\mathcal{O}_{T,t} \to \mathcal{O}_{T,\tau}$ is S-univ. injective.

Let $F = \{$elements of $\mathcal{O}_{T,t}$ that are nonzero divisors in $\mathcal{O}_{T,t} \otimes k(s)\}$.

Then : $\mathcal{O}_{T,\tau} = \mathcal{O}_{T,t}[F^{-1}] = \operatorname{colim}_{a \in F} \mathcal{O}_{T,t}[a^{-1}]$

6)

(Result: (EGA 0_{III} 10.2.4) If $A \to B$ local morphism of local noeth. rings, M, N two B-modules of f.t., with N A-flat, $u: M \to N$ B-morphism:
(i) $M \otimes_A k_A \xrightarrow{u \otimes_A id} N \otimes_A k_A$ injective
$\Updownarrow$
(ii) u injective and $\mathrm{coker}(u)$ is A-flat.)

Apply this with $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{T,t} = M$, $N = \mathcal{O}_{T,t}[a^{-1}]$ $(a \in F)$
and $u: \mathcal{O}_{T,t} \xrightarrow{\times a} \mathcal{O}_{T,t} \longrightarrow \mathcal{O}_{T,t}[a^{-1}]$

We get exact sequence: $0 \longrightarrow \mathcal{O}_{T,t} \longrightarrow \mathcal{O}_{T,t}[a^{-1}] \longrightarrow H_a \longrightarrow 0$
with H_a $\mathcal{O}_{S,s}$-flat.

Passing to the colimit: $0 \to \mathcal{O}_{T,t} \to \mathcal{O}_{T,\tau} \to H \to 0$
with H $\mathcal{O}_{S,s}$-flat.

But $(\mathcal{O}_{T,t}$ and H $\mathcal{O}_{S,s}$-flat$)$ $\Longleftrightarrow$ $\mathcal{O}_{T,t} \to \mathcal{O}_{T,\tau}$ S-univ. injective $\boxtimes$

$\rightsquigarrow$ proof of Thm 2: • (i) $\Leftrightarrow$ (i') because $\mathcal{M}_{x'}$ and $\mathcal{N}_t$ are the same $\mathcal{O}_{S,s}$-module and $\mathcal{M}_{x'}$ $\mathcal{O}_{X',x'}$-flat $\Leftrightarrow$ $\mathcal{M}_x$ $\mathcal{O}_{X,x}$-flat because $(X',x') \to (X,x)$ is an étale neigh.

• (ii) $\Leftrightarrow$ (ii') from the above lemma.

• (ii'') $\Rightarrow$ (ii') is obvious

• (ii') $\Rightarrow$ (i'): we have an exact seq. of $\mathcal{O}_{S',s'}$-modules:
$$0 \longrightarrow \mathcal{L}_t \longrightarrow \mathcal{N}_t \longrightarrow \mathcal{P}_t \longrightarrow 0$$
But $\mathcal{P}_t$ is $\mathcal{O}_{S',s'}$ flat and $\mathcal{L}_t$ as well (because $T \to S'$ smooth so flat)
So $\mathcal{N}_t$ $\mathcal{O}_{S',s'}$-flat.

• (i') $\Rightarrow$ (ii) Let $R = \ker(\alpha)$. There is exact seq. of $\mathcal{O}_{T,\tau}$-mod:
$$0 \longrightarrow R_\tau \longrightarrow \mathcal{L}_\tau \longrightarrow \mathcal{N}_\tau \longrightarrow 0$$ because $\mathcal{P}_\tau = 0$ (Nakayama)

$\mathcal{N}_t$ S'-flat $\Rightarrow$ $\mathcal{N}_\tau$ S'-flat, so it remains exact after $\otimes k(s)$:
$$0 \longrightarrow R_\tau \otimes k(s) \longrightarrow \mathcal{L}_\tau \otimes k(s) \xrightarrow{\sim} \mathcal{N}_\tau \otimes k(s) \longrightarrow 0$$
So $R_\tau \otimes k(s) = 0$, but R is of f.t over T since $\mathcal{N}$ is of f.p.
So we can use Nakayama: $R_\tau = 0$, and α_τ bijective.
Using the lemma, α_t is S'-univ. inj. Since $\mathcal{L}_t$ is S'-flat, this is equivalent to $\mathcal{N}_t/\mathcal{L}_t = \mathcal{P}_t$ S'-flat.

• (ii) $\Rightarrow$ (ii''): R and $\mathcal{P}$ are of finite type over T, so by considering their support, the set $\mathcal{U}$ of T of points where α is bij is open, containing τ.
As $T \to S'$ is open (because smooth), its image $\mathcal{U}'$ is open in S', $s \in \mathcal{U}'$.
But the fibres of $T \to S'$ are integral, so by lemma 2
α is S'-univ. injective over $\mathcal{U}'$.

$\boxtimes$