THE RAYNAUD–GRUSON FLATTENING THEOREM: OVERVIEW

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Let A be a commutative ring, B a finitely presented A-algebra, and M a finitely generated B-module. We say that M is A-*flat* (or *flat over* A) if it is flat as an A-module. In [RG71], Raynaud and Gruson give a convenient characterization of A-flat modules and to use it to prove their flattening theorem.

In this talk, we shall formulate the theorem and its corollaries, and prove it in some special cases.

1. Statement of the flattening theorem

Let S be a scheme and let $\mathcal{I} \subseteq \mathcal{O}_S$ be a quasi-coherent ideal of finite type. The blowup

$$S' = \mathrm{Bl}_{\mathcal{I}}(S) := \mathrm{Proj}_S \bigoplus_{n\geq 0} \mathcal{I}^n \longrightarrow S$$

of S along $\mathcal{I}$ is the final object in the category of S-schemes $T \to S$ such that $\mathcal{I} \cdot \mathcal{O}_T$ is an invertible ideal (locally generated by a nonzerodivisor). It is projective over S and an isomorphism above $U = S - V(\mathcal{I})$. The ideal $\mathcal{I}$, or the closed subscheme $V(\mathcal{I})$ determined by it, is called the *center* of the blowup. It not uniquely determined by the morphism $S' \to S$, see Warning 1.1 below. Given a scheme S and an open subscheme $U \subseteq S$, a morphism $S' \to S$ isomorphic over S to the blowup in an ideal $\mathcal{I}$ with $V(\mathcal{I}) \cap U = \varnothing$ is called a U-*admissible blowup*.

By the universal property, if $X \to S$ is any morphism, the blowup X' of X along $\mathcal{I} \cdot \mathcal{O}_X$ fits into a commutative square

$$\begin{array}{ccc} X' & \longrightarrow & S' \\ \downarrow & & \downarrow \\ X & \longrightarrow & S. \end{array}$$

Typically, this diagram is not cartesian unless $X \to S$ is flat. The induced morphism $X' \to X_{S'} = X \times_S S'$ is a closed immersion, defined by the ideal of sections of $\mathcal{O}_{X_{S'}}$ which vanish on the preimage of U, or equivalently those which are $\mathcal{I}$-torsion. In

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other words, X' is the scheme theoretic closure of X_U in $X_{S'}$. The scheme X' (or the map $X' \to X_{S'}$) is called the *strict transform* of X along $S' \to S$.

We can take strict transforms of modules as well. For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ we define its *strict transform* $\mathcal{M}'$ to be the quotient of the pull-back of $\mathcal{M}$ to X' (or, which is the same, to $X_{S'}$) by its $\mathcal{I}$-torsion submodule (or the submodule of sections which vanish on the preimage of U).

Warning 1.1 (Blowup does not know its center)**.** The blowup morphism $S' \to S$ remembers the ideal $\mathcal{I}$ only partially. For example, if $\mathcal{J}$ is an invertible ideal and $n \geq 1$, then $\mathcal{I}$ and $\mathcal{I}^n \cdot \mathcal{J}$ define the same blowup. The notion of a strict transform depends on the choice of $\mathcal{I}$ and not only on the morphism $S' \to S$. For example, if $S = \operatorname{Spec}(A)$ is affine and $\mathcal{I}$ is principal, defined by a nonzerodivisor $f \in A$, then (using the above notation) we have $X' = S' = S$ but $\mathcal{M}'$ is the quotient of $\mathcal{M}$ by the submodule of sections annihilated by a power of f. However, the above descriptions of X' and $\mathcal{M}'$ show that they are uniquely determined by $S' \to S$ together with the open subset U.

The flattening theorem asserts that for a suitable choice of $\mathcal{I}$, the sheaf $\mathcal{M}'$ will be flat over S'. More precisely, it says the following.

Theorem 1.2 (Flattening theorem, [RG71, 5.2.2])**.** *Let*

- *S be a quasi-compact and quasi-separated scheme,*
- *$f\colon X \to S$ a morphism of finite presentation,*
- *$\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module of finite type,*
- *$U \subseteq S$ a quasi-compact open subset.*

Suppose moreover that $\mathcal{M}_U = \mathcal{M}|_{f^{-1}(U)}$ is flat over U and finitely presented over $\mathcal{O}_{X_U}$.

Then, there exists a U-admissible blowup $S' \to S$ such that the strict transform $\mathcal{M}'$ is flat over S' and finitely presented over $\mathcal{O}_{X'}$, where X' is the strict transform of X.

The following diagram might be helpful in visualizing the theorem.

$$\begin{array}{ccccc} \mathcal{M}_U & & \mathcal{M}' & & \mathcal{M} \\ \vdots & & \vdots & & \vdots \\ X_U & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & S' & \longrightarrow & S \end{array}$$

Example 1.3 (Curves)**.** Suppose that S is a smooth curve, or more generally a Dedekind scheme (Noetherian and its local rings are discrete valuation rings). Then $\mathcal{M}$ is flat over S if and only if it is torsion free over $\mathcal{O}_S$. At the same time, every nonzero ideal of $\mathcal{O}_S$ is invertible, so every nontrivial blowup $S' \to S$ is an isomorphism. But if U is dense, then $S' \to S$ is the identity, and the strict transform $\mathcal{M}'$ of $\mathcal{M}$ is the quotient by its torsion submodule (see Warning 1.1), and hence is flat over S.

In essence, the flattening theorem tries to turn non-flatness into torsion, which then one can get rid of via strict transform.

Example 1.4 (Ideal of a smooth point on a surface)**.** For a less trivial example, let $S = \mathbf{A}^2_k$, let $X = S$, and let $\mathcal{M}$ be the ideal of the origin P in $\mathcal{O}_S$. Then $\mathcal{M}$ is torsion-free, free on $U = S \setminus \{P\}$, but not flat. Let $S' \to S$ be the blowup at P. The pullback $\mathcal{M}_{S'}$ can be computed as follows. Introducing coordinates x, y on $\mathbf{A}^2_k$, $\mathcal{M}$ admits a "Koszul" resolution

$$\mathcal{O}_S \xrightarrow{(y,-x)} \mathcal{O}_S^2 \xrightarrow{(x,y)} \mathcal{M} \longrightarrow 0$$

Then $\mathcal{M}_{S'}$ has a resolution

$$\mathcal{O}_{S'} \longrightarrow \mathcal{O}_{S'}^2 \longrightarrow \mathcal{M}_{S'} \longrightarrow 0$$

and on the open subset $V_x \subseteq S'$ where x divides y, say $y = tx$, we have the relation $x(te_1 - e_2) = 0$, i.e. $te_1 - e_2$ is x-torsion. After dividing by x-torsion (which here coincides with x power torsion), $\mathcal{M}'_{V_X}$ is freely generated by e_2. Similarly on the open V_y where y divides x. Looking at the map $\mathcal{M}_{S'} \to \mathcal{O}_{S'}$, pullback of $\mathcal{M} \hookrightarrow \mathcal{O}_S$, whose image is the ideal sheaf $(x, y)\mathcal{O}_{S'}$ of the exceptional divisor E, we see that $\mathcal{M}' = \mathcal{O}_{S'}(-E)$.

2. Corollaries

Recall that a morphism of schemes $f\colon X \to S$ is *schematically dominant* if the map $\mathcal{O}_S \to f_*\mathcal{O}_X$ is injective. We introduce the following terminology: let S be a scheme and let $U \subseteq S$ be an open subset. A *U-modification of S* is a factorization

$$\begin{array}{ccc} & & S' \\ & \nearrow & \downarrow \\ U & \longrightarrow & S \end{array}$$

of the inclusion $U \to S$ where $U \to S'$ is a schematically dominant open immersion and $S' \to S$ is proper. Equivalently, it is a proper morphism $S' \to S$ which is an isomorphism over U and such that (the preimage of) U is schematically dense in S'.

The notions of a U-modification and of a U-admissible blowup are closely related but logically independent. On one hand, there exist non-projective U-modifications, and projective U-modifications which are not blowups. More, even a U-modification which is a blowup might not admit a center which is disjoint from U, see Example 2.2 (this is impossible e.g. if S is regular). On the other hand, a U-admissible blowup might not be a U-modification if U is not schematically dense in S.

Example 2.1 (Line on a quadric cone)**.** Let $S = \mathrm{Spec}(k[x, y, z]/(xy - z^2))$ be the quadric cone, let $U = S - V(x, y, z)$ be the smooth locus, and let $L \subseteq S$ be the line through the origin cut out by the ideal (x, z). Then $L \cap U$ is an effective Cartier divisor on U but L itself is not Cartier. The blowup $S' \to S$ of S along L is a U-modification, although its center L intersects U, so a priori it might not be a U-admissible blowup. However, the divisor $2L$ is Cartier, with ideal $\mathcal{I}_{2L} = x \cdot \mathcal{O}_S$. Note how $\mathcal{I}_L^2$ is not equal to $\mathcal{I}_{2L}$. Indeed, if this were true, then $\mathcal{I}_L^2$ would be an invertible ideal, forcing $\mathcal{I}_L$ to be invertible as well. Instead, we have

$$\mathcal{I}_L^2 = (x^2, xz, z^2)\mathcal{O}_S = (x^2, xz, xy)\mathcal{O}_S = x(x, y, z)\mathcal{O}_S = \mathcal{I}_{2L} \cdot \mathcal{I}_P$$

where $P = V(x,y,z)$ is the vertex of the cone. Then

$$S' = \mathrm{Bl}_{\mathcal{I}_L}(S) = \mathrm{Bl}_{\mathcal{I}_L^2}(S) = \mathrm{Bl}_{\mathcal{I}_{2L}\cdot\mathcal{I}_P}(S) = \mathrm{Bl}_{\mathcal{I}_P}(S)$$

(see Warning 1.1), so blowing up L has the same as blowing up the vertex P. In particular, S' is smooth. In fact, $\mathcal{I}_P$ coincides with the first Fitting ideal $F_1(\mathcal{I}_L)$ of $\mathcal{I}_L$, see Example 3.1.

Example 2.2 (Plane on the cone over a quadric surface)**.** In the previous example, we got lucky in finding a blowup center disjoint from U, essentially because the divisor L was $\mathbf{Q}$-Cartier. We will produce an example of a blowup which is a U-modification but not a U-admissible blowup. Consider

$$S = \mathrm{Spec}(k[x,y,z,w]/(xy - zw)),$$

the cone over the quadric surface in $\mathbf{P}^3$ (isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$). Let $H = V(x,z) \simeq \mathrm{Spec}(k[y,w])$ be a plane in S, and let $S' \to S$ be the blowup along H. Again, on the open set $U = S - \{P\}$ where $P = V(x,y,z,w)$ is the vertex, the divisor H is Cartier, so $\pi\colon S' \to S$ is a U-modification. We claim that there does not exist a closed subscheme Z supported at P such that $S' \simeq \mathrm{Bl}_Z(S)$. If this were the case, then the preimage of Z in S' would be an effective Cartier divisor with support equal to $E = \pi^{-1}(P)$. However, E is not even a divisor on the threefold S'! Let us explicate S' by looking at the open subset V_x where x divides z, say $z = tx$:

$$V_x = \mathrm{Spec}\left(\frac{k[x,y,t,w]}{x(y-tw)}/(x\text{-torsion})\right) = \mathrm{Spec}(k[x,t,w]) \simeq \mathbf{A}^3.$$

The preimage of $P = V(x,y,z,w)$ in V_X is defined by $(x, tw, tx, w) = (x,w)$ and hence is the line $\mathrm{Spec}(k[t]) \simeq \mathbf{A}^1$ and cannot be the support of a Cartier divisor on $\mathbf{A}^3$.

The first corollary asserts that every U-modification is dominated by a U-admissible blowup.

Corollary 2.3 (Ubiquity of blowups, [RG71, Corollaire 5.7.12])**.** *Let S be a quasi-compact and quasi-separated scheme, let $U \subseteq S$ be a quasi-compact open subscheme, and let $S' \to S$ be a U-modification. Then, there exists a U-admissible blowup $S'' \to S$ such that the composition $S'' \to S$ is a U-admissible blowup.*

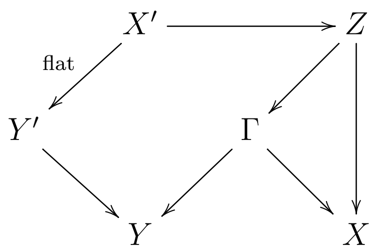
Proof. We prove this assuming that $S' \to S$ is finitely presented for simplicity. In fact, using the flattening theorem we can reduce to this case, see [Stacks Project Tag 081R]. We apply Theorem 1.2 to the morphism $S' \to S$ and the sheaf $\mathcal{O}_{S'}$, obtaining a U-admissible blowup $S'' \to S$ such that the strict transform $S''' \to S''$ of $S' \to S$ is flat:

$$\begin{array}{ccc} S''' & \xrightarrow{\text{blowup}} & S' \\ {\scriptstyle\text{flat}}\downarrow & & \downarrow \\ S'' & \xrightarrow[\text{blowup}]{} & S. \end{array}$$

It is enough to show that $S''' \to S''$ is an isomorphism. But $S''' \to S''$ is proper, flat, and an isomorphism over U (which can be assumed to be schematically dense in S''), and Lemma 2.8 below implies what we want. □

Corollary 2.4 (Chow's lemma, [RG71, Corollaire 5.7.14])**.** *Let S be a quasi-compact and quasi-separated scheme, let $X \to S$ be a separated morphism of finite type, and let $U \subseteq X$ be a quasi-compact open subset which is quasi-projective over S. Then, there exists a U-admissible blowup $X' \to X$ such that X' is quasi-projective over S.*

Proof. Fix an open immersion $U \to Y$ into a projective S-scheme Y, and let Γ be the schematic closure of U in $X \times_S Y$. We apply Corollary 2.3 to $\Gamma \to X$ (which is a projective U-modification), obtaining a U-admissible blowup $Z \to \Gamma$ such that $Z \to \Gamma \to X$ is a U-admissible blowup. Next, we apply Theorem 1.2 to the map $Z \to \Gamma \to Y$, obtaining a U-admissible blowup $Y' \to Y$ such that the strict transform $X' \to X$ of Z in Y' is flat over Y', see the diagram below.



Lemma 2.8 shows that $X' \to Y'$, being flat and an isomorphism over U, is an open immersion. Then $X' \to X$ is a U-admissible blowup and X' is quasi-projective over S, being an open subscheme of the projective S-scheme Y'. ☐

The proof used the following fundamental fact, which we shall use quite a bit, often without mention.

Lemma 2.5 (Composition of U-admissible blowups, [RG71, Lemme 5.1.4])**.** *Let S be a quasi-compact and quasi-separated scheme, let $U \subseteq S$ be a quasi-compact open subset, and let $S' \to S$ be a U-admissible blowup. Let $S'' \to S'$ be a further U-admissible blowup. Then, the composition $S'' \to S$ is a U-admissible blowup as well.*

Remark 2.6. The assertions of Corollary 2.3 and 2.4 also hold for algebraic spaces.

The final corollary presented here is less serious but serves as a good illustration.

Corollary 2.7 (Extending a vector bundle)**.** *Let S be a quasi-compact and quasi-separated scheme, let $U \subseteq S$ be a quasi-compact open subset, and let $\mathcal{M}$ be a locally free $\mathcal{O}_U$-module of finite type. Then, there exists a U-admissible blowup $S' \to S$ and a locally free $\mathcal{O}_{S'}$-module of finite type $\mathcal{M}'$ such that $\mathcal{M}'_U \simeq \mathcal{M}$.*

Instead of a proof. We do not give the proof (it can be obtained by first extending $\mathcal{M}$ to a finite type $\mathcal{O}_S$-module and then applying the flattening theorem with $X = S$). Instead, we give an argument which applies in the case when there exists a locally free $\mathcal{O}_S$-module of finite type $\mathcal{V}$ together with a surjection $\mathcal{V}_U \to \mathcal{M}$ (this should hold at least if S is quasi-projective over an affine scheme). Assume for simplicity that $\mathcal{M}$ and has constant rank r, then $\mathcal{M}$ defines a morphism

$$\alpha\colon U \longrightarrow \mathrm{Gr}_r(\mathcal{V})$$

to the Grassmannian of rank r quotients of $\mathcal{V}$, such that $\alpha^*(\mathcal{Q}) = \mathcal{M}$ where $\mathcal{Q}$ denotes the universal quotient bundle. By projectivity of $\mathrm{Gr}_r(\mathcal{V}) \to S$ and Corollary 2.3, there exists a U-admissible blowup $S' \to S$ such that α extends to a morphism

$\alpha' \colon S' \to \mathrm{Gr}_r(\mathcal{V})$, and we can take $\mathcal{M}' = (\alpha')^*(\mathcal{Q})$. In fact, the use of the full Corollary 2.3 can be avoided, see Lemma 4.2. □

The proofs above relied on the following characterization of open immersions.

Lemma 2.8 ([Stacks Project Tag 081M])**.** *Let $f \colon X \to S$ be a morphism of schemes and $U \subseteq S$ and open. If*

(1) f is separated, locally of finite type, and flat,
(2) $f^{-1}(U) \to U$ is an isomorphism, and
(3) $U \to S$ is quasi-compact and schematically dominant,

then f is an open immersion.

3. Fitting ideals, proof in the case $X = S$

Let A be a ring and let M be a finitely generated A-module. Pick a presentation

$$A^I \xrightarrow{U} A^n \longrightarrow M \longrightarrow 0$$

where I is a possibly infinite index set. Treating U as a $|I| \times n$ matrix, denote by $F_r(M)$ the ideal in A generated by the $(n-r) \times (n-r)$ minors of U (in other words, the entries of $\wedge^{n-r}(U)$). By convention, we set $F_r(M) = A$ for $r \geq n$ and $F_r(M) = 0$ for $r < 0$, so that $F_r(M) \subseteq F_{r+1}(M)$ for all r. As the notation already suggests, the ideals $F_r(M)$ are independent on the choice of a presentation of M. They are called the *Fitting ideals* of M; they are finitely generated if M is finitely presented.

It is clear that the above definition globalizes: for a quasi-coherent sheaf of finite type $\mathcal{M}$ on a scheme S, we can define the quasi-coherent ideals $F_r(\mathcal{M}) \subseteq \mathcal{O}_S$. Their formation commutes with base change $S' \to S$. We have $F_r(\mathcal{M}) = \mathcal{O}_S$ if and only if $\mathcal{M}$ is *locally* generated by r sections. In other words,

$$\mathrm{Supp}(F_r(\mathcal{M})) = \{s \in S \,:\, \dim_{k(s)}(\mathcal{M} \otimes k(s)) > r\}.$$

In particular, $\mathcal{M}$ is locally free of rank r if and only if $F_r(\mathcal{M}) = \mathcal{O}_S$ and $F_{r-1}(\mathcal{M}) = 0$.

Example 3.1 (Line on the quadric cone, revisited)**.** Let $S = \mathrm{Spec}(A)$, $A = k[x,y,z]/(xy - z^2)$ be the quadric cone as in Example 2.1, and let $\mathcal{M} = (x,z)\mathcal{O}_S$, the ideal sheaf of a line through the origin. The generators x and z satisfy the obvious "Koszul" relation $x \cdot z = z \cdot x$ and one extra relation $y \cdot x = z \cdot z$. Consequently, we have a presentation

$$A^2 \xrightarrow{\begin{bmatrix} -y & z \\ -z & x \end{bmatrix}} A^2 \longrightarrow (x,z)A \longrightarrow 0$$

from which we infer that

$$F_0(\mathcal{M}) = (xy - z^2)\mathcal{O}_S = 0, \quad F_1(\mathcal{M}) = (x,y,z)\mathcal{O}_S \quad \text{and} \quad F_2(\mathcal{M}) = \mathcal{O}_S.$$

We observe that after blowing up the only nontrivial Fitting ideal $F_1(\mathcal{M})$, which is the ideal of the origin P, the strict transform of $\mathcal{M}$ is an invertible ideal.

Example 3.2 (Plane on the cone over a quadric surface, revisited)**.** Consider the situation in Example 2.2, i.e. $S = \mathrm{Spec}(k[x,y,z,w]/(xy - zw))$, $P = V(x,y,z,w)$, and $H = V(x,z)$. A calculation entirely similar to the one above shows that the Fitting ideals of $\mathcal{M} = \mathcal{I}_H$ are equal to

$$F_0(\mathcal{M}) = (xy - zw)\mathcal{O}_S = 0, \quad F_1(\mathcal{M}) = (x,y,z,w)\mathcal{O}_S = \mathcal{I}_P \quad \text{and} \quad F_2(\mathcal{M}) = \mathcal{O}_S.$$

However, as we saw in Example 2.2, this time the blowup $S'' \to S$ of $P = V(F_1(\mathcal{M}))$ in S is *not* isomorphic to the blowup $S' \to S$ of H. For the former, the resulting threefold S'' is smooth and isomorphic to the total space of $\mathcal{O}(-1,-1)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ and exceptional divisor (preimage of P in S'') is its zero section, isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. The map $S'' \to S'$ is an isomorphism over $U = S - \{P\}$, and over P the resulting map is one of the projections $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$. If $\tilde{S}' \to S$ is the blowup of the "dual" plane $\tilde{H} = V(y,w)$, then on the special fiber of $S'' \to \tilde{S}'$ we obtain the other projection $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$. The birational transformation $S' \leftarrow S \to \tilde{S}'$ is called the "Atiyah flop."

The following simple observation is the key link between Fitting ideals and the flattening theorem.

Lemma 3.3 ([Ray72, Chapter 4, §3, Lemma 1]). *Let S be a scheme and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_S$-module of finite type. Let $r \geq 0$ be an integer such that $\mathcal{I} = F_r(\mathcal{M})$ is an invertible ideal and $\mathcal{M}$ is locally free of rank r on $U = S \setminus V(I)$. Let $\mathcal{N} \subseteq \mathcal{M}$ denote the $\mathcal{I}$-torsion submodule. Then, $\mathcal{M}' = \mathcal{M}/\mathcal{N}$ is locally free of rank r on S.*

Proof. The question being local, we may assume that $S = \operatorname{Spec}(A)$ and $\mathcal{I}$ is generated by a nonzerodivisor $f \in A$. We may fix a presentation $A^I \xrightarrow{U} A^n \to M \to 0$ and assume that the determinant $\det(u_{ij})$ (with $j \in \{r+1, \ldots, n\}$ and $i \in I_0 \subseteq I$ where $|I_0| = n - r$) equals f. By Cramer's rule (see Lemma 3.4 below) and the fact that the determinant of every minor of size $n - r$ of U is divisible by f, denoting by $e_1, \ldots, e_n$ the images of the basis vectors on A^n in M, we have

$$fe_i = f \sum_{j=1}^{r} b_{ij} e_j \quad \text{for every } i > r$$

for some $b_{ij} \in A$. This implies that the images of $e_1, \ldots, e_r$ generate $\mathcal{M}/\mathcal{N}$, and we have a short exact sequence

$$0 \longrightarrow K \longrightarrow A^r \longrightarrow M/N \longrightarrow 0$$

On the open set $U = D(f)$, the second map $A^r \to M/N$ induces an isomorphism $A[1/f]^r \to M[1/f] = (M/N)[1/f]$, and hence K is annihilated by a power of f. But f is a nonzerodivisor on A and K is a submodule of A^r, so $K = 0$, and M/N is locally free of rank r. □

Lemma 3.4 (Cramer's rule). *Let A be a ring and let $U = [u_{ij}]$ be an $s \times n$ matrix with entries in A, where $n \geq r$. Write*

$$U = \left[\frac{V}{W}\right]$$

where V is the square matrix consisting of the first s rows of U, and W is the rest. For $j \leq s < j'$ let $V_{jj'}$ be the matrix obtained by replacing the j-th row of V with the

j'-th row of U (which is the $(j'-s)$-th row of W). Treating U as a map $A^s \to A^n$, and letting $e_1, \ldots, e_n$ be the standard basis of R^n, we have for every $j \le s$

$$\det(V)e_j + \sum_{j'=s+1}^{n} \det(V_{jj'})e_{j'} \quad \in \quad \mathrm{im}(U).$$

Proof. Let $V' = [v'_{ij}]$ be the adjugate matrix of V, so that $VV' = \det(V) \cdot \mathrm{Id}$ and $v'_{ij} = (-1)^{i+j}\det(V'_{ij})$ where V'_{ij} is obtained from V_{ij} by deleting the i-th column and j-th row. Expanding $\det(V_{jj'})$ via its j-th row, we have

$$\det(V_{jj'}) = \sum_{i=1}^{r} (-1)^{i+j} u_{ij'} \det(V'_{ji}) = \sum_{i=1}^{r} v'_{ji} u_{ij'}.$$

On the other hand, applying U to $V'e_j$ ($j \le s$) we obtain

$$UV'e_j = VV'e_j + WV'e_j = \det(V)e_j + \sum_{j'=s+1}^{n} \left(\sum_{i=1}^{r} v'_{ji} u_{ij'} \right) e_{j'}$$

where the sum in parentheses equals $\det(V_{jj'})$ by the previous observation. □

Corollary 3.5. *The assertion of Theorem 1.2 holds if $X = S$ and S is Noetherian.*

Proof. On the open set U, $\mathcal{M}$ is flat and of finite presentation, and hence locally free. For every $r \ge 0$, let $U_r \subseteq U$ be the locus where $\mathcal{M}_U$ has rank r, so that U is the disjoint union of a finite number of the sets U_r. Let $\overline{U}_r$ be the scheme-theoretic closure of U_r in S. Blowing up their pairwise intersections (which are disjoint from U) we find a U-admissible blow-up on which the closures of the U_r are disjoint. Therefore we reduce to the case $\mathcal{M}_U$ locally free of constant rank r.

Let $S' \to S$ be the blowup along the r-th Fitting ideal $F_r(\mathcal{M})$, which is supported in $S \setminus U$. Then $F_r(\mathcal{M}_{S'})$ is invertible. By Lemma 3.3, the strict transform $\mathcal{M}'$ of $\mathcal{M}$ is locally free of rank r. □

We deduce the following result, which will be used in the subsequent section.

Corollary 3.6 (Extending a line bundle)**.** *Let S be a Noetherian scheme, U and open subset of S, and $\mathcal{L}$ an invertible sheaf on U. Then, there exists a U-admissible blowup $S' \to S$ and a line bundle $\mathcal{L}'$ on S' such that $\mathcal{L}'|_U \simeq \mathcal{L}$.*

Proof. Let $\overline{\mathcal{L}}$ be any extension of $\mathcal{L}$ to a coherent sheaf on S. It exists by [Stacks Project Tag 01PI]. Apply Corollary 3.5 to $\overline{\mathcal{L}}$. □

4. Proof in the projective case

Let $X \to S$ be a morphism of schemes and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. We define the Quot functor

$$\mathrm{Quot}(\mathcal{M}/X/S)\colon (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}, \quad T \mapsto \{T\text{-flat quotients of } \mathcal{M}_T\}/\sim.$$

Here, a T-flat quotient of $\mathcal{M}_T$ is a surjective map $\mathcal{M}_T \to \mathcal{Q}$ to a quasi-coherent $\mathcal{O}_{X_T}$-module $\mathcal{Q}$, and two quotients $\mathcal{M}_T \to \mathcal{Q}_i$ ($i = 0, 1$) are equivalent if there exists an isomorphism $\mathcal{Q}_0 \simeq \mathcal{Q}_1$ under $\mathcal{M}_T$.

Theorem 4.1 (Grothendieck)**.** *Suppose that S is Noetherian and X is projective over S. Then, $\mathrm{Quot}(\mathcal{M}/X/S)$ is represented by the disjoint union of countably many projective S-schemes.*

Lemma 4.2 (Corollary 2.3 in projective case)**.** *Let S be a quasi-compact and quasi-separated scheme, U a quasi-compact open subscheme of S, and $S' \to S$ a projective U-modification of S. Then, there exists a U-admissible blow-up $S'' \to S'$ such that $S'' \to S$ is a U-admissible blowup as well.*

Proof. Step 1. Embed S' in $\mathbf{P}^n_S$ and let

$$(s_0, \dots, s_n) \colon \mathcal{O}_{S'}^{n+1} \longrightarrow \mathcal{O}_{S'}(1)$$

be the corresponding surjection onto a line bundle $\mathcal{O}_{S'}(1)$. Let

$$(s_0, \dots, s_n) \colon \mathcal{O}_{U}^{n+1} \longrightarrow \mathcal{L}$$

be its restriction to U. By Corollary 3.6, replacing S with a U-admissible blowup and S' with its strict transform, we may assume that $\mathcal{L}$ extends to a line bundle $\overline{\mathcal{L}}$ on S.

Step 2. Again replacing S with a U-admissible blowup, we may assume that $S-U$ is the support of an effective Cartier divisor D (in particular, U is schematically dense in S). Then, replacing $\overline{\mathcal{L}}$ with $\overline{\mathcal{L}} \otimes \mathcal{O}_S(rD)$ for large enough r, we may assume that $s_0, \dots, s_n$ extend to sections $\overline{s}_0, \dots, \overline{s}_n$ of $\overline{\mathcal{L}}$.

Step 3. Let Z be the base locus of $(\overline{s}_0, \dots, \overline{s}_n)$, i.e. the support of the cokernel of the map

$$(\overline{s}_0, \dots, \overline{s}_n) \colon \mathcal{O}_{S}^{n+1} \longrightarrow \overline{\mathcal{L}}.$$

By construction, we have $Z \cap U = \varnothing$. Let $S'' \to S$ be the blowup of S along Z. Arguing as in [Har77, Example II 7.17.3], we see that we obtain a morphism $S'' \to \mathbf{P}^n_S$ over S. Since U is schematically dense in S'', the image of this map lands in S'.

Step 4. Finally, let $S''' \to S'$ be the strict transform of S' along $S'' \to S$. The map $S'' \to S'$ induces a section of $S''' \to S''$:

$$\begin{array}{ccc} S''' & \longrightarrow & S' \\ \uparrow\downarrow & \nearrow & \downarrow \\ S'' & \longrightarrow & S. \end{array}$$

Since U is schematically dense in S'' and S''', we have $S''' \simeq S''$, and we conclude that $S'' \to S'$ is a blowup as well.[1] □

Corollary 4.3. *The assertion of Theorem 1.2 holds if S is Noetherian and X is projective over S.*

Proof. Since $\mathcal{M}_U$ is flat over U, we obtain an element of $\mathrm{Quot}(\mathcal{M}/X/S)(U)$. The image of U in (the scheme representing) $\mathrm{Quot}(\mathcal{M}/X/S)(U)$ is quasi-compact and hence is contained in an open subscheme $Q \subseteq \mathrm{Quot}(\mathcal{M}/X/S)$ which is projective over S. Let $\overline{U}$ be the scheme-theoretic closure of U in Q. Then $\overline{U} \to S$ is a projective U-modification, and hence (by Corollary 3.6) is dominated by a U-admissible blowup $S' \to S$. Let $\mathcal{M}'$ be the $\mathcal{O}_{X_{S'}}$-module which is the quotient of $\mathcal{M}_{S'}$ corresponding to the map $S' \to \mathrm{Quot}(\mathcal{M}/X/S)$. By definition of the Quot scheme, $\mathcal{M}'$ is flat over S', and by construction its restriction to X_U equals $\mathcal{M}_U$. It follows that $\mathcal{M}'$ is indeed the strict transform of $\mathcal{M}$. □

[1] Alternatively, we could check that $S'' = \mathrm{Bl}_{\mathcal{I}}(S)$ satisfies the universal property of the blowup of S' along $\mathcal{I} \cdot \mathcal{O}_{S'}$.

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