MATH550 Commutative Algebra — Homework solutions

1 Problem Set 1

Problem 1.1. Let $\mathfrak{p}$ and $\mathfrak{q}$ be two prime ideals in a ring A such that neither $\mathfrak{p} \subseteq \mathfrak{q}$ nor $\mathfrak{q} \subseteq \mathfrak{p}$. Show that the ideal $\mathfrak{p} \cap \mathfrak{q}$ is not prime.

Solution. By assumption, we have elements $a \in \mathfrak{p} \setminus \mathfrak{q}$ and $b \in \mathfrak{q} \setminus \mathfrak{p}$. Then $c = ab$ belongs to $\mathfrak{p} \cap \mathfrak{q}$ but neither a nor b does; thus $\mathfrak{p} \cap \mathfrak{q}$ is not prime. □

Problem 1.2 (Atiyah–Macdonald 1.11)**.** A ring A is Boolean if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that

i) $2x = 0$ for all $x \in A$;

ii) every prime ideal $\mathfrak{p}$ is maximal, and $A/\mathfrak{p}$ is a field with two elements;

iii) every finitely generated ideal in A is principal.

Solution. (a) Taking $x = 2$ gives $4 = 2$, so $2 = 0$ in A. (b) Since $x(x-1) = 0$, either x or $x-1$ belongs to $\mathfrak{p}$ (but not both). Thus A is the disjoint union of $\mathfrak{p}$ and $\mathfrak{p}+1$, which shows $A/\mathfrak{p}$ has two elements, 0 and 1. Thus $A/\mathfrak{p} = \mathbb{F}_2$, which is a field, so $\mathfrak{p}$ is maximal. (c) By induction it is enough to show that for every $x, y \in A$, the ideal (x,y) is principal. Let $z = x + y + xy$, which belongs to x, y, then

$$xz = x^2 + xy + x^2y = x + 2xy = x,$$

showing that $x \in (z)$, and similarly $yz = y$. Thus $(x,y) = (z)$. □

Problem 1.3. Show that a ring A is a domain if and only if it admits an injective homomorphism $A \hookrightarrow K$ into a field K. Show that A is reduced (has no nonzero nilpotent elements) if and only if it admits an injective homomorphism $A \hookrightarrow \prod_{\alpha \in I} K_\alpha$ into a product of (possibly infinitely many) fields.

Solution. Clearly: a field is a domain, every domain is reduced, a subring of a domain is a domain, a subring of a reduced ring is reduced, and the product of reduced rings is reduced. Thus a subring of a field is a domain and a subring of a product of fields is reduced.

If A is a domain, then the set $S = A[S^{-1}]$ of nonzero elements of A is a multiplicative system, and the localization $K = A[S^{-1}]$ is a field, called the field of fractions of A. The map $A \to K$ sending $a \in A$ to $a/1$ is injective.

Recall that the intersection of all prime ideals of any ring is the set of nilpotent elements (the nilradical). Consider the map

$$A \longrightarrow \prod_{\mathfrak{p}} \kappa(\mathfrak{p})$$

where $\kappa(\mathfrak{p})$ is the fraction field of the domain $A/\mathfrak{p}$. The kernel of this map is the intersection of the kernels of the maps $A \to \kappa(\mathfrak{p})$. But since $A/\mathfrak{p} \to \kappa(\mathfrak{p})$ is injective, this equals the intersection of the kernels of $A \to A/\mathfrak{p}$, i.e. the nilradical of A. Thus the above map is injective if A is reduced. □

Problem 1.4 (see Atiyah–Macdonald 1.26)**.** Let X be a compact Hausdorff space and let $A = C(X, \mathbb{R})$ be the ring of continuous functions on X. For $x \in X$, let $\mathfrak{m}_x \subseteq A$ be the set of all $f \in A$ such that $f(x) = 0$. Show that $\mathfrak{m}_x$ is a maximal ideal in A, and that every maximal ideal $\mathfrak{m} \subseteq A$ is of the form $\mathfrak{m}_x$ for a unique $x \in X$.

Solution. Let $x \in X$. The map $A \to \mathbb{R}$ sending f to $f(x)$ is a surjective homomorphism, whose kernel $\mathfrak{m}_x$ is thus a maximal ideal. By Tietze's theorem, for $x \neq y$ in X we can find $f \in A$ with $f(x) = 0$ and $f(y) = 1$, which shows that $\mathfrak{m}_x \neq \mathfrak{m}_y$. Finally, let $\mathfrak{m} \subseteq A$ be a maximal ideal. Suppose that $\mathfrak{m} \neq \mathfrak{m}_x$ for any $x \in X$, then for every $x \in X$ we find $f_x \in \mathfrak{m}$ with $f_x(x) \neq 0$. Let $U_x = \{y \in X : f(y) \neq 0\}$, which is an open neighborhood of x. Since X is compact, there exist $x_1, \ldots, x_n \in X$ such that $X = U_{x_1} \cup \ldots \cup U_{x_n}$. Then the function

$$f = f_{x_1}^2 + \cdots + f_{x_n}^2$$

is everywhere positive, and hence a unit. However, this function belongs to $\mathfrak{m}$, contradiction. □

Problem 1.5. Let again X be a compact Hausdorff space and let $A = C(X,\mathbb{R})$ be the ring of continuous functions on X. Show that every prime ideal in A is contained in a unique maximal ideal.

Solution. In any ring, every proper ideal is contained in a maximal ideal. Suppose that $\mathfrak{p} \subseteq A$ is a prime ideal contained in two maximal ideals $\mathfrak{m}_x$ and $\mathfrak{m}_y$ with $x \neq y$. Let $x \in U$ and $y \in V$ be disjoint open neighborhoods. By Tietze's extension applied to the closed subspace $\{x\} \cup (X \setminus U)$ we find an $f \in A$ with $f(x) = 1$ and $f|_{X\setminus U} = 0$. Similarly, we find $g \in A$ with $g(y) = 1$ and $g|_{X\setminus V} = 0$. Then $fg = 0 \in \mathfrak{p}$, however neither f nor g is in $\mathfrak{p}$. □

2 Problem Set 2

Problem 2.1. Consider the following rings:

$$A_1 = \mathbb{C}[x] \qquad A_2 = \mathbb{C}[x,x^{-1}] \qquad A_3 = \mathbb{C}[x^2,x^3]$$
$$A_4 = \mathbb{C}[x,y] \qquad A_5 = \mathbb{C}[x]/(x^6) \qquad A_6 = \mathbb{C}[x,y]/(x^2,y^3)$$
$$A_7 = \mathbb{C}[x,y]/(xy) \qquad A_8 = \mathbb{C}[x,y]/(xy^2) \qquad A_9 = \mathbb{C}[x,y]/(xy,y^2)$$

Prove that the rings A_i and A_j are not isomorphic as $\mathbb{C}$-algebras for $i \neq j$. Can you show they are also not isomorphic as rings?

Solution. The rings A_1, A_2, A_3, and A_4 are domains and A_5, A_6, A_7, A_8, and A_9 are not.

Among the domains, A_1 and A_2 are PIDs and A_3 and A_4 are not (the ideals (x^2,x^3) and (x,y) are not principal).

The rings A_1 and A_2 are not isomorphic since the group of units in A_1 is $\mathbb{C}^\times$ and the group of units in A_2 is $\mathbb{C}^\times \times x^{\mathbb{Z}}$. These two groups are not isomorphic; for example, the first one is divisible and the second one is not.

The rings A_3 and A_4 are not isomorphic because A_3 is a UFD (unique factorization domain) and A_4 is not (since $(x^2)^3 = (x^3)^2$ contradicts unique factorization). Alternatively one could show by hand that there does not exist a surjective map $A_3 \to \mathbb{C}[t]$, while obviously such a map exists for A_4.

Among the non-domains, the rings A_5 and A_6 are finite-dimensional (in fact six-dimensional) as $\mathbb{C}$-vector spaces, while A_7, A_8, and A_9 are infinite-dimensional.

The six-dimensional A_5 and A_6 are not isomorphic for example because the unique maximal ideal (x) in A_5 is principal, while the maximal ideal (x,y) in A_6 is not.

Among the infinite-dimensional non-domains A_7, A_8, and A_9, only A_7 is reduced and only A_9 is irreducible (meaning that $A_9/\sqrt{0}$ is a domain). □

Problem 2.2 (Atiyah–Macdonald 1.18)**.** Let $X = \operatorname{Spec}(A)$ and $x, y \in X$. Show that

(a) x is a closed point (i.e. $\{x\} \subseteq X$ is a closed subset) if and only if the corresponding prime ideal $\mathfrak{p}_x \subseteq A$ is maximal;

(b) the closure $\overline{\{x\}}$ equals $V(\mathfrak{p}_x)$;

(c) $y \in \overline{\{x\}}$ if and only if $\mathfrak{p}_x \subseteq \mathfrak{p}_y$.

Solution. A closed set $Z = V(I)$ contains x if and only if $I \subseteq \mathfrak{p}_x$. Therefore the closure of $\{x\}$, defined as the intersection of all closed subsets containing x, is equal to $V(\mathfrak{p}_x)$, showing (b). Parts (a) and (c) follow since $V(\mathfrak{p}_x)$ is the set of prime ideals containing $\mathfrak{p}_x$. □

Definition. Let X be a topological space. We say that X is **irreducible** if for every pair of closed subsets $Y_0, Y_1 \subseteq X$ such that $X = Y_0 \cup Y_1$, we have $X = Y_0$ or $X = Y_1$.

Problem 2.3 (Atiyah–Macdonald 1.19)**.** Show that $\operatorname{Spec}(A)$ is irreducible if and only if the nilradical $\sqrt{0} \subseteq A$ is a prime ideal.

Solution. Suppose that $X = Y_0 \cup Y_1$ where $Y_i = V(J_i)$ are two closed subsets, cut out by ideals $J_0, J_1 \subseteq A$. Thus $V(J_0 \cap J_1) = X$, which means that $J_0 \cap J_1 \subseteq \bigcap \mathfrak{p} = \sqrt{0}$. If $\sqrt{0}$ is a prime ideal, let $\eta \in X$ be the corresponding "generic" point. Then $\eta \in Y_i$ for some $i \in \{0,1\}$, which means that $J_i \subseteq \sqrt{0}$, so that $Y_i = X$. Conversely, suppose that X is irreducible, and let $x, y \in A$ be two elements such that xy is nilpotent; we must show that either x or y is nilpotent. Define $Y_0 = V(x)$ and $Y_1 = V(y)$, then

$$Y_0 \cup Y_1 = V(xy) = V((xy)^m)) = V(0) = X,$$

and hence $Y_i = X$ for some i, say $V(x) = X$, showing that x is nilpotent. □

Problem 2.4 (Atiyah–Macdonald 1.20)**.** Let X be a topological space.

1. If $Y \subseteq X$ is an irreducible subspace, then its closure $\overline{Y}$ is irreducible.
2. Every irreducible subspace of X is contained in a maximal irreducible subspace.
3. The maximal irreducible subspaces of X are closed and cover X. They are called the **irreducible components** of X. What are the irreducible components of a Hausdorff space?
4. If A is a ring and $X = \mathrm{Spec}(A)$, then the irreducible components of X are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A.

Solution. Omitted. □

Definition. Let X be a topological space. We say that X is

1. **sober** if every irreducible closed subset $Y \subseteq X$ has a unique generic point, i.e. there exists a unique $\eta_Y \in Y$ such that $Y = \overline{\{\eta_Y\}}$;
2. **quasi-compact** (qc) if every open cover has a finite subcover;
3. **spectral** if it is sober, qc, if the intersection of every two qc open subsets of X is qc, and if its qc open subsets form a base for the topology.

Problem 2.5. Let A be a commutative ring. Prove that $\mathrm{Spec}(A)$ is a spectral space.

Solution. We showed in the lecture that $\mathrm{Spec}(A)$ is quasi-compact. By definition, a base of open subsets is given by $D(f)$ for $f \in A$, and these are stable under intersection since $D(f) \cap D(g) = D(fg)$. We also showed that $D(f)$ is homeomorphic to $\mathrm{Spec}(A[f^{-1}])$, and in particular is quasi-compact.

It remains to show $\mathrm{Spec}(A)$ is sober. Let $Z \subseteq \mathrm{Spec}(A)$ be an irreducible closed subset, and write $Z = V(I)$ for a radical ideal I. Then $Z \simeq \mathrm{Spec}(A/I)$, and A/I is a domain by one of the previous problems. Its unique generic point corresponds to the prime ideal (0). □

⋆ **Problem 2.6** (Converse to Problem 5)**.** Read the relevant part of M. Hochster *Prime ideal structure in commutative rings*[1] Trans. AMS, 142 (1969), pp. 43-60, and explain to me[2] the proof of Hochster's theorem that *every spectral space is homeomorphic to* $\mathrm{Spec}(A)$ *for some commutative ring A*.

Solution. Omitted. □

3 Problem Set 3

Problem 3.1. Let $A = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$. Show that $M(x) = M \otimes_A \kappa(x)$ is zero for every $x \in \mathrm{Spec}(A)$.

Solution. Let $x \in \mathrm{Spec}(\mathbb{Z})$ be a closed point corresponding to a prime p, then $M \otimes_A \kappa(x) = M/pM$. But M is a divisible abelian group, in particular $M = pM$, so $M/pM = 0$. Let $\eta \in \mathrm{Spec}(\mathbb{Z})$ be the generic point, corresponding to the prime ideal (0), then $\kappa(\eta) = \mathbb{Q}$. We have the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow M \longrightarrow 0$$

which after tensoring with $\mathbb{Q}$ becomes

$$0 \longrightarrow \mathbb{Q} \xrightarrow{\mathrm{id}} \mathbb{Q} \longrightarrow M \otimes \mathbb{Q} \longrightarrow 0$$

showing that $M \otimes \mathbb{Q} = 0$. □

[1] https://doi.org/10.2307/1995344 • https://www.jstor.org/stable/1995344
[2] During office hours or by appointment.

Problem 3.2. Let A be a ring and M a finitely presented A-module. (Recall that this means that there **exists** a presentation

$$(\star) \qquad A^m \xrightarrow{\alpha} A^n \xrightarrow{\pi} M \longrightarrow 0,$$

or equivalently that there exists a surjection $A^n \to M$ whose kernel is finitely generated.) Show that the kernel of **every** surjection $A^k \to M$ is finitely generated.

Solution. Let $A^k \to M$ be a surjection, and denote its kernel by K. Using the fact that A^n and A^m are free, we find maps α and β making the diagram with exact rows below commute.

$$\begin{array}{ccccccccc} & & A^m & \longrightarrow & A^n & \longrightarrow & M & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \| & & \\ 0 & \longrightarrow & K & \longrightarrow & A^k & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Applying the snake lemma to this diagram, we obtain an isomorphism $\mathrm{coker}(\alpha) \simeq \mathrm{coker}(\beta)$. In particular, this shows that $\mathrm{coker}(\alpha)$ is finitely generated. Finally, we have a short exact sequence

$$0 \longrightarrow \mathrm{im}(\alpha) \longrightarrow K \longrightarrow \mathrm{coker}(\alpha) \longrightarrow 0$$

where $\mathrm{im}(\alpha)$ and $\mathrm{coker}(\alpha)$ are both finitely generated. It follows easily that K is finitely generated. ☐

Problem 3.3. Let A be a domain. Recall that an A-module M is **torsion-free** if for every nonzero $a \in A$, the map $a\colon M \to M$ is injective.

(a) Show that every flat A-module is torsion-free.

(b) Suppose that A is a principal ideal domain. Show that every torsion-free A-module is flat.

Hint: Treat finitely generated modules first. Try to reduce to this case using filtered colimits.

(c) Give an example of a domain A and a torsion-free A-module which is not flat.

Solution. (a) Let $f \in A$ be a nonzero element and let M be a flat A-module. Tensoring the short exact sequence

$$0 \longrightarrow A \xrightarrow{f} A \longrightarrow A/(f) \longrightarrow 0$$

with M we obtain a short exact sequence

$$0 \longrightarrow M \xrightarrow{f} M \longrightarrow M/fM \longrightarrow 0,$$

in particular $f\colon M \to M$ is injective, so M is torsion-free.

(b) Let M be a torsion-free module over a PID A. If M is finitely generated, we can write

$$M \simeq A^n \oplus \bigoplus_{i=1}^{r} A/(f_i)$$

for some nonzero $f_1, \dots, f_r \in A$. If M is torsion-free, then we must have $r = 0$, so $M \simeq A^n$ is free, in particular flat. In general, let $\{M_\alpha\}$ be the family of all finitely generated submodules of M. This family is filtered: if $M_0, M_1 \subseteq M$ are finitely generated submodules, so is their sum $M_0 + M_1$. Then $M = \varinjlim M_\alpha$ is the filtered colimit of free modules, and in particular is flat (part of Lazard's theorem — easy to show directly using the fact that tensor product commutes with filtered colimits, and filtered colimits are exact).

(c) Let $A = k[X,Y]$ for a field k and let $I = (X,Y)$. This is torsion-free, being a submodule of A. However, it is not flat. To see this, consider the short exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow k \longrightarrow 0. \tag{1}$$

We claim that this is no longer exact on the left after we tensor with I. In other words, we claim that

$$I \otimes_A I \longrightarrow I, \qquad f \otimes g \mapsto fg$$

is not injective. Indeed, the element $w = X \otimes Y - Y \otimes X$ is sent to zero. But why is it nonzero?

To get a handle on this, we find a presentation of I as an A-module. The first step is easy: I is generated as an A-module by X and Y, so we begin with

$$A^2 \xrightarrow{(X,Y)} I \longrightarrow 0.$$

There is an obvious element of the kernel of this surjection, namely $(Y, -X)$. We claim that

$$0 \longrightarrow A \xrightarrow{(Y,-X)^T} A^2 \xrightarrow{X,Y)} I \longrightarrow 0 \tag{2}$$

is exact. Explicitly: if $f, g \in A = k[X,Y]$ is a pair of polynomials such that $Xf = Yg$, then $f = Yh$, $g = Xh$ for some $h \in k[X,Y]$. This is immediate from unique factorization of polynomials.

Tensoring our presentation with I on the right we obtain an exact sequence

$$I \xrightarrow{(Y,-X)^T} I^2 \xrightarrow{(X,Y)} I \otimes_A I \longrightarrow 0$$

and $w = X \otimes Y - Y \otimes X$ is the image of $(Y, -X) \in I^2$. This element is in the image of $(Y,-X)\colon A \to I^2$ but not in the image of $(Y,-X)\colon I \to I^2$, which certifies that $w \neq 0$, as desired.

Alternatively, one could stare at the following diagram obtained by tensoring the resolution (2) with the canonical short exact sequence (1):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & I & \xrightarrow{(Y,-X)} & I^2 & \longrightarrow & I \otimes I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \alpha & & \\
0 & \longrightarrow & A & \xrightarrow{(Y,-X)} & A^2 & \longrightarrow & I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & k & \xrightarrow[0]{} & k^2 & \longrightarrow & I \otimes k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Snake lemma shows that $\ker(\alpha) \simeq k$, and tracing the diagram we can see that $1 \in k$ corresponds to the element $w = X \otimes Y - Y \otimes X$. Interestingly, this detailed analysis shows that w is a torsion element of $I \otimes_k I$ (which of course can be checked directly). □

Problem 3.4. Find an example of a homomorphism between domains $A \to B$ and a torsion-free A-module M such that $M \otimes_A B$ is not torsion-free.

Solution. Take $A = k[X,Y]$, $M = I = (X,Y)$ as in part (c) of the previous problem, and let $B = A[T]/(Y - TX)$ (note that $B \simeq k[X,T]$). Tensoring the presentation for M obtained in the previous problem with B, we obtain the following presentation for $M \otimes_A B$:

$$B \xrightarrow{(TX,-X)} B^2 \longrightarrow M \otimes_A B \longrightarrow 0.$$

Then the element $(T, -1)$ of B^2 maps to a nonzero element v of $M \otimes_A B$. However, $X \cdot v$ is the image of $(TX, -X)$, which is zero. Thus $M \otimes_A B$ is not torsion-free. □

Problem 3.5 (Jelisiejew 11.2)**.**

(a) Let I be a finitely generated ideal in a local ring $(A, \mathfrak{m})$. Prove that if $I = I^2$, then $I = A$ or $I = 0$.

(b) Let $A = C(\mathbb{R}, \mathbb{R})$ be the ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and $\mathfrak{m} = \{f \in A : f(0) = 0\}$. Prove that $\mathfrak{m} = \mathfrak{m}^2$ and conclude that the ideal $\mathfrak{m}$ is not finitely generated.

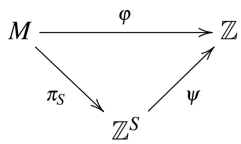
Solution. (a) Suppose that $A \neq I$, then $I \subseteq \mathfrak{m}$. Then $I^2 \subseteq \mathfrak{m}I \subseteq I$, so if $I^2 = I$ then $I = \mathfrak{m}I$. If I is moreover finitely generated, this implies $I = 0$ by Nakayama's lemma.

(b) If $f \in \mathfrak{m}$, then $\sqrt[3]{f}$ is a well-defined continuous function which vanishes at 0. This shows $\mathfrak{m} \subseteq \mathfrak{m}^3$, and hence $\mathfrak{m} = \mathfrak{m}^2$. □

⋆ **Problem 3.6** ($\mathbb{Z}^{\mathbb{N}}$ is not free)**.** Let $M = \mathbb{Z}^{\mathbb{N}}$ be the group of integer-valued sequences. For a subset $S \subseteq \mathbb{N}$, denote by $\pi_S \colon M \to \mathbb{Z}^S$ the projection map. Show that every homomorphism

$$\varphi \colon M \longrightarrow \mathbb{Z}$$

admits a factorization



for a finite subset $S \subseteq \mathbb{N}$. Deduce that the $\mathbb{Z}$-module M is not free (even though it is flat).

Hint: Reduce to the case $\phi(e_n) > 0$ for every $n \geq 0$, where $e_n \in M$ is the element $e_n(m) = 1$ if $m = n$ and 0 otherwise. Then consider an element $x \in M$ with $x(n) = 2^{a_n}$ for a rapidly growing sequence (a_n).

Solution. Omitted. □

4 Problem Set 4

Problem 4.1. Let A be a domain and let M be a finitely presented A-module. Prove that M is flat (equivalently, projective or locally free) if and only if the function

$$\delta_M \colon \operatorname{Spec}(A) \longrightarrow \mathbb{N}, \qquad \delta_M(x) = \dim_{\kappa(x)} M(x)$$

is constant. Find a counterexample to the "if" part with A not a domain.

Hint: Reduce to A local, in which case $\operatorname{Spec}(A)$ has the generic point η (corresponding to the prime ideal (0)) and the closed point s (corresponding to the unique maximal ideal $\mathfrak{m}$). Combine $\delta_M(\eta) = \delta_M(s)$ and Nakayama to show that M is free.

Solution. Since A is a domain, $\operatorname{Spec}(A)$ has a generic point η, corresponding to the prime ideal (0), with $\kappa(\eta) = K$, the fraction field of A. Let $x \in \operatorname{Spec}(A)$ be a point corresponding to a prime ideal $\mathfrak{p} \subseteq A$. Let $A' = A_{\mathfrak{p}}$ (the "local ring at x") and $M' = M_{\mathfrak{p}} = M \otimes_A A'$. Note that $\delta_x(M') = \delta_x(M)$ and $\delta_\eta(M') = \delta_x(M)$ where we identify $\operatorname{Spec}(A')$ with a subset of $\operatorname{Spec}(A)$. Let $x_1, \ldots, x_n \in M'$ be elements mapping to a basis of $M' \otimes \kappa(x)$ (so $n = \delta_x(M')$), so that $x_1, \ldots, x_n$ generate M' by Nakayama's lemma. Let N be the kernel of the induced surjection $A' \to M'$. Tensoring with K (which is flat, being a localization), we obtain a short exact sequence of vector spaces

$$0 \longrightarrow N \otimes_{A'} K \longrightarrow K^n \longrightarrow M' \otimes_{A'} K \longrightarrow 0$$

and we see that $\delta_M(\eta) \leq \delta_M(x)$, and that equality holds if and only if $N \otimes_{A'} K = 0$, which holds if and only if M' is free.

Finally, consider $A = \mathbb{Z}/4$ and $M = 2A$. Then M is not flat over A, but $\operatorname{Spec}(A)$ has only one point, so obviously δ_M is constant. □

Problem 4.2. Prove that the following ring homomorphisms are not flat.

(a) $k[x^2, x^3] \hookrightarrow k[x]$;

(b) $k[x^2, xy, y^2] \hookrightarrow k[x, y]$;

(c) $k[x, xy] \hookrightarrow k[x, y]$.

Solution. (a) We use the criterion of the previous problem. The fiber of this map at the maximal ideal (x^2,x^3) is

$$k[x]/(x^2,x^3) = k[x]/(x^2)$$

which has dimension two. However, the map is an isomorphism after inverting x, so the fiber has dimension 1 at all other points.

(b) We use the criterion of the previous problem. Here the fiber over (x^2,xy,y^2) is three-dimensional, but is two-dimensional everywhere else.

(c) See the solution to Problem 3.4. If $M = (x,xy)$, then the map $x\colon M \to M$, is injective, but is not injective after tensoring with $k[x,y]$. □

Let A be an $\mathbb{F}_p$-algebra (that is, $pA = 0$). The **Frobenius morphism** of A is

$$F\colon A \longrightarrow A, \qquad F(x) = x^p$$

which is a ring homomorphism since $(X+Y)^p = X^p + Y^p$ modulo p.

Problem 4.3. Prove that the Frobenius $F\colon A \to A$ induces the identity map $\operatorname{Spec}(A) \to \operatorname{Spec}(A)$.

Solution. Let $\mathfrak{p} \subseteq A$ be a prime ideal. If $x \in F^{-1}(\mathfrak{p})$, i.e. $x^p \in \mathfrak{p}$, then $x \in \mathfrak{p}$ since $\mathfrak{p}$ is prime. This shows that $F^{-1}(\mathfrak{p}) = \mathfrak{p}$. □

Problem 4.4. (a) Let $A = \mathbb{F}_p[T_1,\dots,T_n]$. Prove that $F\colon A \to A$ is finite and flat.

(b) Let $A = \mathbb{F}_p[T^2,T^3]$. Prove that $F\colon A \to A$ is finite but not flat.

Solution. (a) It suffices to note that the monomials $T_1^{a_1}\cdots T_n^{a_n}$ with $a_1,\dots,a_n \in \{0,\dots,p-1\}$ form a basis of A as a module over $A^p = \mathbb{F}_p[T_1^p,\dots,T_n^p]$.

(b) The map F is integral (for every $\mathbb{F}_p$-algebra), since $x \in A$ satisfies the integral equation $T^p - x^p = 0$. In the situation in (b), this is a map between finite type $\mathbb{F}_p$-algebras, and hence is of finite type. Integral and finite type implies finite. (Alternatively one could check by hand that the monomials T^i where $0 \le i < 3p$, $i \neq 0$ generate $\mathbb{F}_p[T^2,T^3]$ as a module over $\mathbb{F}_p[T^{2p},T^{3p}]$.)

To check the map is not flat, we compute fibers at (T^2,T^3) and at $(T-1)$. For the first, we have $\mathbb{F}_p[T^2,T^3]/(T^{2p},T^{3p}) = \mathbb{F}_p[T^2,T^3]/(T^{2p})$, which is $(2p-1)$-dimensional over $\mathbb{F}_p$. For the second, we have $\mathbb{F}_p[T^2,T^3]/(T^p-1) \simeq \mathbb{F}_p[T]/(T^p-1) \simeq \mathbb{F}_p[X]/(X^p)$, which is p-dimensional. So the map is not flat by the criterion of Problem 4.1. □

Problem 4.5. Let $A = k[X,Y,Z]/(X^2+Y^2+Z^2-1)$ where k is a field of characteristic $\neq 2$. Consider the module of Kähler differentials $M = \Omega^1_{A/k}$. Show that M is locally free (or equivalently projective) and describe it as a direct summand of A^3.

Solution. The module $M = \Omega^1_{A/k}$ is generated by dX, dY, dZ subject to the relation

$$0 = d(X^2+Y^2+Z^2-1) = 2XdX + 2YdY + 2ZdZ.$$

Since $\operatorname{char}(k) \neq 2$, we can cancel the two. Thus M has a presentation

$$0 \longrightarrow A \xrightarrow{(X,Y,Z)} A^3 \longrightarrow M \longrightarrow 0.$$

We note that the map $A^3 \to A$ defined by $(f,g,h) \mapsto Xf+Yg+Zh$ is a section of the left map, since $(X,Y,Z) \mapsto X^2+Y^2+Z^2 = 1$. This allows us to decompose $A^3 \simeq A \oplus M$, showing that M is projective.

The equation $X^2+Y^2+Z^2 = 1$ shows that $(X,Y,Z) = A$. We check that M becomes free after inverting one of the variables, say X. Over the localized ring $A[X^{-1}]$, the element $w = (X,Y,Z)$ of A^3 can be written as $X^{-1}(1,Y/X,Z/X)$, and it follows that $dY = (0,1,0)$ and $dZ = (0,0,1)$ map to a basis of M. □

⋆ **Problem 4.6** (Algebraic "hairy ball theorem")**.** Show that for $k = \mathbb{R}$, the module $M = \Omega^1_{A/k}$ in the above example is not free. Can you treat other base fields k as well?

Solution. Omitted. □

5 Problem Set 5

Problem 5.1 (Variant of Cayley–Hamilton)**.** Let M be a finitely generated A-module and let $\varphi\colon M \to M$ an A-module morphism. Let $I \subseteq A$ be an ideal such that $\varphi(M) \subseteq I \cdot M$. Show that φ satisfies an equation of the form

$$\varphi^n + a_1\varphi^{n-1} + \cdots + a_n = 0$$

where $a_i \in I^i$ for $i = 1, \ldots, n$.

Hint: We proved this in case $I = A$ (see also Atiyah–Macdonald, 2.4). Modify the proof.

Solution. Omitted. □

Problem 5.2. Let k be a field of characteristic $\neq 2$ and let A be a k-algebra. Construct a bijection between the sets of

- **involutions** on A, i.e. k-algebra homomorphisms $f\colon A \to A$ such that $f \circ f = \mathrm{id}_A$;
- $\mathbb{Z}/2$**-gradings** on A, i.e. direct sum decompositions of the underlying abelian group

$$A \simeq A_0 \oplus A_1$$

  such that $k \subseteq A_0$ and $A_i \cdot A_j \subseteq A_{i+j \bmod 2}$.

Solution. Given an involution f, we set

$$A_0 = \{x \in A : f(x) = x\} \quad \text{and} \quad A_1 = \{x \in A : f(x) = -x\}.$$

Then every $x \in A$ can be written as

$$x = \underbrace{\frac{x+f(x)}{2}}_{\in A_0} + \underbrace{\frac{x-f(x)}{2}}_{\in A_1}$$

so $A = A_0 \oplus A_1$. It is clear that this gives a $\mathbb{Z}/2$-grading. Conversely, given a $\mathbb{Z}/2$-grading $A = A_0 \oplus A_1$, we define the corresponding involution f by $f = \mathrm{id}$ on A_0 and $f = -\mathrm{id}$ on A_1. It is straightforward to check that f is an involution, and that these construction establish mutually inverse bijections. □

Problem 5.3. Prove that every unique factorization domain is normal (i.e. integrally closed in its field of fractions).

Solution. Let A be a unique factorization domain and let x/y be an element of its field of fractions ($x, y \in A$, $y \neq 0$). Since A is a UFD, we can clean common factors and assume that x and y are coprime. If x/y is integral over A, we have an equation of the form

$$(x/y)^n + a_1(x/y)^{n-1} + \cdots + a_n = 0, \qquad a_1, \ldots, a_n \in A.$$

Multiplying by y^n and rearranging, we obtain

$$x^n = -y(a_1x^{n-1} + \cdots + a_ny^{n-1}).$$

If y is not a unit, it has an irreducible factor, and by the above equation said factor must divide x, contradicting the coprimality of x and y. Thus y is a unit, and $x/y \in A$. □

Problem 5.4. A topological space X is called **Noetherian** if every increasing chain of open subsets stabilizes.

(a) Let A be a Noetherian ring. Prove that $\mathrm{Spec}(A)$ is a Noetherian topological space.

(b) Does the converse hold?

(c) Prove that a topological space X is Noetherian if and only if every open subset of X is quasi-compact.

Solution. (a) Equivalently we must show every decreasing chain of closed subsets stabilizes. Let $Z_0 \supseteq Z_1 \supseteq \dots$ be a decreasing chain of closed subsets, and let

$$J_n = \mathfrak{I}(Z_n) = \bigcap_{\mathfrak{p} \in Z_n} \mathfrak{p}$$

be the corresponding radical ideal of A. Then $J_0 \subseteq J_1 \subseteq \dots$, and if A is Noetherian, this must stabilize. But $Z_n = V(J_n)$, so the chain of closed subsets stabilizes as well.

(b) No, for example consider a vector space V over a field k, and make $A = k \oplus V\varepsilon$ into a ring by the formula $(x + v\varepsilon)(y + w\varepsilon) = xy + (xw + yv)\varepsilon$. Then every subspace $W \subseteq V$ gives rise to an ideal $W\varepsilon \subseteq A$. Thus A is Noetherian if and only if V is finite-dimensional. However, $\mathrm{Spec}(A)$ is a single point, since it is equal to $\mathrm{Spec}(A/\sqrt{0})$ and $A/\sqrt{0} = A/V\varepsilon = k$.

(c) Let $U_0 \subseteq U_1 \subseteq \dots$ be an increasing chain of open subsets of X and let $U = \bigcup U_n$. If U is quasi-compact, then we can pick a finite subcover, so that $U = U_n$ for some n, and the chain stabilizes. This shows the "if" direction. For "only if," suppose X Noetherian and let $U \subseteq X$ be an open subset, and let $U = \bigcup_{\alpha \in I} U_\alpha$ be an open cover. Suppose no finite subfamily covers U. We define $U'_0 \subseteq U'_1 \subseteq \dots \subseteq U$ (each U'_n being a finite union of the U_α's) inductively: $U'_0 = \emptyset$, and since $U'_{n-1} \neq U$, let $x \in U \setminus U'_{n-1}$, and let $\alpha \in I$ be such that $x \in U_\alpha$. Then set $U'_n = U'_{n-1} \cup U_\alpha$. By construction, we have $U'_n \neq U'_{n-1}$, so the chain $U_0 \subseteq U_1 \subseteq \dots$ does not stabilize, contradiction. □

In the problem below, we use the following construction. Let B be a ring and M a B-module. We make the direct sum $B \oplus M\varepsilon$ (with ε just a symbol) into a ring with multiplication

$$(b + m\varepsilon)(b' + m'\varepsilon) = bb' + (bm' + b'm)\varepsilon$$

The subgroup $I = 0 \oplus M\varepsilon$ is an ideal with $I^2 = 0$ and quotient $(B \oplus M\varepsilon)/I = B$. We denote by $\pi \colon B \oplus M\varepsilon \to B$ the quotient map.

Problem 5.5. Let $A \to B$ be a map of rings and M a B-module. Construct a bijection between the sets of

- A-linear derivations $\delta \colon B \to M$;
- A-algebra homomorphisms $\varphi \colon B \to B \oplus M\varepsilon$ such that $\pi \circ \varphi = \mathrm{id}_A$.

Solution. Let $\delta \colon B \to M$ be an A-linear derivation. Then $\varphi \colon B \to B \oplus M\varepsilon$ defined by

$$\varphi(b) = b + \delta(b)\varepsilon$$

is an A-algebra homomorphism such that $\pi \circ \varphi = \mathrm{id}_A$. We omit the (straightforward) rest of the solution. □

6 Problem Set 6

Problem 6.1. Let $Z = \mathrm{Spec}(\mathbb{Z}) \setminus \{\eta\}$ where η is the generic point. Prove that there does not exist a finitely generated $\mathbb{Z}$-algebra A such that the image of $\mathrm{Spec}(A) \to \mathrm{Spec}(\mathbb{Z})$ is equal to Z.

Solution. By Chevalley's theorem this image is a constructible subset. However, every non-empty open of $\mathrm{Spec}(\mathbb{Z})$ is the complement of a finite set of closed points (corresponding to prime numbers), and it follows that every constructible subset of $\mathrm{Spec}(\mathbb{Z})$ is either open or closed. But $\{\eta\}$ is neither open nor closed. □

Problem 6.2. Let $k = \mathbb{F}_q$ be the field with q elements and let

$$A = k[X,Y]/(X^qY - XY^q).$$

For $a, b \in k$ not both zero, consider the k-algebra map

$$\phi_{a,b} \colon k[T] \longrightarrow A, \qquad \phi_{a,b}(T) = aX + bY.$$

Prove that the map $\phi_{a,b}$ is not finite.

Solution. Omitted. □

Problem 6.3. Let k be a field and A a finitely generated domain over k. Let L be the integral closure of k in A. Show that L is finite over k. *Hint:* First show that L is a field, then pick a maximal ideal of A and apply Nullstellensatz.

Solution. We note that L is a domain (being a subring of A) and integral over k (by definition). This implies it is an increasing union of finite k-algebras which are domains. However, such an algebra R is a field (the multiplication by a nonzero element $R \to R$ is injective, and hence surjective since this is a map between finite-dimensional vector spaces over k). Thus L, being an increasing union of fields, is a field. Let $\mathfrak{m} \subseteq A$ be a maximal ideal, so that $A/\mathfrak{m}$ is a finite extension of k. The map $L \to A \to A/\mathfrak{m}$ is injective (being a map between fields), and hence L finite over k as well. □

Problem 6.4. Let X be a spectral space and let $W \subseteq X$ be a constructible subset. Prove that W is quasi-compact.

Solution. Omitted. □

Problem 6.5. Let A be a ring and let $Z \subseteq \operatorname{Spec}(A)$ be a closed subset. Prove that Z is constructible if and only if there exists a finitely generated ideal $I \subseteq A$ such that $Z = V(I)$.

Solution. The set $Z = V(J)$ is constructible if and only if its complement $U = \bigcup_{f\in J} D(f) = \operatorname{Spec}(A) \setminus Z$ is constructible. If Z is constructible, then U is quasi-compact (previous problem), and hence $U = D(f_1) \cup \ldots \cup D(f_n)$ for $f_1, \ldots, f_n \in J$, and hence $Z = V(I)$ where $I = f_1, \ldots, f_n)$. Conversely, if $Z = V(I)$ with $I = (f_1, \ldots, f_n)$, then $U = D(f_1) \cup \ldots \cup D(f_n)$ is a quasi-compact open. □

⋆ **Problem 6.6.** Let $Z = \operatorname{Spec}(\mathbb{Z}) \setminus \{\eta\}$ where η is the generic point. Prove that there does not exist a $\mathbb{Z}$-algebra A such that the image of $\operatorname{Spec}(A) \to \operatorname{Spec}(\mathbb{Z})$ is equal to Z.

Solution. Omitted. □

⋆ **Problem 6.7.** Let $\sigma\colon \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map satisfying $\sigma \circ \sigma = \mathrm{id}$. Prove that σ has a fixed point. *Hint:* Reduce to the analogous question over a finite field of odd characteristic.

Solution. Omitted. □

7 Problem Set 7

Problem 7.1 (Composition of valuation rings)**.** Let A be a valuation ring with fraction field K and residue field L, and denote by $\pi\colon A \to L$ the quotient map. Let B be a valuation ring with fraction field L and residue field E. Consider the subring $C \subseteq A$ defined by

$$C = \{x \in A \,:\, \pi(x) \in B\}.$$

Prove that C is a valuation ring with fraction field K and residue field E.

Solution. Let $x \in K^\times$, we want to show either $x \in C$ or $x^{-1} \in C$. Suppose without loss of generality that $x \in A$, and let $y = \pi(x) \in L$. If $y \in B$, then $x \in C$. If $y \notin B$, then $y \in L^\times$ and $y^{-1} \in B$, so $x \in A^\times$ (since L is the residue field of the local ring A). Thus $x^{-1} \in A$ and $\pi(x^{-1}) = y^{-1} \in B$, so that $x^{-1} \in C$. We have proved that C is a valuation subring of K. Moreover, since $\pi\colon A \to L$ is surjective, so is the induced map $C \to B$. Composing with $B \to E$, we obtain a surjection $C \to E$, whose kernel is the unique maximal ideal, showing that E is the residue field of C. □

Problem 7.2 (Integers in the fifth cyclotomic field)**.** Let $\zeta = e^{2\pi i/5}$ be the primitive root of unity of order 5, and let $K = \mathbb{Q}(\zeta)$. Prove that the ring of integers $\mathcal{O}_K$ (the integral closure of $\mathbb{Z}$ in K) is equal to $\mathbb{Z}[\zeta]$.

Solution. Omitted ;) □

Problem 7.3. Let K be an algebraically closed field and let R be a finitely generated K-algebra. Suppose that for every maximal ideal $\mathfrak{m} \subseteq R$ we have $R_\mathfrak{m} = K$. Prove that $R \simeq K^n$ for some $n \geq 0$.

Solution. We begin with a simple but powerful general observation: Let A be a ring, let $S \subseteq A$ be a multiplicative system, and let M be a finitely generated A-module. Suppose that $M[S^{-1}] = 0$. Then $M[f^{-1}] = 0$ for some $f \in S$. To show this, let $x_1, \dots, x_m$ generate M. Their images $x_i/1$ generate $M[S^{-1}] = 0$. Since this is zero, we must have $f_i x_i = 0$ for some $f_i \in S$. Set $f = f_1 \cdots f_n$, so that $f x_i = 0$ for all i. Then $x_i/1 = 0$ in $M[f^{-1}]$, so that $M[f^{-1}] = 0$.

Armed with this, let us note that since $R_\mathfrak{m}/\mathfrak{m} \cdot R_\mathfrak{m} = R/\mathfrak{m} = K$, the assumption $R_\mathfrak{m} = K$ means that $\mathfrak{m}_\mathfrak{m} = \mathfrak{m} \cdot R_\mathfrak{m} = 0$. Applying the observation above to the R-module $\mathfrak{m}$ we obtain an $f \notin \mathfrak{m}$ such that $f\mathfrak{m} = 0$. This means that $D(f) = V(\mathfrak{m}) = \{x\}$ is a single isolated point.

Finally, we check the following claim: suppose that A is a ring and $x \in \operatorname{Spec}(A)$ is an isolated point. Then there exists a ring A' such that $A \simeq A_x \times A'$ where $A_x = A_\mathfrak{p}$ where $\mathfrak{p} \subseteq A$ is the prime (in fact maximal) ideal corresponding to x. (I omit this since I ran out of time.) □

Problem 7.4. Let k be a field and let $R = k[[T_1, \dots, T_n]]$ be the ring of formal power series in $n \geq 1$ variables. Prove that R is a local ring with maximal ideal $\mathfrak{m} = (T_1, \dots, T_n)$.

Solution. We must show that a power series $f \in k[[T_1, \dots, T_n]]$ with nonzero constant term is invertible. Write $f = a_0 + g$ where $g \in \mathfrak{m}$ and $a_0 \in k^\times$. Then

$$f^{-1} = \frac{1}{a_0 + g} = \frac{a_0^{-1}}{1 + a_0^{-1} g} = a_0^{-1} - a_0^{-2} g + a_0^{-3} g^2 + \cdots$$

where the sum converges since $a_0^{-(n+1)} g^n \in \mathfrak{m}^n$. □

Problem 7.5 (Nodal curve is analytically reducible)**.** Prove that the element $Y^2 - X^2(X+1)$ of the power series ring $\mathbb{C}[[X,Y]]$ is the product of two non-units.

Solution. Omitted (see Hartshorne *Algebraic Geometry*, Chapter I, Example 5.6.3, p. 34). □

For the next problem, recall the following definitions: for a prime p

- the ring of p-adic integers $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n$ is the p-adic completion of $\mathbb{Z}$ (it is a discrete valuation ring with maximal ideal (p));
- the field of p-adic (rational) numbers $\mathbb{Q}_p$ is the fraction field $\operatorname{Frac}(\mathbb{Z}_p) = \mathbb{Z}_p[1/p]$;
- the p-adic norm of an element $x \in \mathbb{Q}_p$ is the non-negative real number

$$|x|_p = p^{-\nu_p(x)}, \qquad \nu_p(x) = \max\{n \in \mathbb{Z} : p^{-n} x \in \mathbb{Z}_p\}$$

where we use the convention $|0|_p = p^{-\infty} = 0$.

Problem 7.6 (p-adic analytic functions)**.** Let p be a prime and let

$$A^\circ = \mathbb{Z}_p\langle T \rangle = \varprojlim_n (\mathbb{Z}/p^n)[T]$$

be the p-adic completion of $\mathbb{Z}[T]$. Let

$$A = \mathbb{Q}_p\langle T \rangle = A^\circ[1/p].$$

Prove that

$$A \simeq \left\{ f = \sum_{n \geq 0} a_n T^n \in \mathbb{Q}_p[[T]] \, : \, |a_n|_p \to 0 \text{ as } n \to \infty \right\}.$$

(*FYI:* This is the ring of power series which converge on the closed p-adic unit disc.)

Solution. We shall first construct an isomorphism:

$$\theta \colon A^\circ \longrightarrow \left\{ f = \sum_{n \geq 0} a_n T^n \in \mathbb{Z}_p[[T]] \, : \, |a_n|_p \to 0 \text{ as } n \to \infty \right\}.$$

To construct the map note that $(p) \subseteq (p,T)$, so that we obtain a map from the p-adic completion of $\mathbb{Z}[T]$, which is A° to its (p,T)-adic completion, which is $\mathbb{Z}_p[[T]]$. Concretely, given $f \in A^\circ$, for every $n \geq 0$ we consider its image

$f_n \in \mathbb{Z}/p^n[T]$, and then its image $g_n \in \mathbb{Z}/p^n[\![T]\!]$. The elements g_n are compatible and give rise to a power series $g \in \mathbb{Z}_p[\![T]\!]$, which is the image of f under our map θ.

It is clear from the construction that the map θ is injective. Indeed, if f maps to zero, then it maps to zero in $\mathbb{Z}/p^n[T]$ for all n (this uses injectivity of $\mathbb{Z}/p^n[T] \to \mathbb{Z}/p^n[\![T]\!]$), so $f = 0$.

To describe the image of θ, we note that g is in the image if and only if for every n, the image $g_n \in \mathbb{Z}/p^n[\![T]\!]$ is in the image of $\mathbb{Z}/p^n[T]$, i.e. a polynomial. Now, write $g = \sum_{n\geq 0} a_n T^n$ with $a_n \in \mathbb{Z}_p$. This is a polynomial modulo p^m if a_n are divisible by p^m for all but finitely many m. Thus, being a polynomial modulo p^m for all m means that the a_n are divisible by higher and higher powers of p, i.e. that $|a_n|_p \to 0$ as $n \to \infty$.

Finally, applying the localization $(-)[1/p]$ to both sides of θ we obtain the required isomorphism. □