MATH550 Commutative Algebra — Problem Set 7

Due Dec 16, 2025.

Choose five out of the six (solving all six counts as extra credit).

Problem 1 (Composition of valuation rings)**.** Let A be a valuation ring with fraction field K and residue field L, and denote by $\pi\colon A \to L$ the quotient map. Let B be a valuation ring with fraction field L and residue field E. Consider the subring $C \subseteq A$ defined by

$$C = \{x \in A \,:\, \pi(x) \in B\}.$$

Prove that C is a valuation ring with fraction field K and residue field E.

Problem 2 (Integer in the fifth cyclotomic field)**.** Let $\zeta = e^{2\pi i/5}$ be the primitive root of unity of order 5, and let $K = \mathbb{Q}(\zeta)$. Prove that the ring of integers $\mathcal{O}_K$ (the integral closure of $\mathbb{Z}$ in K) is equal to $\mathbb{Z}[\zeta]$.

Problem 3. Let K be an algebraically closed field and let R be a finitely generated K-algebra. Suppose that for every maximal ideal $\mathfrak{m} \subseteq R$ we have $R_{\mathfrak{m}} = K$. Prove that $R \simeq K^n$ for some $n \geq 0$.

Problem 4. Let k be a field and let $R = k[[T_1, \ldots, T_n]]$ be the ring of formal power series in $n \geq 1$ variables. Prove that R is a local ring with maximal ideal $\mathfrak{m} = (T_1, \ldots, T_n)$.

Problem 5 (Nodal curve is analytically reducible)**.** Prove that the element $Y^2 - X^2(X+1)$ of the power series ring $\mathbb{C}[[X,Y]]$ is the product of two non-units.

For the next problem, recall the following definitions: for a prime p

- the ring of p-adic integers $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n$ is the p-adic completion of $\mathbb{Z}$ (it is a discrete valuation ring with maximal ideal (p));
- the field of p-adic (rational) numbers $\mathbb{Q}_p$ is the fraction field $\mathrm{Frac}(\mathbb{Z}_p) = \mathbb{Z}_p[1/p]$;
- the p-adic norm of an element $x \in \mathbb{Q}_p$ is the non-negative real number

$$|x|_p = p^{-\nu_p(x)}, \qquad \nu_p(x) = \max\{n \in \mathbb{Z} \,:\, p^{-n}x \in \mathbb{Z}_p\}$$

  where we use the convention $|0|_p = p^{-\infty} = 0$.

Problem 6 (p-adic analytic functions)**.** Let p be a prime and let

$$A^\circ = \mathbb{Z}_p\langle T\rangle = \varprojlim_n (\mathbb{Z}/p^n)[T]$$

be the p-adic completion of $\mathbb{Z}[T]$. Let

$$A = \mathbb{Q}_p\langle T\rangle = A^\circ[1/p].$$

Prove that

$$A \simeq \left\{ f = \sum_{n\geq 0} a_n T^n \in \mathbb{Q}_p[[T]] \,:\, |a_n|_p \to 0 \text{ as } n \to \infty \right\}.$$

(*FYI:* This is the ring of power series which converge on the closed p-adic unit disc.)