**MATH550 Commutative Algebra — Problem Set 6**

Due Dec 4, 2025.

**Problem 1.** Let $Z = \operatorname{Spec}(\mathbb{Z}) \setminus \{\eta\}$ where $\eta$ is the generic point. Prove that there does not exist a finitely generated $\mathbb{Z}$-algebra $A$ such that the image of $\operatorname{Spec}(A) \to \operatorname{Spec}(\mathbb{Z})$ is equal to $Z$.

**Problem 2.** Let $k = \mathbb{F}_q$ be the field with $q$ elements and let

$$A = k[X,Y]/(X^qY - XY^q).$$

For $a, b \in k$ not both zero, consider the $k$-algebra map

$$\phi_{a,b} \colon k[T] \longrightarrow A, \qquad \phi_{a,b}(T) = aX + bY.$$

Prove that the map $\phi_{a,b}$ is not finite.

**Problem 3.** Let $k$ be a field and $A$ a finitely generated domain over $k$. Let $L$ be the integral closure of $k$ in $A$. Show that $L$ is finite over $k$. *Hint:* First show that $L$ is a field, then pick a maximal ideal of $A$ and apply Nullstellensatz.

**Problem 4.** Let $X$ be a spectral space and let $W \subseteq X$ be a constructible subset. Prove that $W$ is quasi-compact.

**Problem 5.** Let $A$ be a ring and let $Z \subseteq \operatorname{Spec}(A)$ be a closed subset. Prove that $Z$ is constructible if and only if there exists a finitely generated ideal $I \subseteq A$ such that $Z = V(I)$.

$\star$ **Problem 6.** Let $Z = \operatorname{Spec}(\mathbb{Z}) \setminus \{\eta\}$ where $\eta$ is the generic point. Prove that there does not exist a $\mathbb{Z}$-algebra $A$ such that the image of $\operatorname{Spec}(A) \to \operatorname{Spec}(\mathbb{Z})$ is equal to $Z$.

$\star$ **Problem 7.** Let $\sigma \colon \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map satisfying $\sigma \circ \sigma = \mathrm{id}$. Prove that $\sigma$ has a fixed point. *Hint:* Reduce to the analogous question over a finite field of odd characteristic.