MATH550 Commutative Algebra — Problem Set 5

Due Nov 25, 2025.

Problem 1 (Variant of Cayley–Hamilton)**.** Let M be a finitely generated A-module and let $\varphi\colon M \to M$ an A-module morphism. Let $I \subseteq A$ be an ideal such that $\varphi(M) \subseteq I \cdot M$. Show that φ satisfies an equation of the form

$$\varphi^n + a_1\varphi^{n-1} + \cdots + a_n = 0$$

where $a_i \in I^i$ for $i = 1, \ldots, n$.

Hint: We proved this in case $I = A$ (see also Atiyah–Macdonald, 2.4). Modify the proof.

Problem 2. Let k be a field of characteristic $\neq 2$ and let A be a k-algebra. Construct a bijection between the sets of

- **involutions** on A, i.e. k-algebra homomorphisms $f\colon A \to A$ such that $f \circ f = \mathrm{id}_A$;
- $\mathbb{Z}/2$**-gradings** on A, i.e. direct sum decompositions of the underlying abelian group

  $$A \simeq A_0 \oplus A_1$$

  such that $k \subseteq A_0$ and $A_i \cdot A_j \subseteq A_{i+j \bmod 2}$.

Problem 3. Prove that every unique factorization domain is normal (i.e. integrally closed in its field of fractions).

Problem 4. A topological space X is called **Noetherian** if every increasing chain of open subsets stabilizes.

(a) Let A be a Noetherian ring. Prove that $\mathrm{Spec}(A)$ is a Noetherian topological space.

(b) Does the converse hold?

(c) Prove that a topological space X is Noetherian if and only if every open subset of X is quasi-compact.

In the problem below, we use the following construction. Let B be a ring and M a B-module. We make the direct sum $B \oplus M\varepsilon$ (with ε just a symbol) into a ring with multiplication

$$(b + m\varepsilon)(b' + m'\varepsilon) = bb' + (bm' + b'm)\varepsilon$$

The subgroup $I = 0 \oplus M\varepsilon$ is an ideal with $I^2 = 0$ and quotient $(B \oplus M\varepsilon)/I = B$. We denote by $\pi\colon B \oplus M\varepsilon \to B$ the quotient map.

Problem 5. Let $A \to B$ be a map of rings and M a B-module. Construct a bijection between the sets of

- A-linear derivations $\delta\colon B \to M$;
- A-algebra homomorphisms $\varphi\colon B \to B \oplus M\varepsilon$ such that $\pi \circ \varphi = \mathrm{id}_A$.