MATH550 Commutative Algebra — Problem Set 4

Due Nov 18, 2025.

Problem 1. Let A be a domain and let M be a finitely presented A-module. Prove that M is flat (equivalently, projective or locally free) if and only if the function

$$\delta_M \colon \operatorname{Spec}(A) \longrightarrow \mathbb{N}, \qquad \delta_M(x) = \dim_{\kappa(x)} M(x)$$

is constant. Find a counterexample to the "if" part with A not a domain.

Hint: Reduce to A local, in which case $\operatorname{Spec}(A)$ has the generic point η (corresponding to the prime ideal (0)) and the closed point s (corresponding to the unique maximal ideal $\mathfrak{m}$). Combine $\delta_M(\eta) = \delta_M(s)$ and Nakayama to show that M is free.

Problem 2. Prove that the following ring homomorphisms are not flat.

(a) $k[x^2, x^3] \hookrightarrow k[x]$;

(b) $k[x^2, xy, y^2] \hookrightarrow k[x,y]$;

(c) $k[x, xy] \hookrightarrow k[x,y]$.

The Frobenius morphism

Let A be an $\mathbb{F}_p$-algebra (that is, $pA = 0$). The **Frobenius morphism** of A is

$$F \colon A \longrightarrow A, \qquad F(x) = x^p$$

which is a ring homomorphism since $(X+Y)^p = X^p + Y^p$ modulo p.

Problem 3. Prove that the Frobenius $F \colon A \to A$ induces the identity map $\operatorname{Spec}(A) \to \operatorname{Spec}(A)$.

Problem 4. (a) Let $A = \mathbb{F}_p[T_1, \ldots, T_n]$. Prove that $F \colon A \to A$ is finite and flat.

(b) Let $A = \mathbb{F}_p[T^2, T^3]$. Prove that $F \colon A \to A$ is finite but not flat.

Kähler differentials

Problem 5. Let $A = k[X,Y,Z]/(X^2+Y^2+Z^2-1)$ where k is a field of characteristic $\neq 2$. Consider the module of Kähler differentials $M = \Omega^1_{A/k}$. Show that M is locally free (or equivalently projective) and describe it as a direct summand of A^3.

$\star$ **Problem 6** (Algebraic "hairy ball theorem")**.** Show that for $k = \mathbb{R}$, the module $M = \Omega^1_{A/k}$ in the above example is not free. Can you treat other base fields k as well?