MATH550 Commutative Algebra — Problem Set 3

Due Nov 11, 2025.

Problem 1. Let $A = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$. Show that $M(x) = M \otimes_A \kappa(x)$ is zero for every $x \in \mathrm{Spec}(A)$.

Problem 2. Let A be a ring and M a finitely presented A-module. (Recall that this means that there **exists** a presentation

$$(\star) \qquad A^m \xrightarrow{\alpha} A^n \xrightarrow{\pi} M \longrightarrow 0,$$

or equivalently that there exists a surjection $A^n \to M$ whose kernel is finitely generated.) Show that the kernel of **every** surjection $A^k \to M$ is finitely generated.

Hint: First show that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence and M' and M'' are finitely generated, then so is M. Next, build a map from the given finite presentation $(\star)$ to the sequence $0 \to K \to A^k \to M \to 0$ and apply the Snake lemma.

Problem 3. Let A be a domain. Recall that an A-module M is **torsion-free** if for every nonzero $a \in A$, the map $a\colon M \to M$ is injective.

(a) Show that every flat A-module is torsion-free.

(b) Suppose that A is a principal ideal domain. Show that every torsion-free A-module is flat.

Hint: Treat finitely generated modules first. Try to reduce to this case using filtered colimits.

(c) Give an example of a domain A and a torsion-free A-module which is not flat.

Problem 4. Find an example of a homomorphism between domains $A \to B$ and a torsion-free A-module M such that $M \otimes_A B$ is not torsion-free.

Problem 5 (Jelisiejew 11.2)**.**

(a) Let I be a finitely generated ideal in a local ring $(A, \mathfrak{m})$. Prove that if $I = I^2$, then $I = A$ or $I = 0$.

(b) Let $A = C(\mathbb{R}, \mathbb{R})$ be the ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and $\mathfrak{m} = \{f \in A : f(0) = 0\}$. Prove that $\mathfrak{m} = \mathfrak{m}^2$ and conclude that the ideal $\mathfrak{m}$ is not finitely generated.

$\star$ **Problem 6** ($\mathbb{Z}^{\mathbb{N}}$ is not free)**.** Let $M = \mathbb{Z}^{\mathbb{N}}$ be the group of integer-valued sequences. For a subset $S \subseteq \mathbb{N}$, denote by $\pi_S\colon M \to \mathbb{Z}^S$ the projection map. Show that every homomorphism

$$\varphi\colon M \longrightarrow \mathbb{Z}$$

admits a factorization

$$\begin{tikzcd} M \arrow[rr, "\varphi"] \arrow[dr, "\pi_S"'] & & \mathbb{Z} \\ & \mathbb{Z}^S \arrow[ur, "\psi"'] & \end{tikzcd}$$

for a finite subset $S \subseteq \mathbb{N}$. Deduce that the $\mathbb{Z}$-module M is not free (even though it is flat).

Hint: Reduce to the case $\phi(e_n) > 0$ for every $n \geq 0$, where $e_n \in M$ is the element $e_n(m) = 1$ if $m = n$ and 0 otherwise. Then consider an element $x \in M$ with $x(n) = 2^{a_n}$ for a rapidly growing sequence (a_n).