

MATH550 Commutative Algebra — Problem Set 2

Due Oct 28, 2025.

Warm-up

Problem 1. Consider the following rings:

$$A_1 = \mathbb{C}[x]$$

$$A_2 = \mathbb{C}[x, x^{-1}]$$

$$A_3 = \mathbb{C}[x^2, x^3]$$

$$A_4 = \mathbb{C}[x, y]$$

$$A_5 = \mathbb{C}[x]/(x^6)$$

$$A_6 = \mathbb{C}[x, y]/(x^2, y^3)$$

$$A_7 = \mathbb{C}[x, y]/(xy)$$

$$A_8 = \mathbb{C}[x, y]/(xy^2)$$

$$A_9 = \mathbb{C}[x, y]/(xy, y^2)$$

Prove that the rings A_i and A_j are not isomorphic as $\mathbb{C}$ -algebras for $i \neq j$. Can you show they are also not isomorphic as rings?

Problems about Spec and irreducible subsets

Problem 2 (Atiyah–Macdonald 1.18). Let $X = \operatorname{Spec}(A)$ and $x, y \in X$. Show that

- (a) x is a closed point (i.e. $\{x\} \subseteq X$ is a closed subset) if and only if the corresponding prime ideal $\mathfrak{p}_x \subseteq A$ is maximal;
- (b) the closure $\overline{\{x\}}$ equals $V(\mathfrak{p}_x)$;
- (c) $y \in \overline{\{x\}}$ if and only if $\mathfrak{p}_x \subseteq \mathfrak{p}_y$.

Definition. Let X be a topological space. We say that X is **irreducible** if for every pair of closed subsets $Y_0, Y_1 \subseteq X$ such that $X = Y_0 \cup Y_1$, we have $X = Y_0$ or $X = Y_1$.

Problem 3 (Atiyah–Macdonald 1.19). Show that $\operatorname{Spec}(A)$ is irreducible if and only if the nilradical $\sqrt{0} \subseteq A$ is a prime ideal.

Problem 4 (Atiyah–Macdonald 1.20). Let X be a topological space.

1. If $Y \subseteq X$ is an irreducible subspace, then its closure $\overline{Y}$ is irreducible.
2. Every irreducible subspace of X is contained in a maximal irreducible subspace.
3. The maximal irreducible subspaces of X are closed and cover X . They are called the **irreducible components** of X . What are the irreducible components of a Hausdorff space?
4. If A is a ring and $X = \operatorname{Spec}(A)$, then the irreducible components of X are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A .

Spectral spaces

We work with the following definition.

Definition. Let X be a topological space. We say that X is

1. **sober** if every irreducible closed subset $Y \subseteq X$ has a unique generic point, i.e. there exists a unique $\eta_Y \in Y$ such that $Y = \overline{\{\eta_Y\}}$;
2. **quasi-compact** (qc) if every open cover has a finite subcover;
3. **spectral** if it is sober, qc, if the intersection of every two qc open subsets of X is qc, and if its qc open subsets form a base for the topology.

Problem 5. Let A be a commutative ring. Prove that $\operatorname{Spec}(A)$ is a spectral space.

Extra credit

The deadline does not apply to the bonus problem.

★ **Problem 6** (Converse to Problem 5). Read the relevant part of M. Hochster *Prime ideal structure in commutative rings*¹ Trans. AMS, 142 (1969), pp. 43-60, and explain to me² the proof of Hochster's theorem that *every spectral space is homeomorphic to $\text{Spec}(A)$ for some commutative ring A .*

¹<https://doi.org/10.2307/1995344> • <https://www.jstor.org/stable/1995344>

²During office hours or by appointment.