MATH550 Commutative Algebra — Problem Set 1

Due Oct 21, 2025.

Comments:

- All rings are commutative and unital.
- Send your solutions in PDF form to `pachinger@kse.org.ua`
- Solutions to the exercises in Atiyah–Macdonald are too easy to find online. I encourage you not to look for them — the point of the homework is to get familiar with the new notions, which is difficult without trying to solve some homework problems.

Problem 1. Let $\mathfrak{p}$ and $\mathfrak{q}$ be two prime ideals in a ring A such that neither $\mathfrak{p} \subseteq \mathfrak{q}$ nor $\mathfrak{q} \subseteq \mathfrak{p}$. Show that the ideal $\mathfrak{p} \cap \mathfrak{q}$ is not prime.

Problem 2 (Atiyah–Macdonald 1.11)**.** A ring A is Boolean if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that

i) $2x = 0$ for all $x \in A$;

ii) every prime ideal $\mathfrak{p}$ is maximal, and $A/\mathfrak{p}$ is a field with two elements;

iii) every finitely generated ideal in A is principal.

Problem 3. Show that a ring A is a domain if and only if it admits an injective homomorphism $A \hookrightarrow K$ into a field K. Show that A is reduced (has no nonzero nilpotent elements) if and only if it admits an injective homomorphism $A \hookrightarrow \prod_{\alpha \in I} K_\alpha$ into a product of (possibly infinitely many) fields.

For the next two exercises you might want to recall Tietze's extension theorem.

Problem 4 (see Atiyah–Macdonald 1.26)**.** Let X be a compact Hausdorff space and let $A = C(X, \mathbb{R})$ be the ring of continuous functions on X. For $x \in X$, let $\mathfrak{m}_x \subseteq A$ be the set of all $f \in A$ such that $f(x) = 0$. Show that $\mathfrak{m}_x$ is a maximal ideal in A, and that every maximal ideal $\mathfrak{m} \subseteq A$ is of the form $\mathfrak{m}_x$ for a unique $x \in X$.

Problem 5. Let again X be a compact Hausdorff space and let $A = C(X, \mathbb{R})$ be the ring of continuous functions on X. Show that every prime ideal in A is contained in a unique maximal ideal.