MATH550 Commutative Algebra — Practice exam 1

Theory

Problem 1 (10 pts.). State and prove the Artin–Tate lemma.

Problem 2 (10 pts.). Let $I, J \subseteq A$ be ideals in a ring A. Prove that

$$V(I) \cup V(J) = V(I \cap J) = V(I \cdot J).$$

Practice

Problem 3 (10 pts.). Let A be a ring and let M be a finitely generated A-module. Prove that the set

$$\operatorname{supp}(M) = \{x \in \operatorname{Spec}(A) : M_x \neq 0\}$$

is a closed subset of $\operatorname{Spec}(A)$. Here $M_x = M \otimes_k A_{\mathfrak{p}_x}$.

Problem 4 (10 pts.). Let $A = k[X,Y,Z]/(X^2 - YZ)$, let $I = (X,Y)$, and let $\mathfrak{m} = (X,Y,Z)$. Compute $I \otimes_A A/\mathfrak{m}$. Show that I is not principal. Calculate $\Omega^1_{A/k}$ and show it is not free.

Problem 5 (10 pts.). Let V be a valuation ring and let $\mathfrak{p} \subseteq V$ be a prime ideal. Prove that both $V_{\mathfrak{p}}$ and $V/\mathfrak{p}$ are valuation rings.

MATH550 Commutative Algebra — Practice exam 2

Theory

Problem 1 (10 pts.). Define constructible subsets of $\operatorname{Spec}(A)$ and state Chevalley's theorem.

Problem 2 (10 pts.). Show that an element $x \in A$ which belongs to every prime ideal of A is nilpotent.

Practice

Problem 3 (10 pts.). Calculate the following tensor products of algebras:

(a) $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$;

(b) $\mathbb{C}[T] \otimes_{\mathbb{C}[T^2]} \mathbb{C}[T]$ and show it is not a domain.

Problem 4 (10 pts.). Let $X = \operatorname{Spec}(k[T_1, T_2, \ldots])$ be the spectrum of the polynomial ring in infinitely many variables over a field k. Find an open subset $U \subseteq X$ which is not quasi-compact.

Problem 5 (10 pts.). Let k be a field and let

$$A = \{f \in k[X,Y] : f(X,0) \in k\} = k \oplus Yk[X,Y]$$

be the ring of polynomials in X and Y whose image in $k[X,Y]/(Y) = k[X]$ is a constant polynomial. Prove that the ring A is not Noetherian.