MATH550 Commutative Algebra — List of material

This list is not exhaustive, and it might change later, but not by much.

Definitions

1. Unit, nilpotent element, idempotent element. Ideal, prime ideal, maximal ideal. Radical ideal, the radical of an ideal, nilradical. Localization with respect to a multiplicative system. Finite type, and finitely presented maps of rings.
2. Spectrum of a commutative ring and its topology.
3. Module over a ring. Tensor product, extension of scalars, coproduct of algebras. Localization of a module. Flat module. Kähler differentials.
4. Integral elements and integral closure. Finite and integral morphisms of rings. Valuation rings and valuations.
5. Local rings and local homomorphisms.
6. Noetherian rings.
7. Graded rings, homogeneous ideals.

Results

You should know all of the statements and at least the sketch of proof (unless stated otherwise).

1. The nilradical is the intersection of all primes.
2. The spectrum of a ring is quasi-compact.
3. Tensor product is right-exact.
4. Structure of modules over PID (without proof).
5. A ring is local if and only if non-units form an ideal.
6. Nakayama's lemma.
7. A finitely presented flat module over a local ring is free.
8. Going-up: $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is closed if $A \to B$ is integral.
9. Artin–Tate lemma.
10. Hilbert's basis theorem.
11. Nullstellensatz (several forms), without proof.
12. Noether normalization lemma.
13. Chevalley's theorem (without proof).
14. Equivalent conditions for separable field extensions (proof of at least three of the implications).
15. Finiteness of integral closure (without proof).
16. Criterion for a graded ring to be Noetherian.

Methods

1. Computing tensor product of modules using a module presentation of one of the factors.
2. Computing the tensor product of algebras using an algebra presentation of one of the factors.
3. Computing $\Omega^1_{A/k}$ for $A = k[T_1, \ldots, T_n]/(f_1, \ldots, f_r)$.
4. Computing the fiber of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ above x as $\mathrm{Spec}(B \otimes_A \kappa(x))$.