MATH550 Commutative Algebra — Final exam (corrected) + solutions December 18, 2025

Theory

Problem 1 (10 pts.)**.** Let A be a ring. Prove that $\mathrm{Spec}(A)$ is quasi-compact, and that if A is Noetherian, then every open subset of $\mathrm{Spec}(A)$ is quasi-compact.

Problem 2 (10 pts.)**.** Give the definition of a local ring. Prove that a ring is local if and only if it is nonzero and the set of its non-invertible elements is closed under addition.

Practice

Choose three out of four.

Problem 3 (10 pts.)**.** Consider $K = \mathbb{Q}(\sqrt{7})$, and let $A \subseteq K$ be the integral closure of $\mathbb{Z}$ in K.

(a) Prove that an element $w = a + b\sqrt{7}$ of K (where $a, b \in \mathbb{Q}$) belongs to A if and only if $a, b \in \mathbb{Z}$.

(b) For which primes p does the ring A/pA have nonzero nilpotent elements?

(c) Compute $\Omega^1_{A/\mathbb{Z}}$ and decompose it as a direct sum of cyclic groups.

Solution. (a) Let $\bar{w} = a - b\sqrt{7}$, so that $w + \bar{w} = 2a$ and $w\bar{w} = a^2 - 7b^2 2$. Then w satisfies the quadratic equation

$$(T - w)(T - \bar{w}) = T^2 - 2aT + (a^2 - 7b^2).$$

Thus, if $a, b \in \mathbb{Z}$, then this equation has integer coefficients, and w is therefore integral over $\mathbb{Z}$. Conversely, suppose that w is integral over $\mathbb{Z}$, then either w is rational (i.e. $w = \bar{w} = a$), in which case it is an integer, or w is irrational, in which case the above equation is the minimal polynomial of w, and hence its coefficients are integers. It remains to show that if $2a$ and $a^2 - 7b^2$ are integers, then so are a and b. We have $a = a'/2$ with $a' \in \mathbb{Z}$, and $a^2 - 7b^2 \in \mathbb{Z}$ implies that $b = b'/2$ with $b' \in \mathbb{Z}$. Moreover, the integer $(a')^2 - 7(b')^2$ is divisible by 4. Since 7 is not a square modulo 4, this shows that a' and b' are both even, so that $a, b \in \mathbb{Z}$.

(b) By (a) the map $\mathbb{Z}[X]/(X^2 - 7) \to A$ sending X to $\sqrt{7}$ is an isomorphism. Therefore A/pA is isomorphic to $\mathbb{F}_p[X]/(X^2 - 7)$. If $p \neq 2, 7$, the polynomial $X^2 - 7$ is either irreducible (so that A/pA is a field) or has two distinct roots (so that $A/pA \simeq \mathbb{F}_p \times \mathbb{F}_p$), and in either case A/pA is reduced. For $p = 2$ and $p = 7$ we have

$$A/2A = \mathbb{F}_2[X]/(X^2 - 7) = \mathbb{F}_2[X]/((X-1)^2), \qquad A/7A = \mathbb{F}_7[X]/(X^2 - 7) = \mathbb{F}_7[X]/(X^2)$$

which are both non-reduced.

(c) Using the same presentation as in (b) we write

$$\Omega^1_{A/\mathbb{Z}} = A \cdot dX / A \cdot (2X dX) \simeq A/(2\sqrt{7}).$$

Now $w = a + b\sqrt{7}$ belongs to the ideal $(2\sqrt{7})$ if and only if 14 divides a and 2 divides b. Thus

$$\Omega^1_{A/\mathbb{Z}} \simeq \mathbb{Z}/14 \oplus \mathbb{Z}/2 \simeq \mathbb{Z}/7 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$ □

Problem 4 (10 pts.)**.** Let A be a Noetherian domain and let M be a finitely generated A-module. Show that there exists a nonzero $f \in A$ such that $M[f^{-1}]$ is a free $A[f^{-1}]$-module.

Proof. Let K be the fraction field of A and let $x_1, \ldots, x_n \in M \otimes_A K$ be a basis of the K-vector space $M \otimes_A K$. Since $M \otimes_A K = M[S^{-1}]$ where $S = A \setminus \{0\}$, we can write $x_i = m_i/f_i$ with $m_i \in M$ and $f_i \in A \setminus \{0\}$. Let $g = f_1 \cdots f_n \neq 0$, and replace A with $A[g^{-1}]$ and M with $M[g^{-1}]$, so that $x_i = f_i^{-1} m_i$ are in M. Consider the A-module map $A^n \to M$ sending the i-th basis vector e_i to x_i, and let L and Q be its kernel and cokernel. They are both finitely generated (Q being a quotient of M and L since A is Noetherian). Tensoring the exact sequence

$$0 \longrightarrow L \longrightarrow A^n \longrightarrow M \longrightarrow Q \longrightarrow 0$$

with K, we obtain an exact sequence (since K is flat over A, being a localization of A)

$$0 \longrightarrow L \otimes_A K \longrightarrow K^n \longrightarrow M \otimes_A K \longrightarrow Q \otimes_A K \longrightarrow 0$$

in which the middle map is an isomorphism (as $x_1, \ldots, x_n$ are a basis of $M \otimes_A K$). It follows that

$$L \otimes_A K = 0 = Q \otimes_A K.$$

It remains to show that if N is a finitely generated module over a domain A with fraction field K such that $N \otimes_A K = 0$, then $N[h^{-1}] = 0$ for some $h \in A \setminus \{0\}$. To show this, let $y_1, \ldots, y_r$ generate N. Thus the elements $y_i/1$ of $N \otimes_A K = N[(A \setminus \{0\})^{-1}]$ are zero, which means that $h_i y_i = 0$ for some nonzero $h_i \in A$. Then $h = h_1 \cdots h_n$ does the job.

To finish, we apply this observation to $N = L$ and $N = Q$, obtaining h and h', and setting $f = hh'$ we obtain $N[f^{-1}] = 0 = Q[f^{-1}]$. Tensoring our exact sequence with $A[f^{-1}]$ we obtain that $A[f^{-1}]^n \to M[f^{-1}]$ is an isomorphism, and $M[f^{-1}]$ is free. □

Problem 5 (10 pts.)**.** Let $A = \mathbb{C}[X,Y]$ and $B = \mathbb{C}[U,V]$. Consider the $\mathbb{C}$-algebra map $A \to B$ sending X to U and Y to UV.

(a) Find a presentation for the B-module $\Omega^1_{B/A}$. Is it torsion-free?

(b) Find a presentation for the $\mathbb{C}$-algebra $C = B \otimes_A B$. Is it a domain?

Solution. (a) We begin with a presentation of B over A:

$$B = \mathbb{C}[X,Y,U,V]/(X-U, Y-UV) = A[U,V]/(X-U, Y-UV).$$

Then $\Omega^1_{B/A}$ is generated by dU and dV modulo the relations $d(X-U) = dU = 0$ and $d(Y-UV) = -UdV - VdU = -UdV = 0$. So we have one generator dV and one relation $UdV = 0$. Thus

$$\Omega^1_{B/A} \simeq B/(V) = \mathbb{C}[U].$$

This module is not torsion-free, as it is nonzero and annihilated by V.

(b) We use the above presentation for B over A, then "the same" formula gives a presentation of $C = B \otimes_A B$ over B, namely

$$C = B[U',V']/(X-U', Y-U'V')$$

(where we use the primes to distinguish the two factors B). Expanding $B = \mathbb{C}[U,V]$ and using $X = U$, $Y = UV$ this yields:

$$C = \mathbb{C}[U,V,U',V']/(U-U', UV-U'V') = \mathbb{C}[U,V,V']/(U(V-V')).$$

We can simplify this even further by substituting $V - V' = W$:

$$C = \mathbb{C}[U,V,W]/(UW).$$

This is not a domain since U, W are both nonzero while $UW = 0$. □

Problem 6 (10 pts.)**.** Let $A = \bigoplus_{n \geq 0} A_n$ be a Noetherian graded ring and let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finitely generated graded A-module. Prove that M_n is a finitely generated A_0-module for every $n \in \mathbb{Z}$.

Solution. Let $f_1, \ldots, f_r \in A_+$ be a set of generators of the ideal $A_+ = \bigoplus_{n>0} A_n$. Decomposing each f_i into its homogeneous components we may assume that the f_i are homogeneous, say $f_i \in A_{d_i}$ with $d_i > 0$. Recall (from the characterization of Noetherian graded rings) that this implies that the A_0-algebra map

$$A_0[T_1, \ldots, T_r] \longrightarrow A, \qquad T_i \mapsto f_i$$

is a surjection of graded rings, where the source is graded by placing T_i in degree d_i.

Similarly, let $m_1, \ldots, m_s \in M$ generate M over A, and again we may assume that they are homogeneous, say $m_j \in M_{e_j}$. For $n \in \mathbb{Z}$, consider the elements

$$m_{a,j} = f_1^{a_1} \cdots f_r^{a_r} \cdot m_j \in M_n$$

for $a = (a_1, \dots, a_r) \in \mathbb{N}^r$ and $j \in \{1, \dots, s\}$ such that

$$n = \sum_{i=1}^{r} a_i d_i + e_j.$$

For a given n, there are only finitely such choices of a and j. We claim that the elements $m_{a,j}$ generate M_n as a module over A_0. Indeed, let $m \in M_n$. Since the m_j generate M, we can write

$$m = \sum_{j=1}^{s} g_j m_j$$

for some $g_j \in A$. We may assume $g_j \in A_{n-e_j}$ is homogeneous. Thus each g_j is a polynomial in $f_1, \dots, f_r$ with coefficients in A_0. Decompose g_j into a sum of monomials

$$g_j = \sum_a u_{a,j} f_1^{a_1} \dots f_r^{a_r}$$

where the sum is over all $a = (a_1, \dots, a_r) \in \mathbb{N}^r$ with $n - e_j = \sum a_i d_i$ and $u_{a,j} \in A_0$. This shows that

$$m = \sum_{j=1}^{s} g_j m_j = \sum_{j=1}^{s} \sum_a u_{a,j} f_1^{a_1} \dots f_r^{a_r} m_j = \sum_{j=1}^{s} \sum_a u_{a,j} m_{a,j}. \qquad \square$$