MATH551 Algebraic Geometry — Problem Set 3

Due Mar 1, 2026.

Algebraic groups

Problem 1. Let G be an algebraic group. Show that G is

(a) separated,

(b) non-singular.

Hint for (a): Use the multiplication map $\mu \colon G \times G \to G$ to describe the diagonal.

Problem 2. Let $G = \mathbb{G}_m^a$ and $H = \mathbb{G}_m^b$.

(a) Prove that every map $f \colon G \to H$ with $f(1) = 1$ is a group homomorphism.

(b) Prove that group homomorphisms $G \to H$ form a group isomorphic to $\mathrm{Hom}(\mathbb{Z}^b, \mathbb{Z}^a)$.

Blow-ups

Problem 3 (cf. Hartshorne Ex. I 5.7)**.** Let $Y \subseteq \mathbb{P}^2$ be a nonsingular projective plane curve defined by the homogeneous equation $f(x,y,z) = 0$ (for $f \in k[x,y,z]$ square-free, homogeneous of degree $d > 1$). Let $X = V(f) \subseteq \mathbb{A}^3$ be the "affine cone" of Y.

(a) Show that the origin $0 \in \mathbb{A}^3$ is the unique singular point of X.

(b) Show that the strict transform $\tilde{X}$ in the blowup of $\mathbb{A}^3$ at the origin is non-singular.

(c) Show that the tangent cone $\tilde{X} \cap \pi^{-1}(0) \subseteq \mathbb{P}^2$ is equal to Y.

Sheaves

Problem 4 (Flasque/flabby sheaves, cf. Hartshorne Ex. II 1.16)**.** A sheaf $\mathcal{F}$ on a topological space X is called *flasque* (a.k.a. *flabby*) if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective.

(a) Show that a constant sheaf on an irreducible topological space is flasque.

(b) Suppose that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves and that $\mathcal{F}''$ is flasque. Show that for every open $U \subseteq X$, the sequence

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U) \longrightarrow 0$$

is exact.

(c) Let $f \colon Y \to X$ be a continuous map and let $\mathcal{F}$ be a flasque sheaf on Y. Show that the sheaf $f_*\mathcal{F}$ on X is flasque.

(d) For any sheaf $\mathcal{F}$, the sheaf $D(\mathcal{F})$ of discontinuous sections of $\mathcal{F}$ is flasque.

Problem 5 (cf. Hartshorne Ex. II 1.21)**.** Let X be an algebraic set and let $\mathcal{O}_X$ be its sheaf of regular functions. Let $Y \subseteq X$ be a closed subset and let $i \colon Y \to X$ be the inclusion.

(a) For each open subset $U \subseteq X$, let $\mathcal{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathcal{I}_Y(U)$ is a sheaf. It is called the *sheaf of ideals* $\mathcal{I}_Y$ of Y, and it is a subsheaf of the sheaf of rings $\mathcal{O}_X$.

(b) Show that the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Y$ is isomorphic to $i_*\mathcal{O}_Y$, so that we have a short exact sequence

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y \longrightarrow 0.$$

(c) Consider the case $X = \mathbb{P}^1$ and $Y = \{x, y\}$ for distinct points $x, y \in X$. Show that $\mathcal{O}_X(X) \to (i_*\mathcal{O}_Y)(X)$ is not surjective (even though $\mathcal{O}_X \to i_*\mathcal{O}_Y$ is surjective by (b)).