MATH551 Algebraic Geometry — Problem Set 2

Due Feb 22, 2026.

Problem 1. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set endowed with an action of a finite group G. Let $A = \mathcal{O}(X)$ be the corresponding algebra, so that G acts on A by k-algebra automorphisms. Let

$$B = A^G = \{f \in A \, : \, g * f = f \text{ for all } g \in G\} \subseteq A$$

be the subring of invariants. Last semester, we deduced from the Artin–Tate lemma that B is a finitely generated k-algebra. This allows us to consider $Y = \mathrm{MSpec}(B)$, the affine algebraic set with $\mathcal{O}(Y) = B$. Let $\pi \colon X \to Y$ be the morphism induced by the inclusion $B \hookrightarrow A$. Show that

(a) $Y = X/G$ is the orbit space (that is, $\pi(x) = \pi'(x)$ iff $x = g(x')$ for some $g \in G$);

(b) for every morphism $f \colon X \to Z$ to an affine algebraic set Z such that $f \circ g = f$ for every $g \in G$ there exists a unique $\bar{f} \colon Y \to Z$ such that $f = \bar{f} \circ \pi$.

Hint for (a): First show that for any two disjoint closed subsets $Z_0, Z_1 \subseteq X$ there exists an $f \in A$ such that $f(z) = 0$ for all $z \in Z_0$ and $f(z) = 1$ for all $z \in Z_1$. Apply this to a pair of G-orbits, then use the G-action to obtain elements of A^G.

Problem 2. Let $k = \mathbb{C}$. For which values of the parameter $\lambda \in k$ is the hypersurface $X \subseteq \mathbb{P}^n$ defined by

$$\lambda(X_0^{n+1} + \cdots + X_n^{n+1}) = (n+1)X_0 \cdots X_n$$

nonsingular?

Info: This is the Dwork family of Calabi–Yau hypersurfaces of dimension $n-1$. The case $n = 3$ (quintic threefolds) has been extensively studied in the context of mirror symmetry.

Problem 3. Let X be a variety and let $X_1, \ldots, X_n \subseteq X$ be constructible subsets such that $X = X_1 \cup \cdots \cup X_n$. Show that one of the X_i contains a non-empty open subset of X.

Problem 4. Let $X = V(f) \subseteq \mathbb{P}^2$ be a conic (so $f \in k[X,Y,Z]$ is an irreducible homogeneous polynomial of degree two). Show that $X \simeq \mathbb{P}^1$.

Problem 5. Assume that $\mathrm{char}(k) \neq 2$. Let $X = V(f) \subseteq \mathbb{P}^3$ be an irreducible quadric surface (so $f \in k[T_0, \ldots, T_3]$ is an irreducible homogeneous polynomial of degree two). Show that X is rational (i.e., has a non-empty open subset isomorphic to an open subset of $\mathbb{P}^2$).

Hint: first show that up to a linear change of coordinates, f can be put in the form $T_0T_1 - T_2T_3$ or $T_0T_1 - T_2^2$.

$\star$ **Problem 6.** Suppose that k is uncountable. Let X be a variety and let $X_n \subseteq X$ ($n \geq 1$) be a sequence of constructible subsets such that $X = \bigcup_{n=1}^{\infty} X_n$. Show that one of the X_n contains a non-empty open subset of X.