9. Lecture 9 (Feb 19): More on schemes

Recommended reading: Hartshorne II.2–3

9.1. Exactness of the Čech and Amitsur complexes

In Lecture 8, in the course of constructing the structure sheaf on an affine scheme $\mathrm{Spec}(A)$, we gave a slightly tedious proof that that for $f_1, \dots, f_r \in A$ generating the unit ideal, the sequence

$$0 \longrightarrow A \longrightarrow \prod_{i=1}^r A[f_i^{-1}] \longrightarrow \prod_{i,j=1}^r A[(f_i f_j)^{-1}]$$

is exact (see Lemma 8.2.1). The following result generalizes this in two ways: we replace A and its localizations by an A-module M and its localizations, and we extend the sequence to the right. Curiously, the proof does not involve tedious calculations but rather general ideas surrounding the notion of a faithfully flat morphism (introduced below).

Lemma 9.1.1. *Let A be a ring, let $f_1, \dots, f_r \in A$ with $A = (f_1, \dots, f_r)$, and let M be an A-module. Then, the following complex (called the augmented Čech complex of M) is exact*

$$0 \longrightarrow M \longrightarrow \prod_{i_0} M[f_{i_0}^{-1}] \longrightarrow \prod_{i_0,i_1} M[(f_{i_0} f_{i_1})^{-1}] \longrightarrow \cdots$$

(Here M is put in degree -1 and the differential

$$d\colon \prod_{i_0,\dots,i_p} M[(f_{i_0} \cdots f_{i_p})^{-1}] \longrightarrow \prod_{i_0,\dots,i_p,i_{p+1}} M[(f_{i_0} \cdots f_{i_p} f_{i_{p+1}})^{-1}]$$

is given by the formula

$$d((m_{i_0 \dots i_p}))_{j_0 \dots j_{p+1}} = \sum_{k=0}^{p+1} (-1)^k m_{j_0 \dots \widehat{j_k} \dots j_{p+1}}$$

where $\widehat{j_k}$ means that j_k is omitted.)

Observations. We precede the proof with a sequence of observations.

(1) For a ring A and $f, g \in A$, we have

$$A[f^{-1}] \otimes_A A[g^{-1}] \simeq A[(fg)^{-1}]$$

(universal properties give the most immediate proof of this.)

(2) Let $B = \prod_{i=1} A[f_i^{-1}]$. Then the p-th term in the augmented Čech complex of M can be written as

$$\prod_{i_0,\dots,i_p} M[(f_{i_0} \cdots f_{i_p})^{-1}] \simeq \underbrace{B \otimes B \otimes \cdots \otimes B}_{p+1 \text{ times}} \otimes M$$

(all $\otimes$ are $\otimes_A$). The differentials in this complex are given by the formula

$$d(b_0 \otimes \cdots \otimes b_p \otimes m) = \sum_{k=0}^{p} (-1)^k b_0 \otimes \cdots \otimes 1 \otimes \cdots \otimes b_p \otimes m \quad (\text{introduce } 1\otimes \text{ before } b_k).$$

This motivates us to make the following definition: let $A \to B$ be a map of rings and let M be an A-module. Then the augmented **Amitsur complex** of M is the complex

$$C_a^\bullet(B/A, M) = \left[\, 0 \longrightarrow M \longrightarrow B \otimes M \longrightarrow B \otimes B \otimes M \longrightarrow \cdots \,\right]$$

with M in degree -1 and differential given by the formula above. Thus the augmented Čech complex is the augmented Amitsur complex for $B = \prod_{i=1}^r A[f_i^{-1}]$.

(3) The formation of the augmented Amitsur complex is compatible with extension of scalars in the following way. Let $A \to B$ be a map of rings, let M be an A-module, and let $A \to A'$ be a map of rings. Let $B' = A' \otimes_A B$ and $M' = A' \otimes_A M$. Then

$$C_a^\bullet(B/A,M) \otimes_A A' \simeq C_a^\bullet(B'/A',M').$$

(4) For the next observation we introduce the following definition. A map of rings $A \to B$ is **faithfully flat** if B is flat over A and the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. One verifies easily that

- a flat map $A \to B$ is faithfully flat if and only if for every A-module M, we have $M = 0 \Leftrightarrow M \otimes B = 0$;
- a map $A \to B$ is faithfully flat if and only if for every complex of A-modules $C^\bullet$, we have $C^\bullet$ is exact $\Leftrightarrow C^\bullet \otimes B$ is exact.

Then the observation is that $A \to B = \prod_{i=1}^r A[f_i^{-1}]$ is faithfully flat. Indeed, every localization is flat, so $A \to B$ is flat, and it is faithfully flat since the map

$$\mathrm{Spec}(B) = \coprod_{i=1}^{r} \mathrm{Spec}(A[f_i^{-1}]) = \coprod_{i=1}^{r} D(f_i) \longrightarrow \mathrm{Spec}(A)$$

is surjective, as $A = (f_1, \ldots, f_r)$ is equivalent to $\mathrm{Spec}(A) = \bigcup_{i=1}^r D(f_i)$.

(5) Finally, let $f: A \to B$ be a map of rings admitting a section $s: B \to A$, i.e. a ring homomorphism satisfying $s \circ f = \mathrm{id}_A$. We claim that for every A-module M, the augmented Amitsur complex $C_a^\bullet(B/A,M)$ is exact. To see this, we introduce the maps

$$h^p: C_a^p(B/A;M) = \underbrace{B \otimes B \otimes \cdots \otimes B}_{p+1 \text{ times}} \otimes M \longrightarrow \underbrace{B \otimes B \otimes \cdots \otimes B}_{p \text{ times}} \otimes M = C_a^{p-1}(B/A;M),$$

$$b_0 \otimes \cdots \otimes b_p \otimes m = b_0 \otimes \cdots \otimes b_{p-1} \otimes s(b_p)m.$$

We then verify easily that on $C_a^p(B/A;M)$ we have

$$d^{p-1}h^p + h^{p+1}d^p = \mathrm{id},$$

i.e. that h^p define a nullhomotopy. This immediately implies exactness: if $x \in \ker(d^p)$, then the above equality yields

$$x = d^{p-1}h^p(x) + h^{p+1}d^p(x) = d^{p-1}h^p(x) \in \mathrm{im}(d^{p-1}).$$

We are now ready to state and prove the following strengthening of Lemma 9.1.1.

Lemma 9.1.2. *Let $A \to B$ be a faithfully flat map of rings and let M be an A-module. Then the augmented Amitsur complex of M*

$$C_a^\bullet(B/A,M) = \left[\, 0 \longrightarrow M \longrightarrow B \otimes M \longrightarrow B \otimes B \otimes M \longrightarrow \cdots \,\right]$$

is exact.

Proof. Since $A \to B$ is faithfully flat, it is enough to check that $C_a^\bullet(B/A,M) \otimes_A B$ is exact. By Observation (3), we have

$$C_a^\bullet(B/A,M) \otimes_A B \simeq C_a^\bullet(B \otimes_A B/B, B \otimes_A M).$$

The map $B \to B \otimes_A B$ admits a "tautological" (or "diagonal") section $s: B \otimes_A B \to B$ given by $s(b \otimes b') = bb'$. Thus Observation (5) implies that $C_a^\bullet(B \otimes_A B/B, B \otimes_A M)$ is exact. □

Remarks 9.1.3. 1. The word "augmented" refers to starting the complex at -1. The non-augmented version is

$$C^\bullet(B/A,M) = \left[\, B\otimes M \longrightarrow B\otimes B\otimes M \longrightarrow \cdots \,\right],$$

which is then a resolution of M.

2. Just as the exactness of the Čech complex in degrees $-1,0$ allows us to construct the structure sheaf $\mathcal{O}_X$ on $X = \mathrm{Spec}(A)$ by associating $A[f^{-1}]$ to $D(f)$, the variant with an A-module M allows us to construct a sheaf of $\mathcal{O}_X$-modules $\tilde{M}$ on $\mathcal{O}_X$ with $\tilde{M}(D(f)) = M[f^{-1}]$. Such sheaves of modules on X are called **quasi-coherent** and will be discussed in greater detail soon.

3. The exactness of the entire Čech complex, not just its beginning, will be used later to show that for a quasi-coherent sheaf $\mathcal{F}$ on an affine scheme X we have the vanishing of sheaf cohomology

$$H^p(X,\mathcal{F}) = 0 \qquad \text{for all } p>0.$$

4. The generality offered by Lemma 9.1.2 can be useful in many situations. Examples of faithfully flat maps are given by surjective étale maps, which are used to build the étale topology. Another (though related) example relates this complex to Galois cohomology:

   Let L/K be a finite Galois field extension and let $G = \mathrm{Gal}(L/K)$ be its Galois group. Then the map

$$L\otimes_K L \longrightarrow \prod_{\sigma\in G} L, \qquad x_0\otimes x_1 \mapsto (x_0\sigma(x_1))_{\sigma\in G}$$

   is an isomorphism of L-algebras. More generally, we have isomorphisms

$$\underbrace{L\otimes_K L\otimes_K \cdots \otimes_K L}_{p+1} \simeq \prod_{\sigma_1,\ldots,\sigma_p\in G} L, \qquad x_0\otimes\cdots\otimes x_p \quad\mapsto\quad x_0\cdot\sigma_1(x_1)\cdot(\sigma_1\sigma_2)(x_2)\cdot\ldots\cdot(\sigma_1\ldots\sigma_p)(x_p).$$

   Thus the Amitsur complex for $A = M = K$ and $B = L$ can be written in the form

$$0 \longrightarrow K \longrightarrow L \longrightarrow L^G \longrightarrow L^{G\times G} \longrightarrow \cdots$$

   which one can verify to be the (augmented) bar complex computing the group cohomology $H^*(G,L)$. We deduce Hilbert's Theorem 90:

$$H^p(G,L) = \begin{cases} K & p=0, \\ 0 & p>0. \end{cases}$$

9.2. Maps to affine schemes

We give a proof of the following result stated last time.

Proposition 9.2.1. *Let X be an affine scheme and let Y be a locally ringed space. Then the map*

$$\phi \mapsto \phi^* : \mathrm{Hom}_{\mathrm{LRS}}(Y,X) \longrightarrow \mathrm{Hom}_{\mathrm{Rings}}(\mathcal{O}(X),\mathcal{O}(Y))$$

is bijective.

Proof. Let $A = \mathcal{O}(X)$ so that $X = \operatorname{Spec}(A)$. We first show **injectivity**. Let $\phi, \psi: Y \to X$ be maps of locally ringed spaces such that $\phi^* = \psi{*}: A \to \mathcal{O}(Y)$. We first show that $\phi(y) = \psi(y)$ for every $y \in Y$. Suppose otherwise: $\phi(y) = x \neq x' = \psi(y)$. Thus the prime ideals corresponding to $x, x' \in X = \operatorname{Spec}(A)$ are distinct, and hence (up to swapping x and x') there exists $f \in A$ such that $f \in \mathfrak{p}_x \setminus \mathfrak{p}_{x'}$. Since $\mathcal{O}_{X,x} = A_{\mathfrak{p}_x}$ and $\mathcal{O}_{X,x'} = A_{\mathfrak{p}_{x'}}$, this means that $f(x) = 0$ and $f(x') \neq 0$. Since ϕ and ψ are maps of locally ringed spaces, the homomorphisms

$$\phi^*: A_{\mathfrak{p}_x} = \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y} \quad \text{and} \quad \psi^*: A_{\mathfrak{p}_{x'}} = \mathcal{O}_{X,x'} \to \mathcal{O}_{Y,y}$$

are local, and we conclude that $\phi^*(f)(y) = 0$ and $\psi^*(f)(y) \neq 0$, contradicting $\phi^*(f) = \psi^*(f)$.

Next, we need to show that for every open $U \subseteq X$, the two maps

$$\phi^*, \psi^*: \mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(\phi^{-1}(U)) = \mathcal{O}_Y(\psi^{-1}(U))$$

are equal. It suffices to check this for a basic open $U = D(f)$ for some $f \in A$. In this case we have

$$\phi^{-1}(U) = \psi^{-1}(U) = D(g), \qquad g = \phi^*(f) = \psi^*(f)$$

Both maps in question are the dotted arrow making the square below commute.

$$\begin{array}{ccc} A = \mathcal{O}_X(X) & \xrightarrow{\phi^* = \psi^*} & \mathcal{O}_Y(Y) \\ \downarrow & & \downarrow \\ A[f^{-1}] = \mathcal{O}_X(D(f)) & \dashrightarrow & \mathcal{O}_Y(D(g)) \end{array}$$

and hence they are equal (by universal property of the localization $A[f^{-1}]$).

We turn to proving **surjectivity**. Let $\phi^*: A \to \mathcal{O}(Y)$ be a ring homomorphism for which we seek to construct the corresponding map $Y \to \operatorname{Spec}(A)$. We first construct the map of sets: for $y \in Y$, let $\phi(y) \in X = \operatorname{Spec}(A)$ be the point corresponding to the prime ideal

$$\mathfrak{p} = \ker\left(A \xrightarrow{\phi^*} \mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_{Y,y} \longrightarrow \kappa(y)\right) = \{f \in A : \phi^*(f)(y) = 0\}.$$

We verify easily that $\phi^{-1}(D(f)) = D(\phi^*(f))$. We have shown previously that this is an open subset. Since the $D(f)$ form a base of the topology on $\operatorname{Spec}(A)$. this shows that ϕ is continuous.

In order to upgrade this to a map of locally ringed spaces, it suffices to construct compatible maps, for every $f \in A$,

$$\phi^*: A[f^{-1}] = \mathcal{O}_X(D(f)) \longrightarrow \mathcal{O}_Y(\phi^{-1}(D(f)) = \mathcal{O}_Y(D(\phi^* f)).$$

We have shown earlier that $\phi^* f$ is a unit in $\mathcal{O}_Y(D(\phi^* f))$. Thus (again, universal property of localization) there exists a unique A-algebra map above. We omit the easy verification that this gives a map of ringed spaces.

Finally, for $y \in Y$ and $x = \phi(y)$, the induced map on stalks is

$$A_{\mathfrak{p}_x} = \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y,y}$$

which is local by our definition of the prime ideal $\mathfrak{p}_x$. Thus ϕ is a map of locally ringed spaces inducing ϕ^* on global sections of structure sheaves. □

9.3. Properties of schemes and morphisms of schemes

Today, we just introduce the vocabulary.

Definition 9.3.1. Let X be a scheme.

(a) We denote by $|X|$ the underlying topological space of X. We say that X is **connected**, **quasi-compact**, or **irreducible** if $|X|$ has this property.

(b) We say that X is **reduced** if $\mathcal{O}_X(U)$ is reduced for every open $U \subseteq X$, and that X is **integral** if it is reduced and irreducible (equivalently, if $\mathcal{O}_X(U)$ is a domain for every non-empty open $U \subseteq X$).

(c) We say that X is **noetherian** if it admits a finite open cover $X = U_1 \cup \dots \cup U_r$ with $U_i \simeq \mathrm{Spec}(A_i)$ where the rings $A_1, \dots, A_r$ are noetherian.

For a ring A, by a **scheme over** A we mean a scheme X endowed with a map $X \to \mathrm{Spec}(A)$, or equivalently (by Proposition 9.2.1) one for which $\mathcal{O}_X(U)$ is an A-algebra for every $U \subseteq X$ (and such that the restriction maps are homomorphisms of A-algebras).

Definition 9.3.2. Let $f: Y \to X$ be a morphism of schemes. We say that f is

(a) **quasi-compact** if for every quasi-compact open $U \subseteq X$, the preimage $f^{-1}(U) \subseteq Y$ is quasi-compact;

(b) **affine** if for every affine open $U \subseteq X$, the preimage $f^{-1}(U) \subseteq Y$ is affine;

(c) an **open immersion** if it induces an isomorphism from Y onto an open subscheme of X;

(d) a **closed immersion** if it induces a homeomorphism from Y onto a closed subset of X and the map $\mathcal{O}_X \to f_* \mathcal{O}_Y$ is surjective;

(e) **finite** if it is affine and for every affine open $U \subseteq X$, the map $\mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U))$ is finite;

(f) **locally of finite type** if it is locally of the form $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ for a map of rings $A \to B$ such that B is a finitely generated A-algebra (more precisely: for every $y \in Y$ there exist affine open neighborhoods $y \in V$ and $f(y) \in U$ such that $f(V) \subseteq U$ and the map $f^*: \mathcal{O}_X(U) \to \mathcal{O}_Y(V)$ is of finite type);

(g) of **finite type** if it is locally of finite type and quasi-compact.

We issue the following warning. While an open immersion is uniquely determined by its image, there may be multiple closed immersions $Y \to X$ whose image is the same closed subset $Z \subseteq X$. For example, if $X = \mathrm{Spec}(A)$, then closed immersions $Y \to X$ are of the form $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$, and the corresponding closed subset is $Z = V(I)$. Thus knowing Z we only know the radical $\sqrt{I}$.