8. Lecture 8 (Feb 17): Schemes

Recommended reading: Hartshorne II.2–3

8.1. Sheaves on a base

Let X be a topological space. A **base** of the topology on X is a family of open subsets $\mathcal{B} \subseteq \mathbf{Opens}(X)$ such that every open of X is the union of a family of elements of $\mathcal{B}$.

Examples 8.1.1. (a) Let $X = \mathbb{R}^n$ and let $\mathcal{B}$ be the family of all open balls.

(b) Let X be an algebraic set and let $\mathcal{B}$ be the family of all affine opens.

(c) Let $X = \operatorname{Spec}(A)$ for a ring A and let $\mathcal{B}$ be the family of all open subsets of the form $D(f)$ for $f \in A$.

Definition 8.1.2 (Sheaf on $\mathcal{B}$)**.** Let X be a topological space and let $\mathcal{B}$ be a base of its topology. A **sheaf** on $\mathcal{B}$ is a functor $\mathcal{F}: \mathcal{B}^{\mathrm{op}} \to \mathbf{Ab}$ such that for every $U \in \mathcal{B}$ and every family $\{U_\alpha\}_{\alpha \in I}$ of elements of $\mathcal{B}$ such that $U = \bigcup_{\alpha \in I} U_\alpha$, and every family of sections $s_\alpha \in \mathcal{F}(U_\alpha)$ such that for every $\alpha, \beta \in I$ and every $V \in \mathcal{B}$ contained in $U_\alpha \cap U_\beta$ we have

$$s_\alpha|_V = s_\beta|_V$$

there exists a unique section $s \in \mathcal{F}(U)$ such that for every $\alpha \in I$ we have

$$s|_{U_\alpha} = s_\alpha.$$

Remark 8.1.3. Suppose that $\mathcal{B}$ is closed under intersections (as is the case in Example 8.1.1(b) (if X is separated) and Example 8.1.1(c). In this case, we can take $V = U_\alpha \cap U_\beta$ in the above definition.

Proposition 8.1.4. *Let X be a topological space and let $\mathcal{B}$ be a base of its topology. Then the restriction functor* $\mathbf{PSh}(X) \to \mathbf{PSh}(\mathcal{B})$ *induces an equivalence of categories*

$$\mathbf{Sh}(X) \simeq \mathbf{Sh}(\mathcal{B}).$$

Proof (sketch). We first note that by definition of a base, every open neighborhood of a point $x \in X$ contains a neighborhood which belongs to the base $\mathcal{B}$. Therefore, the stalk $\mathcal{F}_x$ of a presheaf $\mathcal{F}$ on X depends only on its restriction to $\mathcal{B}$:

$$\mathcal{F}_x = \varinjlim_{x \in U \in \mathcal{B}} \mathcal{F}(U).$$

The same formula defines the stalk $\mathcal{F}_x$ of a presheaf $\mathcal{F}$ on $\mathcal{B}$. Our construction of the sheafification (Lecture 7) can therefore be adapted to give a functor from presheaves on $\mathcal{B}$ to sheaves on X. That is, for a presheaf $\mathcal{F}$ on $\mathcal{B}$ we define the presheaf $\mathcal{F}^a$ on X associating to $U \subseteq X$ the set of all $(s(x)) \in \prod_{x \in U} \mathcal{F}_x$ such that for every $x \in U$ there exists an open neighborhood $V \subseteq X$ of x <u>such that $V \in \mathcal{B}$</u> and a section $t \in \mathcal{F}(V)$ such that for every $y \in V$ we have $s(y) = t_y$. The proof of the existence and basic properties of sheafification adapts easily to finish the current proof. □

8.2. Basic lemma

Lemma 8.2.1. *Let A be a ring and let $f_1, \ldots, f_r \in A$ be elements generating the unit ideal in A. Then the map $A \mapsto \prod_{i=1}^r A[f_i^{-1}]$ sending $g \in A$ to $(g, \ldots, g)$ induces an isomorphism*

$$A \xrightarrow{\sim} \ker\left(\prod_{i=1}^r A[f_i^{-1}] \xrightarrow{\delta} \prod_{i,j=1}^r A[(f_i f_j)^{-1}] \right)$$

where the map δ is defined by

$$\delta(g_1,\dots,g_r)_{ij} = g_i - g_j \in A[(f_if_j)^{-1}].$$

Proof. We handle injectivity first. Suppose that $g \in A$ maps to zero in $\prod_{i=1}^r A[f_i^{-1}]$. This means that for each $i = 1,\dots,r$ we have $f_i^{n_i} g = 0$ for some $n_i \geq 1$. Now $A = (f_1,\dots,f_r)$ implies $A = (f_1^{n_1},\dots,f_r^{n_r})$. Write $1 = \sum_{i=1}^r h_i f_i^{n_i}$ for some $h_1,\dots,h_r \in A$. Multiply the last equality by g to get

$$g = \sum_{i=1}^r h_i f_i^{n_i} g = 0.$$

Surjectivity is easily seen to be equivalent to the following claim: given $g_1,\dots,g_r \in A$ and an integers $n,m \geq 0$ such that

$$(f_if_j)^m(f_i^n g_j - f_j^n g_i) = 0$$

for all $i,j \in \{1,\dots,r\}$, there exists an integer $k \geq 0$ such that

$$f_i^k(gf_i^n - g_i) = 0$$

for all $i = 1,\dots,r$. We will show this with $k = m$. Now $A = (f_1,\dots,f_r)$ implies $A = (f_1^{n+m},\dots,f_r^{n+m}$, so we can write $1 = \sum_{j=1}^r h_j f_j^{n+m}$ for some $h_1,\dots,h_r \in A$. Let

$$g = \sum_{j=1}^r h_j f_j^n g_j.$$

Then for $i = 1,\dots,r$ we have

$$g_i f_i^m = \sum_{j=1}^r h_j f_j^{n+m} g_i f_i^m = \sum_{j=1}^r h_j f_i^{n+m} g_j f_j^m = f_i^{n+m} \sum_{j=1}^r h_j f_j^n g_j = f_i^{n+m} g,$$

as desired. □

Corollary 8.2.2. *Let A be a ring and let $X = \operatorname{Spec}(A)$. There exists a unique sheaf of rings $\mathcal{O}_X$ on X such that for every $f \in A$ we have*

$$\mathcal{O}_X(D(f)) = A[f^{-1}]$$

(and such that for $f,g \in A$, the restriction map $\mathcal{O}_X(D(f)) \to \mathcal{O}_X(D(fg))$ coincides with the natural map $A[f^{-1}] \to A[(fg)^{-1}]$). Moreover, for every $x \in X$ corresponding to a prime ideal $\mathfrak{p}_x \subseteq A$, we have

$$\mathcal{O}_{X,x} \simeq A_{\mathfrak{p}_x}.$$

Proof. We first check that the ring $A[f^{-1}]$ depends only on the open subset $U = D(f)$. To this end, let

$$S_U = \{g \in A : D(g) \subseteq U\} = \bigcap_{x \in U} (A \smallsetminus \mathfrak{p}_x).$$

Then $A[f^{-1}] = A[S_U^{-1}]$.

Now consider the base $\mathcal{B}$ of the topology on X consisting of all opens of the form $D(f)$. The association $D(f) \mapsto A[f^{-1}]$ makes sense thanks to the previous paragraph, and defines a presheaf on the base $\mathcal{B}$.

We now use Lemma 8.2.1 to check that this presheaf is a sheaf on the base $\mathcal{B}$. Let thus $U = D(f)$ be covered by $U_\alpha = D(f_\alpha)$, and let $s_\alpha \in A[f_\alpha^{-1}]$ be elements such that for every $\alpha,\beta \in I$, the images of s_α and s_β in $A[(f_\alpha f_\beta)^{-1}]$ are equal. Since $D(f) \simeq \operatorname{Spec}(A[f^{-1}])$ is quasi-compact, there exists a finite subset $I_0 = \{1,\dots,r\} \subseteq I$ such that $U = \bigcup_{i=1}^r U_i$. In other words, $f_1,\dots,f_r$ generate the unit ideal in $A[f^{-1}]$. We are now in position to apply Lemma 8.2.1 to the ring $A[f^{-1}]$, obtaining a unique $s \in A[f^{-1}]$ such that s_i is the image of s in $A[f_i^{-1}]$ for $i = 1,\dots,r$.

Since possibly $I_0 \neq I$, we still need to verify that s_α is the image of s in $A[f^{-1}]$ for all $\alpha \in I$. But this is easy, simply label α as $r+1$ and run the previous argument for $I_1 = I \cup \{\alpha\} = \{1,\dots,r,r+1\}$. □

8.3. Schemes

Definition 8.3.1. (1) A **ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. (Often, we shall write X meaning the pair $(X, \mathcal{O}_X)$.)

(2) A **morphism** of ringed spaces $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a pair (ϕ, ϕ^*) where $\phi : Y \to X$ is a continuous map and where $\phi^* : \mathcal{O}_X \to \phi_* \mathcal{O}_Y$ is a homomorphism of sheaves of rings on X (in other words, a compatible system of ring homomorphisms

$$\phi^* : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(\phi^{-1}(U))).$$

(Again, we shall simply write $\phi : Y \to X$ instead of $(\phi, \phi^*) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$.)

(3) A **locally ringed space** is a ringed space $(X, \mathcal{O}_X)$ such that for every $x \in X$, the stalk $\mathcal{O}_{X,x} = \varinjlim_{x \in U} \mathcal{O}_X(U)$ is a local ring.

(4) A **morphism** of locally ringed spaces $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a map of ringed spaces $\phi : Y \to X$ such that for every $y \in Y$, the induced map

$$\phi^* : \mathcal{O}_{X,\phi(x)} \longrightarrow \mathcal{O}_{Y,y}$$

is a local ring homomorphism (maps the maximal ideal into the maximal ideal).

Examples 8.3.2. 1. Let X be a topological space and let A be a ring. Then the pair $(X, \underline{A})$ of X with the constant sheaf with value A is a ringed space which is a locally ringed space if and only if A is local or $X = \varnothing$.

2. Every swf can be regarded as a locally ringed space, by forgetting that the structure sheaf consists of k-valued functions. Morphisms of swf's induce morphisms of locally ringed spaces. (See Problem Set 4.)

3. Let $X = \mathbb{C}$ be the complex plane and let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. Then X is a locally ringed space (this is a special case of (2)).

4. Let $X = \operatorname{Spec}(A)$ for a ring A, and let $\mathcal{O}_X$ be the structure sheaf constructed in Corollary 8.2.2. Then $(X, \mathcal{O}_X)$ is a locally ringed space.

Let X be a locally ringed space. We introduce the following notation, familiar from our treatment of $\operatorname{Spec}(A)$ last semester:

- for $x \in X$, we denote by $\kappa(x)$ the residue field of the local ring $\mathcal{O}_{X,x}$ and call it the **residue field** of x;
- for an open $U \subseteq X$, an $f \in \mathcal{O}_X(U)$, and $x \in U$, we denote by $f(x)$ the image of $f_x \in \mathcal{O}_{X,x}$ in $\kappa(x)$.

We note that the notion of a map of locally ringed spaces $\phi : Y \to X$ is rigged so that for $y \in Y$ and an open neighborhood U of $x = \phi(y)$ we have a unique map of residue fields $\kappa(x) \to \kappa(y)$ fitting inside the commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_Y(\phi^{-1}(U)) & \longrightarrow & \mathcal{O}_{Y,y} & \longrightarrow & \kappa(y) \\ \uparrow{\scriptstyle \phi^*} & & \uparrow{\scriptstyle \phi^*} & & \vdots \\ \mathcal{O}_X(U) & \longrightarrow & \mathcal{O}_{X,x} & \longrightarrow & \kappa(x). \end{array}$$

Thus, implicitly treating $\kappa(x)$ as a subfield of $\kappa(y)$ via this map, we have the equality in $\kappa(y)$:

$$\phi^*(f)(y) = f(\phi(y)),$$

which lets us pretend that ϕ^* is defined by composing regular functions with ϕ, as in the case of swf's.

Lemma 8.3.3. *Let X be a locally ringed space, let $U \subseteq X$ be an open subset, and let $f \in \mathcal{O}_X(U)$. Then the subset*

$$D(f) = \{x \in U : f(x) \neq 0\} \subseteq U$$

is open and $f|_{D(f)} \in \mathcal{O}_X(D(f))^\times$.

Proof. If $x \in D(f)$, then $f_x = (U, f) \in \mathcal{O}_{X,x}$ is not in the maximal ideal, and hence is invertible. Therefore there exists a germ (V, g) (where $x \in V$ and $g \in \mathcal{O}_X(V)$) such that $(U, f) \cdot (V, g) = 1$. This equality means that there exists a $W \subseteq U \cap V$ containing x such that $f|_W \cdot g|_W = 1$ in $\mathcal{O}_X(W)$. But then for every $y \in W$, we have $f_y \cdot g_y = 1$ in $\mathcal{O}_{X,y}$, so that $f_y \in \mathcal{O}_{X,y}$ is invertible, i.e. $f(y) \neq 0$. This shows that $x \in W \subseteq D(f)$, and hence that $D(f)$ is open (as x was arbitrary). Moreover, we have constructed an open cover $D(f) = \bigcup_{\alpha \in I} W_\alpha$ and elements $g_\alpha \in \mathcal{O}_X(W_\alpha)$ such that $f|_{W_\alpha} \cdot g_\alpha = 1$. These elements form a compatible family: on $W_\alpha \cap W_\beta$, both g_α and g_β give inverses of $f|_{W_\alpha \cap W_\beta}$ and hence are equal. By the sheaf condition, we have a unique $g \in \mathcal{O}_X(D(f))$ with $g_\alpha = g|_{W_\alpha}$. Then $f|_{D(f)} \cdot g = 1$, as checked after restriction to each W_α. □

Definition 8.3.4. (a) An **affine scheme** is a locally ringed space $(X, \mathcal{O}_X)$ isomorphic to $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec} A})$ for some ring A (note that then $A = \mathcal{O}_X(X)$).

(b) A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme; more precisely, there exists an open cover $X = \bigcup_{\alpha \in I} U_\alpha$ such that each $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ is an affine scheme.

(c) A morphism of schemes is a morphism of locally ringed spaces.

Lemma 8.3.5. *Let A be a ring, let $X = \mathrm{Spec}(A)$, and let $f \in A$. Then*

$$(D(f), \mathcal{O}_X|_{D(f)}) \simeq \mathrm{Spec}(A[f^{-1}])$$

is an affine scheme as well.

Proof. The identification of underlying spaces is known from commutative algebra, and the identification of structure sheaves is obvious. □

Corollary 8.3.6. *Let X be a scheme.*

(a) Every open subspace of X (endowed with the restriction of $\mathcal{O}_X$) is a scheme.

(b) Affine open subsets form a base of the topology on X.

Proof. (a) Let $V \subseteq X$ be an open subspace and let $X = \bigcup U_\alpha$ be an affine open cover. Then $V = \bigcup (U_\alpha \cap V)$. However, the $U_\alpha \cap V$ might not be affine. To deal with this, note that since $U_\alpha \cap V$ is an open of $U_\alpha = \mathrm{Spec}(A_\alpha)$, $A_\alpha = \mathcal{O}_X(U_\alpha)$, it is covered by standard opens $W_{\alpha\beta} = D(f_{\alpha\beta})$ which are isomorphic to $\mathrm{Spec}(A_\alpha[f_{\alpha\beta}^{-1}])$ by the previous lemma.

(b) Again let $X = \bigcup U_\alpha$ be an affine open cover. Then for every α the standard opens $D(f) \simeq \mathrm{Spec}(\mathcal{O}(U_\alpha)[f^{-1}]) \subseteq U_\alpha$, $f \in \mathcal{O}(U_\alpha)$ form a basis of the topology on U_α, and hence varying α and f we obtain a base of affine opens on X. □

We will prove the following universal property of $\mathrm{Spec}(A)$ next time:

Proposition 8.3.7. *Let X be an affine scheme and let Y be a locally ringed space. Then the map*

$$\phi \mapsto \phi^*: \mathrm{Hom}_{\mathrm{LRS}}(Y,X) \longrightarrow \mathrm{Hom}_{\mathrm{Rings}}(\mathcal{O}(X),\mathcal{O}(Y))$$

is bijective.

Corollary 8.3.8. *The functor $A \mapsto \mathrm{Spec}(A)$ establishes an arrow-reversing equivalence between commutative rings and affine schemes, with inverse functor $X \mapsto \mathcal{O}_X(X)$.*

8.4. Problem session (Feb 17)

We discussed the following problems.

1. *Prove Lemma 8.3.3.*

   Done, see the proof of Lemma 8.3.3.

2. *More motivation for the definition of a morphism of LRS [Hartshorne, Example II 2.3.2]*

   Let R be a discrete valuation ring, and write $X = \mathrm{Spec}(R) = \{\eta, s\}$ where s is closed. Let $K = \mathrm{Frac}(R)$ and $Y = \mathrm{Spec}(K) = \{y\}$. The map $R \to K$ induces a map of schemes $\phi: Y \to X$ sending y to η. We constructed a different map of ringed spaces (but not on locally ringed spaces) $\psi: Y \to X$ which sends y to s. This shows the importance of the extra condition in the definition of a morphism of locally ringed spaces, and that the assertion of Proposition 8.3.7 is false if we take maps of ringed spaces on the source.

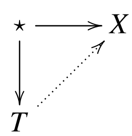
3. *Tangent vectors as maps from a fat point [Hartshorne Exercise II 2.8]*

   As one motivation for the notion of a scheme we show a very geometric way of treating tangent vectors, made possible by the admission of nilpotents.

   Let k be an algebraically closed field and let A be a finitely generated and reduced k-algebra. Let $x \in \mathrm{MSpec}(A) = \mathrm{Hom}_k(A,k)$ be a point of the corresponding affine algebraic set. The **Zariski cotangent space** at x is the vector space over k

   $$T_x = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee = \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k).$$

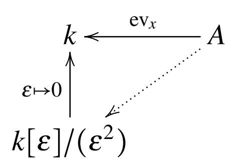
   Let $X = \mathrm{Spec}(A)$, $T = \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ and $\star = \mathrm{Spec}(k)$. Let $\star \to X$ be the map induced by $\mathrm{ev}_x: A \to k$ and let $\star \to T$ be the map induced by $\varepsilon \mapsto 0$. Consider the problem of finding a map $v: T \to X$ of k-schemes fitting inside the commutative diagram

   In other words, we seek maps from the "fat point" $T \to X$ which send the underlying "classical" point $\star \subseteq T$ to x.

   Claim. *Such maps v are in bijection with the Zariski cotangent space T_x.*

   To see this, we use the equivalence between affine schemes and rings to translate the diagram into a diagram of k-algebras

The diagonal map $v^*: A \to k[\varepsilon]/(\varepsilon^2)$ sends thus an element $f \in A$ to

$$v^*(f) = f(x) + \delta_v(f)\varepsilon$$

for some map $\delta_v(f): A \to k$. We verify easily that v^* is a ring homomorphism if and only if $\delta_v(f)$ is derivation, where the target k is regarded as the A-module $A/\mathfrak{m}_x$.

It remains to identify $\mathrm{Der}_k(A,k)$ with $T_x = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$. To this end, we have a direct sum decomposition $A \simeq k \oplus \mathfrak{m}_x$ of A as a k-vector space (decomposing $f \in A$ as $f(x) + (f - f(x))$). A k-linear derivation $\delta: A \to k$ vanishes on k, and hence is determined by its restriction $\mathfrak{m}_x \to k$. Further, the Leibniz rule

$$\delta(fg) = f(x) \cdot \delta(g) + g(x) \cdot \delta(f)$$

shows that δ vanishes on $\mathfrak{m}_x^2$. We thus obtain a map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k$ (i.e. an element of T_x) which uniquely determines δ.

4. *Reduced schemes.* Let X be a scheme. Show that the following conditions are equivalent:

   (a) X is reduced, i.e. for every open $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is reduced,

   (b) for every affine open $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is reduced,

   (c) there exists an affine open cover $X = \bigcup U_\alpha$ where $\mathcal{O}_X(U_\alpha)$ is reduced for each α,

   (d) for each $x \in X$, the stalk $\mathcal{O}_{X,x}$ is reduced.

   The implications (a)⇒(b)⇒(c) are obvious. To show (c)⇒(d), let $x \in X$ and let α be such that $x \in U_\alpha$. If $\mathfrak{p}_x \subseteq \mathcal{O}(U_\alpha)$ is the corresponding prime ideal, we have

$$\mathcal{O}_{X,x} = \mathcal{O}_{U_\alpha,x} = \mathcal{O}(U_\alpha)_\mathfrak{p}.$$

   Now, a localization $A[S^{-1}]$ of a reduced ring is reduced: if $(a/s)^n = 0$, we have $ta^n = 0$ for some $t \in S$, and then $(ta)^n = t^{n-1} \cdot ta^n = 0$, implying $ta = 0$, so $a/s = 0$. Thus $\mathcal{O}_{X,x}$ is reduced. Finally, to show (d)⇒(a), note that we have an injection

$$\mathcal{O}_X(U) \longrightarrow \prod_{x \in U} \mathcal{O}_{X,x}$$

   and a subring of the product of a family of reduced rings is reduced.