8. Lecture 7 (Feb 12): Sheaves

Recommended reading: Hartshorne II.1, Kempf §4, Vakil §2

8.1. Review of sheaves

We review the basics of sheaf theory, without proofs.

Definition 8.1.1. Let X be a topological space.

(a) A **presheaf** (of abelian groups) $\mathcal{F}$ on X consists of the following data:

- for every open $U \subseteq X$, an abelian group $\mathcal{F}(U)$;
- for every pair of opens $U, V \subseteq X$ such that $V \subseteq U$, a homomorphism $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ (called the *restriction map*)

such that

i. $\rho_{U,U} = \mathrm{id}$ for every open $U \subseteq X$;
ii. $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$ for every triple of opens $U, V, W \subseteq X$ such that $W \subseteq V \subseteq U$.

(In other words, $\mathcal{F}$ is a contravariant functor $\mathcal{F} : \mathbf{Opens}(X)^{\mathrm{op}} \to \mathbf{Ab}$ from the category (poset) of open subsets of X to the category of abelian groups.)

(b) A **map of presheaves** $f : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open $U \subseteq X$ which are compatible with the maps in the sense that for $V \subseteq U$ we have

$$f_V \circ \rho^{\mathcal{F}}_{U,V} = \rho^{\mathcal{G}}_{U,V} \circ f_U.$$

(In other words, f is a natural transformation between the two functors $\mathcal{F}, \mathcal{G} : \mathbf{Opens}(X)^{\mathrm{op}} \to \mathbf{Ab}$.

(c) We introduce the following notation and terminology.

- We denote the **category of presheaves** on X by $\mathbf{PSh}(X)$.
- Elements of $\mathcal{F}(U)$ are called **sections** of $\mathcal{F}$ over U.
- Elements of $\mathcal{F}(X)$ are called the **global sections** of $\mathcal{F}$.
- For $s \in \mathcal{F}(U)$ and $V \subseteq U$, we write $s|_V$ for $\rho_{U,V}(s) \in \mathcal{F}(V)$ and call it the **restriction** of s to V.
- For $f : \mathcal{F} \to \mathcal{G}$ we denote the maps $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ simply by f. (In particular, the condition in (b) can be more succintly written as $f(s|_V) = f(s)|_V$.)

(d) A presheaf $\mathcal{F}$ on X is a **sheaf** if for every open $U \subseteq X$ and every family $\{U_\alpha\}_{\alpha \in I}$ of opens with $U = \bigcup U_\alpha$, and every family of sections $s_\alpha \in \mathcal{F}(U_\alpha)$ such that for every $\alpha, \beta \in I$ we have

$$s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta} \quad \text{in} \quad \mathcal{F}(U_\alpha \cap U_\beta)$$

there exists a unique section $s \in \mathcal{F}(U)$ such that for every $\alpha \in I$ we have

$$s|_{U_\alpha} = s_\alpha.$$

We denote by $\mathbf{Sh}(X)$ the category of sheaves (a full subcategory of $\mathbf{PSh}(X)$).

We have defined (pre)sheaves of abelian groups, but in the same way one defines (pre)sheaves of sets, groups, or rings.

Examples 8.1.2. (a) Let X be a topological space and let G be a topological abelian group (e.g. $G = \mathbb{R}$). We define a presheaf $\mathcal{C}_G$ on X by

$$\mathcal{C}_G(U) = \{\text{continuous maps } U \to G\}$$

(with the obvious restriction maps). Then $\mathcal{C}_G$ is a sheaf.

(b) Special case of (1): let G be an abelian group, and make it into a topological group by giving it the discrete topology. Then $\mathcal{C}_G(U)$ consists of *locally constant* functions $U \to G$, and the sheaf $\mathcal{C}_G$ is called the **constant sheaf** with value G and denoted by $\underline{G}$.

(c) Another special case of (1): let G be an abelian group, and this time make it into a topological group by giving it the indiscrete topology. Then $\mathcal{C}_G(U)$ consists of all maps of sets $U \to G$.

(d) Let $X = \mathbb{C}$ be the complex plane and let $\mathcal{O}(U)$ denote the ring of holomorphic functions on an open $U \subseteq X$. This is a sheaf of rings on X.

(e) Let X be a space with functions (over an implicitly chosen field k). Then the association $U \mapsto \mathcal{O}(U) = \{\text{regular functions on } U\}$ defines a sheaf on X, denoted by $\mathcal{O}_X$ and called its **structure sheaf**.

(f) Let $f: Y \to X$ be a map of topological spaces. Let $\mathcal{S}_f$ be the presheaf associating to $U \subseteq X$ the set of all sections of $f|_U$, i.e. of all continuous maps $s: U \to Y$ such that the composition $U \to Y \to X$ equals the inclusion map $U \to X$. This is a sheaf of sets on X, called the **sheaf of sections** of the map f.

Definition 8.1.3. Let X be a topological space.

(a) Let $\mathcal{F}$ be a presheaf on X and let $x \in X$. We define the **stalk** of $\mathcal{F}$ at x to be the direct limit

$$\mathcal{F}_x = \varinjlim_U \mathcal{F}(U)$$

over all open neighborhoods U of x. (Concretely, elements of $\mathcal{F}_x$ are represented by pairs (U,s) where $x \in U$ and $s \in \mathcal{F}(U)$, where we identify (U,s) with (U',s') if there exists an open $U'' \subseteq U \cap U'$ with $x \in U''$ such that $s|_{U''} = s'|_{U''}$. We denote the image of (U,s) in $\mathcal{F}_x$ by s_x and call it the **germ** of s at x.)

(b) The sheaf of **discontinuous sections** $D(\mathcal{F})$ of $\mathcal{F}$ is the presheaf given by associating to $U \subseteq X$ the set of all functions $s: U \to \coprod_{x \in U} \mathcal{F}_x$ such that $s(x) \in \mathcal{F}_x$ for every $x \in U$, or equivalently the product $\prod_{x \in U} \mathcal{F}_x$.

(c) The **sheafification** of $\mathcal{F}$ is the presheaf $\mathcal{F}^a$ given by associating to $U \subseteq X$ the set of all functions $s: U \to \coprod_{x \in U} \mathcal{F}_x$ such that

   i. $s(x) \in \mathcal{F}_x$ for every $x \in U$;

   ii. for every $x \in U$ there exists an open neighborhood $V \subseteq U$ of x and a section $t \in \mathcal{F}(V)$ such that for every $y \in V$, we have $s(y) = t_y$.

Note that $\mathcal{F}^a$ is a sub-presheaf of $D(\mathcal{F})$, and that there is a map of presheaves $i: \mathcal{F} \to \mathcal{F}^a$ mapping $s \in \mathcal{F}(U)$ to the function $x \mapsto s_x$. These constructions are functorial in $\mathcal{F}$ in the obvious way.

Proposition 8.1.4. *Let $\mathcal{F}$ be a presheaf on a topological space X.*

(a) The presheaf $D(\mathcal{F})$ is a sheaf.

(b) The presheaf $\mathcal{F}^a$ is a subsheaf of $D(\mathcal{F})$.

(c) Every map $f: \mathcal{F} \to \mathcal{G}$ into a sheaf $\mathcal{G}$ factors uniquely through $i: \mathcal{F} \to \mathcal{F}^a$.

(In other words, $\mathcal{F} \mapsto \mathcal{F}^a$ defines a functor $\mathbf{PSh}(X) \to \mathbf{Sh}(X)$*, called the* **sheafification functor***, which is left adjoint to the inclusion functor* $\mathbf{Sh}(X) \to \mathbf{PSh}(X)$*.)*

(d) $\mathcal{F}$ is a sheaf if and only if the map $i: \mathcal{F} \to \mathcal{F}^a$ is an isomorphism.

Lemma 8.1.5. *A map of sheaves $\mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if for every $x \in X$, the map on stalks $\mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism.*

Operations on sheaves. Let X be a topological space.

(a) **Binary coproducts.** Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Then $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ is a sheaf (denoted $\mathcal{F} \oplus \mathcal{G}$) which is the categorical coproduct of $\mathcal{F}$ and $\mathcal{G}$. It coincides with the binary product $\mathcal{F} \times \mathcal{G}$, and for $x \in X$ we have $(\mathcal{F} \oplus \mathcal{G})_x = \mathcal{F}_x \oplus \mathcal{G}_x$.

(b) **Kernels.** Let $f: \mathcal{F} \to \mathcal{G}$ be a map of sheaves on X. Then $U \mapsto \ker(\mathcal{F}(U) \to \mathcal{G}(U))$ is a subsheaf of $\mathcal{F}$ denoted by $\ker(f)$. It is the kernel (equalizer of f and the zero map) in the category of sheaves on X, and for $x \in X$ we have $\ker(f)_x = \ker(\mathcal{F}_x \to \mathcal{G}_x)$.

(c) **Products.** Let $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ be a family of sheaves on X. Then $U \mapsto \prod_{\alpha \in I} \mathcal{F}_\alpha(U)$ is a sheaf on X, which is the categorical product $\prod_{\alpha \in I} \mathcal{F}_\alpha$ of the family of sheaves $\{\mathcal{F}_\alpha\}_{\alpha \in I}$. In general, for $x \in X$, $(\prod_{\alpha \in I} \mathcal{F}_\alpha)_x$ does not coincide with $\prod_{\alpha \in I} (\mathcal{F}_\alpha)_x$.

(d) **Cokernels.** Let $f: \mathcal{F} \to \mathcal{G}$ be a map of sheaves on X. The presheaf $\mathrm{cok}_{\mathrm{pre}}(f)$ defined by $U \mapsto \mathrm{cok}(\mathcal{F}(U) \to \mathcal{G}(U))$ is not a sheaf in general. We define the cokernel $\mathrm{cok}(f) = \mathrm{cok}_{\mathrm{pre}}(f)^a$ to be the sheafification of this presheaf. It is the categorical cokernel of f (the coequalizer of f and the zero map) and for every $x \in X$, we have $\mathrm{cok}(f)_x = \mathrm{cok}(\mathcal{F}_x \to \mathcal{G}_x)$.

(e) **Quotient.** For a sheaf $\mathcal{G}$ and a subsheaf $\mathcal{F} \subseteq \mathcal{G}$, the quotient $\mathcal{G}/\mathcal{F}$ is the cokernel of the inclusion $\mathcal{F} \to \mathcal{G}$, i.e. the sheafification of the presheaf $U \mapsto \mathcal{G}(U)/\mathcal{F}(U)$.

(f) **Image.** The image $\mathrm{im}(f)$ of a map of sheaves $f: \mathcal{F} \to \mathcal{G}$ is the sheafification of the presheaf $U \mapsto \mathrm{im}(\mathcal{F}(U) \to \mathcal{G}(U))$ (which is a subsheaf of $\mathcal{G}$), or equivalently the quotient $\mathcal{F}/\ker(f)$ or the kernel of $\mathcal{G} \to \mathrm{cok}(\mathcal{F})$. (In particular, the category $\mathbf{Sh}(X)$ is an abelian category.)

(g) **Cohomology and exactness.** A complex of sheaves $\mathcal{F}^\bullet$ is a sequence of sheaves $\mathcal{F}^n$ ($n \in \mathbb{Z}$) and maps $d^n: \mathcal{F}^n \to \mathcal{F}^{n+1}$ such that $d^n \circ d^{n-1} = 0$ for all n. Its n-th cohomology sheaf $\mathcal{H}^n(\mathcal{F}^\bullet)$ is the quotient $\ker(d^n)/\mathrm{im}(d^{n-1})$. We say that $\mathcal{F}^\bullet$ is exact if $\mathcal{H}^n(\mathcal{F}^\bullet) = 0$ for all n. The stalk $\mathcal{H}^n(\mathcal{F}^\bullet)_x$ is the n-th cohomology of the complex of abelian groups $\mathcal{F}_x^\bullet$, and $\mathcal{F}^\bullet$ is exact if and only if the complexes $\mathcal{F}_x^\bullet$ are exact for all $x \in X$.

(h) **Coproducts.** Let $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ be a family of sheaves on X. Then the presheaf $U \mapsto \bigoplus_{\alpha \in I} \mathcal{F}_\alpha(U)$ is not a sheaf in general, and we define $\bigoplus_{\alpha \in I} \mathcal{F}_\alpha$ to be its sheafification, which is the categorical coproduct of the family of sheaves $\{\mathcal{F}_\alpha\}_{\alpha \in I}$.

(i) **Restriction.** Let $\mathcal{F}$ be a sheaf on X and let $U \subseteq X$ be an open subset. Then the presheaf on U given by $V \mapsto \mathcal{F}(V)$ (for opens $V \subseteq U$) is a sheaf on U, denoted by $\mathcal{F}|_U$.

(j) **Tensor product.** Let A be a ring and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of A-modules on X. The presheaf $U \mapsto \mathcal{F}(U) \otimes_A \mathcal{G}(U)$ is in general not a sheaf, and we define $\mathcal{F} \otimes_A \mathcal{G}$ to be its sheafification. There is an A-bilinear map $\mathcal{F} \times \mathcal{G} \to \mathcal{F} \otimes_A \mathcal{G}$ sending $(s,t) \in \mathcal{F}(U) \times \mathcal{G}(U)$ to the image of $s \otimes t \in \mathcal{F}(U) \otimes_A \mathcal{G}(U)$ in $(\mathcal{F} \otimes_A \mathcal{G})(U)$, which has the usual universal property: every A-bilinear map $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$ into a sheaf of A-modules $\mathcal{H}$ factors uniquely through an A-linear map $\mathcal{F} \otimes_A \mathcal{G} \to \mathcal{H}$. Moreover, we have $(\mathcal{F} \otimes_A \mathcal{G})_x = \mathcal{F}_x \otimes_A \mathcal{G}_x$.

(k) **Sheaf of homomorphisms.** Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of A-modules on X (there are obvious variants for sheaves of sets etc.). Then the presheaf

$$U \mapsto \mathrm{Hom}_A(\mathcal{F}|_U, \mathcal{G}|_U)$$

sending U to the A-module of all A-linear maps $\mathcal{F}|_U \to \mathcal{G}|_U$ is a sheaf denoted by $\mathcal{H}om_A(\mathcal{F}, \mathcal{G})$. It is right-adjoint to $\otimes_A$ in the sense that

$$\mathrm{Hom}_A(\mathcal{F} \otimes_A \mathcal{G}, \mathcal{H}) \simeq \mathrm{Hom}_A(\mathcal{F}, \mathcal{H}om_A(\mathcal{G}, \mathcal{H})).$$

(l) **Direct image.** Let $\phi: Y \to X$ be a map of topological spaces and let $\mathcal{G}$ be a sheaf on Y. Then the presheaf on X defined by $U \mapsto \mathcal{G}(\phi^{-1}(U))$ is a sheaf denoted by $\phi_*(\mathcal{G})$.

(m) **Inverse image.** Let $\phi: Y \to X$ be a map of topological spaces and let $\mathcal{F}$ be a sheaf on X. We define $\phi^*(\mathcal{F})$ to be the sheafification of the following presheaf on Y:

$$U \mapsto \varinjlim_{\phi(U) \subseteq V} \mathcal{F}(V)$$

(filtered colimit over all opens $V \subseteq X$ containing the image $\phi(U)$).

The functors ϕ_* and ϕ^* form an adjoint pair:

$$\mathrm{Hom}(\phi^* \mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F}, \phi_* \mathcal{G}).$$

Moreover, ϕ_* is a left exact functor and ϕ^* is exact.