6. Lecture 6 (Feb 10): Nonsingular varieties

Recommended reading: Hartshorne I.4, I.5, I.6

6.1. More on birational equivalence

We complement our discussion of birational equivalences from last time.

Lemma 6.1.1 (Hartshorne Corollary I 4.5)**.** *Let X and Y be varieties. The following are equivalent:*

(a) There exist dominant rational maps $\phi: Y \dashrightarrow X$ and $\psi: Y \dashrightarrow X$ such that $\phi \circ \psi = \mathrm{id}_X$ and $\psi \circ \phi = \mathrm{id}_Y$;

(b) $k(X) \simeq k(Y)$ as extensions of k;

(c) there exist non-empty opens $U \subseteq X$ and $V \subseteq Y$ such that $U \simeq V$ as varieties.

Proof. (a)⇔(b): Proved earlier.

(a)⇒(c): Represent ϕ by a map $\phi: V' \to X$ and ψ by a map $\psi: U' \to Y$. Set $U = \psi^{-1}(\phi^{-1}(U'))$ and $V = \phi^{-1}(\psi^{-1}(V')$. Then ϕ and ψ restrict to inverse isomorphisms $U \simeq V$.

(c)⇒(a): obvious. □

The following algebraic fact is a bit complicated to show, but not too difficult.

Theorem 6.1.2 (Zariski–Samuel vol. I, Chapter II, Theorem 31, p. 105)**.** *Let k be a perfect field and let K be a finitely generated field extension of k. Then K is* **separably generated** *over k, i.e. it can be written as a finite separable*[1] *extension of $k(T_1, \dots, T_n)$ where $n = \mathrm{trdeg}(K/k)$.*

Recall from CA that every finite separable extension is generated by a single element (the "primitive element theorem"). More precisely, in the situation in the theorem, we can write $K = k(T_1, \dots, T_n)[T]/(f)$ where $f \in k(T_1, \dots, T_n)[T]$ is an irreducible separable polynomial (that is, $f' \neq 0$).

Corollary 6.1.3. *Every variety is birational to an irreducible hypersurface in $\mathbb{P}^{n+1}$.*

Proof. Let X be a variety. Apply Theorem 6.1.2 to $K = k(X)$, writing $k(X)$ as a finite separable extension of $k(T_1, \dots, T_n)$, $n = \mathrm{trdeg}(k(X)/k) = \dim X$. By the primitive element theorem (see discussion above), we can write

$$k(X) = k(T_1, \dots, T_n)[T]/(f)$$

for a separable irreducible polynomial $f \in k(T_1, \dots, T_n)[T]$. Clearing denominators, we obtain an irreducible polynomial $f \in k[T_1, \dots, T_n, T]$. Let $g \in k[T_0, T_1, \dots, T_n, T]$ be its homogenization (which is again irreducible) and let $Y = V_{\mathbb{P}}(g) \subseteq \mathbb{P}^{n+1}$. Then $k(Y) \simeq k(X)$ by construction, and so X and Y are birational. □

6.2. Nonsingular varieties

Recall that a Noetherian local ring A is **regular** if

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(A).$$

Here $\mathfrak{m} \subseteq A$ is the unique maximal ideal and $k = A/\mathfrak{m}$ is its residue field. In general, we have $\geq$ instead of equality, and the left-hand side coincides with the minimal number of generators of $\mathfrak{m}$. Every regular ring is a UFD (this is not so easy to show).

[1] See the note on separable extensions from last semester, available at https://achinger.impan.pl/ca2025/sep.pdf

Definition 6.2.1. Let X be an algebraic set and let $x \in X$. We say that X is **nonsingular** at x if $\mathcal{O}_{X,x}$ is a regular local ring.

We shall also call x a **smooth** or **regular** point. If every point is nonsingular, we say that X itself is nonsingular/smooth/regular.

Examples 6.2.2. (a) If $\dim(X) \leq 1$, the ring $\mathcal{O}_{X,x}$ is regular if and only if it is a discrete valuation ring (or equal to k in case of isolated points), if and only if it is integrally closed.

(b) The affine space $\mathbb{A}^n$ is nonsingular. Indeed, for every $x = (x_1, \ldots, x_n) \in \mathbb{A}^n$ we have $\dim(\mathcal{O}_{X,x}) = n$ and the maximal ideal $\mathfrak{m}$ is generated by n elements $T_i - x_i$.

(c) Let $X = V(f) \subseteq \mathbb{A}^n$ be a hypersurface (where $f \neq 0$) and let $x \in X$. Then X is nonsingular at x if and only if $(\partial f/\partial T_i)(x) \neq 0$ for some i. Proof: Since $\dim(\mathcal{O}_{X,x}) = n-1$, we want $\dim(\mathfrak{m}/\mathfrak{m}^2) = n-1$. If $\mathfrak{n} = (T_1 - x_1, \ldots, T_n - x_n) \subseteq k[T_1, \ldots, T_n]$ is the ideal corresponding to x, then

$$\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{n}/((\mathfrak{n}^2 + (f))$$

which has dimension either n or $n-1$, and the latter precisely when $f \notin \mathfrak{n}^2$. We can write

$$f(T_1, \ldots, T_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i}(x) \cdot (T_i - x) + R$$

where $R \in \mathfrak{n}^2$. Thus $f \in \mathfrak{n}^2$ precisely when $(\partial f/\partial T_i)(x) = 0$ for all i.

Theorem 6.2.3. *Let $X \subseteq \mathbb{A}^n$ be an affine variety and let $I = \mathcal{I}(X) = (f_1, \ldots, f_r)$ be its ideal. Then a point $x \in X$ is nonsingular if and only if*

$$\operatorname{rank}\left[\frac{\partial f_j}{\partial T_i}(x)\right] = n - \dim(X).$$

Proof. We did the case $r = 1$. For the general case see Theorem I 5.1 in Hartshorne. □

In general, for two nonsingular points $x \in X$ and $y \in Y$, the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,y}$ are often non-isomorphic even if they have the same dimension. Indeed, we have $k(X) = \operatorname{Frac}(\mathcal{O}_{X,x})$, so $\mathcal{O}_{X,x} \simeq \mathcal{O}_{Y,y}$ only if X and Y are birational. So our intuition from differential geometry that a smooth variety should locally look like $\mathbb{A}^n$, taken too literally, is false. This is because $\mathcal{O}_{X,x}$ is defined in terms of Zariski open neighborhoods of x, which are very large. One can resolve this issue either by considering the étale topology (which we might cover later) or by completing the local ring.

Theorem 6.2.4 (Special case of Cohen's structure theorem)**.** *Let X be an algebraic set and let $x \in X$. Denote by*

$$\widehat{\mathcal{O}}_{X,x} = \varprojlim_n \mathcal{O}_{X,x}/\mathfrak{m}_x^n$$

the **completion** *of the local ring $\mathcal{O}_{X,x}$ with respect to its maximal ideal. Then x is a nonsingular point of X if and only if*

$$\widehat{\mathcal{O}}_{X,x} \simeq k[\![T_1, \ldots, T_n]\!].$$

More precisely, if $f_1, \ldots, f_n \in \mathfrak{m}_x$, we have a unique continuous homomorphism

$$\theta : k[\![T_1, \ldots, T_n]\!] \longrightarrow \widehat{\mathcal{O}}_{X,x}, \qquad \theta(T_i) = f_i.$$

The map θ is surjective if and only if $\mathfrak{m}_x = (f_1, \ldots, f_n)$, and an isomorphism if and only if in addition

$$n = \dim(k[\![T_1, \ldots, T_n]\!]) = \dim(\mathcal{O}_{X,x}),$$

which happens precisely if $\mathcal{O}_{X,x}$ is regular.

Theorem 6.2.5. *Let X be a variety. Then the set $\mathrm{Sm}(X)$ of nonsingular points of x is a non-empty open subset of X.*

Proof. Theorem 6.2.3 implies that $\mathrm{Sm}(X)$ is open. We need to show it is non-empty. For this, we are going to follow [Hartshorne, Theorem I 3.5] which uses Theorem 6.1.2 above. See [Kempf, Lemma 6.1.6] for a fun and elementary proof of the same fact.

By Corollary 6.1.3 we may assume that $X = V(f) \subseteq \mathbb{A}^{n+1}$ is an affine hypersurface. The construction ensures that the projection $X \to \mathbb{A}^n$ onto the first n coordinates is "separable," more precisely that $(f, f') = 1$ in $k(T_1, \ldots, T_n)[T]$ where $f' = df/dT$. This means that we can write $1 = \alpha f + \beta f'$ in $k(T_1, \ldots, T_n)[T]$. Take $t = (t_1, \ldots, t_n) \in \mathbb{A}^n$ where the denominators of α and β do not vanish and a point $x \in X$ above t. Then $\alpha f + \beta f' = 1$ implies that $f'(x) \neq 0$, and so X is smooth at x. □

Remark 6.2.6 (Resolution of singularities)**.** One of the most important tools in algebraic geometry is the ability to reduce questions about arbitrary varieties to the case of nonsingular varieties by means of "resolution of singularities," established in 1970 by Hironaka. Hironaka's theorem says the following: let X be a projective variety and assume that k has **characteristic zero**, then there exists a birational map $\pi: X' \to X$ such that X' is a nonsingular projective variety. We will learn how to do this for curves (by means of "normalization"). Already for surfaces, the problem becomes nontrivial (but was resolved by Zariski in the 1940s). Nobody knows how to resolve singularities in positive characteristic beyond dimension 3, despite continued effort.

6.3. Nonsingular curves

By a **curve** we shall mean a separated one-dimensional variety.

Lemma 6.3.1. *Let C be a curve and let $x \in C$. The following are equivalent:*

(a) C is non-singular at x i.e. $\mathcal{O}_{C,x}$ is a regular local ring;

(b) $\mathcal{O}_{C,x}$ is a discrete valuation ring;

(c) $\mathcal{O}_{C,x}$ is integrally closed;

(d) the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_{C,x}$ is principal.

Proof. See Atiyah–Macdonald, Proposition 9.2. □

Remark 6.3.2. The algebra behind this proof can be used to show more generally that if X is a variety of dimension d which is normal (i.e. $\mathcal{O}(U)$ is integrally closed for every non-empty affine open $U \subseteq X$), then the set of singular points of X has dimension at most $\dim(X) - 2$. (We say that X is "regular in codimension one.")

Our next goal is to globalize the construction of integral closure, which in particular will give us a way of desingularizing curves.

The discussion below until the end of §5.3 was not covered on Feb 10 and will be done in the future.

Definition 6.3.3. A morphism of algebraic sets $\phi: Y \to X$ is **finite** if for every affine open $U \subseteq X$, its preimage $V = \phi^{-1}(U) \subseteq Y$ is affine and the map of rings $\phi^*: \mathcal{O}(U) \to \mathcal{O}(V)$ is finite (meaning that $\mathcal{O}(V)$ is a finitely generated module over $\mathcal{O}(U)$).

Examples 6.3.4. (a) A closed immersion $i: Y \to X$ is finite.

(b) By the "going-up theorem," a finite map is closed.

(c) A morphism between affine algebraic sets $\phi: Y \to X$ is finite if and only if $\phi^*: \mathcal{O}(X) \to \mathcal{O}(Y)$ is finite.

(d) Let A be a finitely generated domain over k and let A' be its normalization (integral closure in its field of fractions). By "finiteness of integral closure," the map $\mathrm{MSpec}(A') \to \mathrm{MSpec}(A)$ is finite.

(e) By Noether normalization, every affine algebraic set X admits a finite dominant map $X \to \mathbb{A}^n$ where $n = \dim(X)$.

Corollary 6.3.5. *Let X be a curve. Then there exists a finite and birational map of curves $X' \to X$ where X' is nonsingular.*

Proof (sketch). If $X = \mathrm{MSpec}(A)$ is affine, we take $X' = \mathrm{MSpec}(A')$ where A' is the normalization of A. In general, cover X by affines $X_i = \mathrm{MSpec}(A_i)$ and set $X_i' = \mathrm{MSpec}(A_i')$ where A_i' is the normalization of A_i. Then $X_i \cap X_j$ is affine, and its preimage in either X_i' or X_j' is equal to $\mathrm{MSpec}(A_{ij}')$ where A_{ij}' is the normalization of $A_{ij} = \mathcal{O}(X_i \cap X_j)$. This allows us to glue the $X_i' \to X_i$ into a global $X' \to X$. □

Remark 6.3.6. If X is projective then so is X'. We currently lack the tools to show this.

Lemma 6.3.7. *Let C be a non-singular curve. Then every rational map from C to $\mathbb{P}^n$ is everywhere defined.*

Proof. Represent $\phi: C \dashrightarrow \mathbb{P}^n$ by a map $U = C \smallsetminus \{x_1, \dots, x_r\} \to \mathbb{P}^n$. We may assume U is the largest set on which ϕ is defined, and we aim to show $r = 0$. Otherwise, let $x \in \{x_1, \dots, x_r\}$, and we aim to extend the map over x. This is local around x, and so we may assume that $C = \mathrm{MSpec}(A)$ is affine and $x = V(\pi)$ for some $\pi \in A$ (recall that $\mathcal{O}_{C,x}$ is a dvr), and that there exist $f_0, \dots, f_n \in A[1/\pi]$ which generate the unit ideal such that ϕ is on $D(\pi) = C \smallsetminus \{x\}$ given by $(f_0 : \cdots : f_n)$. Clearing denominators, we may assume $f_0, \dots, f_n \in A$ and $f_0 \notin \pi A$. But then $(f_0 : \cdots : f_n)$ defines an extension of ϕ to C. □

Corollary 6.3.8. *Let X and Y be projective non-singular curves. If X and Y are birational, then they are isomorphic.*

6.4. Problem session (Feb 10), part one: blow-ups

Dislaimer: The write-ups from the problem sessions are supposed to be rough transcripts of what has been discussed; they are not meant to be complete.

During the first half of the problem session, we discussed the following construction. The projective space $\mathbb{P}^n$ can be regarded as the space of lines through the origin $0 \in \mathbb{A}^{n+1}$. If $a = (a_0 : \cdots : a_n) \in \mathbb{P}^n$ is a point, then the corresponding line $\ell_a \subseteq \mathbb{A}^{n+1}$ is cut out by the system of linear equations

$$a_i T_j - a_j T_i = 0, \qquad 0 \le i < j \le n$$

(if say $a_0 \neq 0$, then we can consider only the n equations $a_0 T_j - a_j T_0$ with $j = 1, \dots, n$). Note that these are the determinants of the 2×2 minors of the $(n+1) \times 2$ matrix with rows $(a_0, \dots, a_n)$ and $(T_0, \dots, T_n)$.

The **blowup** of the affine space $\mathbb{A}^n$ is the space of pairs

$$\mathcal{B} = \{(a, x) \in \mathbb{P}^n \times \mathbb{A}^{n+1} : x \in L_a\} = V(X_i T_j - X_j T_i) \subseteq \mathbb{P}^n \times \mathbb{A}^{n+1}$$

(where $X_0, \dots, X_n$ are now the homogeneous coordinates on $\mathbb{P}^n$) which we can regard as the set pairs (L, x) where $x \in \mathbb{A}^{n+1}$ and $L \subseteq \mathbb{A}^{n+1}$ is a line through the origin which contains x. The space $\mathcal{B}$ is thus an algebraic set (in fact, a non-singular variety).

We denote by $\pi\colon \mathcal{B} \to \mathbb{A}^{n+1}$ the projection map $(L,x) \mapsto x$. Note that for $x \neq 0$, we have $\pi^{-1}(x) = \{(L_x, x)\}$ where L_x is the unique line through 0 and x. On the other hand, we have $\pi^{-1}(0) = \mathbb{P}^n \times 0$. So all fibers are singletons except one fiber which has dimension n. The map π induces an isomorphism over $\mathbb{A}^{n+1} \setminus 0$ and hence is birational.

Recall that $\mathbb{P}^n$ is the union of $n+1$ copies $U_0, \dots, U_i$ of the affine space $\mathbb{A}^n$. Thus $\mathbb{P}^n \times \mathbb{A}^{n+1}$ is the union of $n+1$ copies $U_i \times \mathbb{A}^{n+1}$ of $\mathbb{A}^{2n+1}$, and $\mathcal{B}$ is the union of $n+1$ open affine subvarieties $\mathcal{B}_i = \mathcal{B} \cap (U_i \times \mathbb{A}^{n+1}$. In order to avoid notational nightmare, let us explicate this in case $n = 1$. Let now the coordinates on $\mathbb{A}^2$ be x, y and on $\mathbb{P}^1$ be U, V.

- The coordinates on $\mathcal{B}_0 = \{U \neq 0\}$ are x, y, and $v = V/U$. There is a single equation $xV = yU$ cutting out $\mathcal{B}$, which translates to $\mathcal{B}_0 = \{xv = y\} \subseteq \mathbb{A}^3$. Note that we have $\mathcal{B}_0 \simeq \mathbb{A}^2$ with coordinates x, v and the map $\pi|_{\mathcal{B}_0}$ is the map $(x, v) \mapsto (x, xv)$.
- Similarly, $\mathcal{B}_1 \simeq \mathbb{A}^2$ with coordinates $y, u = U/V$ with $x = uy$.

See the illustration in Hartshorne I §4.

Definition 6.4.1. Let $X \subseteq \mathbb{A}^{n+1}$ be a closed subset. We define its **strict transform** to be the closed subset $\tilde{X} \subseteq \mathcal{B}$ defined as

$$\tilde{X} = \overline{\pi^{-1}(X \setminus 0)}.$$

The **tangent cone** of X at 0 is $C_0(X) = \tilde{X} \cap \pi^{-1}(0) \subseteq \mathbb{P}^n$.

We computed the strict transform of the following two singular plane curves

$$C : y^2 = x^3, \qquad D : y^2 = x^3 + x^2.$$

For this we used the description of the closure from Lecture 1, section 1.5.

Curve C: on $\mathcal{B}_0$, we substitute $y = vx$ in the equation of C to get

$$y^2 - x^3 = v^2x^2 - x^3 = x^2(v^2 - x).$$

Since we are computing the closure of this minus the line $\{x = 0\}$, we can divide out the x and conclude that $\tilde{C} \cap \mathcal{B}_0$ is the parabola $x = v^2$. It meets the line $\{x = 0\} = \pi^{-1}(0) \cap \mathcal{B}_0$ in the single point 0 but with multiplicity two (it is tangent to the line). A similar computation in the other chart $\mathcal{B}_1$ gives that $\tilde{C} \cap \mathcal{B}_1$ is cut out by $1 = u^3y$, which is a smooth curve which does not meet the line $\{y = 0\}$ and in fact $\tilde{C}$ is contained in $\mathcal{B}_0$. Conclusion: $\tilde{C}$ is a smooth curve, in fact isomorphic to $\mathbb{A}^1$ with coordinate $v = y/x$, and $\mathcal{O}(\tilde{C}) = k[v]$ is the normalization of $\mathcal{O}(C) = k[x,y]/(y^2 - x^3) = k[v^2, v^3]$. Its tangent cone consists of one point $(1:0)$ with multiplicity two. Geometrically this corresponds to the fact that at 0, the curve C is singular but seems to have one tangent direction, the horizontal axis $y = 0$.

Curve D: on $\mathcal{B}_0$, we substitute $y = vx$ in the equation of D to get

$$y^2 - x^3 - x^2 = x^2(v^2 - x - 1),$$

and again $\tilde{D} \cap \mathcal{B}_0$ is cut out by $x = v^2 - 1$, a parabola meeting $\{x = 0\}$ at two points $v = \pm 1$. Geometrically, these points correspond to the two slopes ± 1 of the two "branches" of D at the origin. We skip the remainder of the computation. Conclusion: $\tilde{D}$ is smooth and its tangent cone consist of two points.

We also noted how for the curve $x^p = y^q$ for a coprime pair (p, q) successive blowups perform the Euclid algorithm on (p, q), ultimately desingularizing the curve.

6.5. Problem session (Feb 10), part two: What is algebraic geometry?

We discussed the following short list of problems (easy, hard, and unsolved) studied in algebraic geometry. We managed to cover (a)–(d) and part of (e), and we shall return to this review later.

(a) (Enumerative geometry) How many solutions does the system

$$\begin{cases} f(X,Y) = 0 \\ g(X,Y) = 0 \end{cases}$$

have, for a pair of coprime square-free polynomials $f, g \in k[X,Y]$? The answer (Bezout's theorem) is that this number is at most $\deg(f) \cdot \deg(g)$, and that if k is algebraically closed, it will typically be equal to this bound. More precisely, we will get exactly $\deg(f) \cdot \deg(g)$ if we count the solutions with correct multiplicity, and if we also count points at infinity (in the projective plane $\mathbb{P}^2$), which correspond to common asymptotes of the curves $f = 0$ and $g = 0$.

For a less trivial question of enumerative kind: how many conics are tangent to five randomly chosen conics in $\mathbb{P}^2$? The answer is 3264 (with probability 1). Why?

(b) (Real geometry) How many connected components does the set of real solutions of the equation

$$Y^2 = X^3 + aX + b$$

have? Answer: one if $\Delta = 4a^3 + 27b^2 \geq 0$ and two if $\Delta < 0$.

In general, according to Harnack's theorem for an equation $f(X,Y) = 0$ defining a nonsingular curve, the set of real solutions has at most $\binom{\deg(f)}{2} + 1$ connected components.

For a surprisingly still unsolved problem, consider Maxwell's "mystery of point charges" []. Consider 3 point charges q_1, q_2, q_2 located at three distinct points $p_i = (x_i, y_i)$ in the plane $\mathbb{R}^2$. According to Coulomb's law, the electrostatic force acting charge $c \neq 0$ located at a varying point $p = (x,y)$ equals

$$F(p) = \sum_{i=1}^{3} \frac{q q_i (p - p_i)}{|p - p_i|^3}.$$

We say that $p \in \mathbb{R}^2$ is an equilibrium point if $F(p) = 0$. How many equilibrium points can there be?

According to Maxwell, at most four (and in general, for n charges in $\mathbb{R}^3$, at most $(n-1)^2$). Nobody knows how Maxwell would justify this conjecture. According to Bezout's theorem and my hands-on calculation, the number of complex solutions is likely equal 1600. Currently, the best known bound, due to Gabrielov–Novikov–Shapiro [] (2004), is 12.

(c) (Algebraic topology) What is the "shape" of the set of complex solutions of the equation

$$f(X,Y) = 0?$$

We assume that this set is nonsingular, i.e. that f, $\partial f/\partial X$, and $\partial f/\partial Y$ do not have a common zero, (and, to be completely precise, that the homogenous form of f of highest degree is square-free). The answer is: it is an orientable surface of genus

$$g = \frac{(\deg(f) - 1)(\deg(f) - 2)}{2}$$

with $\deg(f)$ punctures.

(d) (Number theory) For f as above, how many solutions does $f(X,Y) = 0$ have over $\mathbb{Q}$? The answer (Faltings' resolution of the Mordell conjecture): finitely many as long as $g > 1$. In the special case $f(X,Y) = X^n + Y^n - 1$ (with $n \geq 3$) this question is Fermat's last theorem, proved by Wiles and Taylor in 1995.

(e) (Point counting) How many solutions does the equation

$$E \quad : \quad Y^2 = X^3 - X$$

have over the finite field $\mathbb{F}_7 = \mathbb{Z}/7$? Answer: 7, namely $(0,0)$, $(\pm 1, 0)$, $(5, \pm 1)$, $(6, \pm 2)$.

What do we learn from this? The equation describes an elliptic curve E, and the solution set over $\mathbb{F}_p$ (with the addition of a single point at infinity, which acts as the neutral element) is a finite abelian group $\overline{E}(\mathbb{F}_p)$, which in this particular case is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/4$. These finite groups are important in cryptography, and their orders $n_p = \#E(\mathbb{F}_p)$ (for a varying prime p) make up in the corresponding L-function

$$L(E,s) = \prod_{p>2} \frac{1}{1 - a_p p^{-s} + p^{1-2s}}, \qquad a_p = p + 1 - n_p$$

featured in the **BSD** (Birch and Swinnerton-Dyer) conjecture, which states that the order of vanishing of $L(E,s)$ at $s = 1$. One of the early results in the arithmetic of elliptic curves is the **Hasse bound**

$$|p + 1 - n_p| = |a_p| \leq 2\sqrt{p}.$$

The **Sato–Tate conjecture** (proved in 2011) predicts the distribution of the numbers $a_p/2\sqrt{p}$ in the interval $[-1,1]$ as $p \to \infty$.

The Hasse bound was generalized by Schmidt and Weil to arbitrary (smooth and projective) curves over finite fields:

$$|q + 1 - \#C(\mathbb{F}_q)| \leq 2g\sqrt{q}.$$

Curiously, this statement is equivalent to a kind of Riemann hypothesis for the corresponding zeta function $\zeta(C,s)$. We shall prove Weil's result in the final part of the course.

(f) (Moduli theory) Describe the set of all (smooth and projective) complex curves of genus $g \geq 2$. Answer (Deligne–Mumford): they are parametrized by an irreducible and smooth algebraic variety of dimension $3g - 3$.

(g) (Birational geometry) Consider a homogeneous equation of degree $d \geq 1$ in $n + 2 \geq 3$ variables:

$$X \quad : \quad f(X_0, \ldots, X_{n+1}) = 0.$$

Let us suppose that the corresponding set of (complex) solutions in $\mathbb{P}^{n+1}$ is non-singular (i.e. the only possible singular point of X in k^{n+2} is the origin). For which values of d and n can the variety X be rational? By definition, this means that there exists a one-to-one map from an open subset of $\mathbb{A}^n$ into X.

It turns out that X is rational for $d \leq 2$, and for $d = 3$ when $n = 2$ (a cubic surface is rational). For $d = 3$ and $n = 3$, the general cubic threefold is not rational, by a celebrated result of Clemens and Griffiths. For $d = 3$ and $n = 4$, the general cubic fourfold is not rational, according to a recent (summer 2025) preprint of Kontsevich, Katzarkov, Pantev, and Yu.

(h) (Singularity theory) Consider once again a hypersurface (single equation) over $\mathbb{C}$

$$X \quad : \quad f(T_1, \ldots, T_n) = 0$$

and suppose that $f(0, \ldots, 0) = 0$ and that $(0, \ldots, 0)$ is an isolated point of the zero set of f and all $\partial f / \partial T_i$ (that is, the origin is an isolated singular point of X). What can we say about the topology of the singularity of C around p? For a beautiful answer, read Milnor's book *Singular Points of Complex Hypersurfaces*. For a recent development in this area, see J. Fernández de Bobadilla, T. Pełka *Symplectic monodromy at radius zero and equimultiplicity of μ-constant families* (Ann. Math. 2024).