5. Lecture 5 (Feb 5): Local rings, nonsingular varieties

Recommended reading: Hartshorne I.4, I.5, I.6

5.1. Local rings, function fields, and rational maps

Let X be an affine algebraic set with coordinate ring $A = \mathcal{O}(X)$. Recall that irreducible closed subsets $Z \subseteq X$ correspond to prime ideals $\mathfrak{p} \subseteq A$. If Z is an irreducible subset, then every non-empty open subset is dense, and the intersection of two non-empty opens is non-empty.

Definition 5.1.1. Let X be a (not necessarily affine) algebraic set and let $Z \subseteq X$ be an irreducible closed subset. We define the **stalk at** Z (also called the **local ring** at Z) as the filtered colimit

$$\mathcal{O}_{X,Z} = \varinjlim_{U \cap Z \neq \emptyset} \mathcal{O}(U)$$

over all open subsets $U \subseteq X$ which intersect Z (this is a filtered colimit since Z is irreducible, by the previous remark).

In plain terms, by the basic properties of filtered colimits, an element of $\mathcal{O}_{X,Z}$ is an equivalence class of pairs (U, f) where $U \subseteq X$ is an open intersecting Z and where $f \in \mathcal{O}(U)$, where we identify (U, f) with (U', f') if there exists an open $U'' \subseteq U \cap U'$ intersecting Z such that $f|_{U''} = f'|_{U''}$.

Remark 5.1.2. In the definition, we can allow Z to be only *locally* closed. Simply pass to an open of X in which Z is closed.

Examples 5.1.3. (a) Let $x \in X$. Then $\{x\} \subseteq X$ is closed and irreducible, and we denote the ring $\mathcal{O}_{X,\{x\}}$ more simply by $\mathcal{O}_{X,x}$. Its elements are **germs** of regular functions defined in a neighborhood of x.

(b) At the other extreme, suppose that X is itself irreducible (i.e. a "variety"). In this case we can take $Z = X$. We denote the ring $\mathcal{O}_{X,X}$ more simply by $k(X)$ and call it the **function field** of X. Its elements are represented by regular functions defined on a non-empty open subset of X.

Lemma 5.1.4. *Let X be an algebraic set and let $Z \subseteq X$ be an irreducible closed subset.*

(a) Let $U \subseteq X$ be an affine open intersecting Z, so that $Z \cap U$ is an irreducible closed subset of U. Let $A = \mathcal{O}(U)$ and let $\mathfrak{p} = \mathcal{I}(Z \cap U) \subseteq A$ be the prime ideal corresponding to $Z \cap U$. Then

$$\mathcal{O}_{X,Z} = \mathcal{O}_{U,U\cap Z} = A_{\mathfrak{p}}.$$

(Recall that $A_{\mathfrak{p}} = A[(A \setminus \mathfrak{p})^{-1}]$.) In particular, $\mathcal{O}_{X,Z}$ is a local ring with maximal ideal consisting of germs of functions (U, f) which vanish on $Z \cap U$.

(b) If X is irreducible, then $k(X)$ is a field, equal to the fraction field of $\mathcal{O}(U)$ (which is a domain) for every non-empty affine open $U \subseteq X$.

(c) The residue field of the local ring $\mathcal{O}_{X,Z}$ is the function field $k(Z)$ of Z.

Lemma 5.1.5. *Let $f \colon Y \to X$ be a morphism between algebraic sets and let $Z \subseteq Y$ be an irreducible closed subset. Let $W \subseteq X$ be the closure of $f(Z)$. Then W is also irreducible, and f induces a local homomorphism of local rings*

$$f^* \colon \mathcal{O}_{X,W} \longrightarrow \mathcal{O}_{Y,Z}.$$

Proof. That $f(Z)$ and its closure are both irreducible is easy general topology. Moreover, it follows from Chevalley's theorem (§4.4) that $f(Z)$ contains a dense open subset of W. Therefore, if $U \subseteq X$ is an open intersecting W, it has to intersect $f(Z)$, and thus $f^{-1}(U) \subseteq Y$ is an open intersecting Z. The pull-back maps $f^*\colon \mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U))$ for varying U induce map on filtered colimits

$$f^*\colon \mathcal{O}_{X,W} = \varinjlim_U \mathcal{O}_X(U) \longrightarrow \varinjlim_U \mathcal{O}_Y(f^{-1}(U))$$

(both colimits over opens $U \subseteq X$ meeting W) which we compose with the natural map (from the universal property of direct limit!)

$$\varinjlim_U \mathcal{O}_Y(f^{-1}(U)) \longrightarrow \varinjlim_V \mathcal{O}_Y(V) = \mathcal{O}_{Y,Z}$$

(second colimit over opens $V \subseteq Y$ meeting Z) to obtain the desired map $f^*\colon \mathcal{O}_{X,W} \to \mathcal{O}_{Y,Z}$. This homomorphism is local thanks to Lemma 5.1.4(a): if $g \in \mathcal{O}_X(U)$ vanishes on $U \cap W$ then $f^*(g) = g \circ f \in \mathcal{O}_Y(f^{-1}(U))$ vanishes on $f^{-1}(U \cap W) \supseteq f^{-1}(U) \cap Z$. □

We shall now consider rational maps between varieties. For this, let us note the following straightforward corollary of Lemma 5.1.5. For this, let us call a map $f\colon Y \to X$ **dominant** if $f(Y)$ is dense in X (and hence, by Chevalley, contains a dense open subset of X).

Corollary 5.1.6. *A dominant map between varieties $f\colon Y \to X$ induces an extension of function fields $f^*\colon k(X) \hookrightarrow k(Y)$.*

Proof. Apply Lemma 5.1.5 to $Z = Y$, so $\mathcal{O}_{Y,Z} = k(Y)$. Since f is dominant, $f(Z) = f(Y)$ is dense, and we have $W = X$, so $\mathcal{O}_{X,W} = k(X)$. □

A rational map is a germ of a function between varieties.

Definition 5.1.7. Let X and Y be varieties (i.e. irreducible algebraic sets). A **rational map** from Y to X is an equivalence class of pairs (U, f) where $U \subseteq Y$ is a non-empty open subset and where $f\colon U \to X$ is a map of varieties, where we identify (U, f) and (U', f') if $f = f'$ on some non-empty open $U'' \subseteq U \cap U'$. We call a rational map (U, f) **dominant** if $f(U)$ is dense in X (this condition depends only on the equivalence class of (U, f)).

Remark 5.1.8. (a) A dominant rational map (U, f) from Y to X induces a pull-back map $f^*\colon k(X) \to k(Y)$ between the function fields.

(b) Conversely, let $k(X) \to k(Y)$ be a map of k-algebras. Then there exists a unique dominant rational map $Y \to X$ inducing this field extension.

(c) Dominant rational maps can be composed. The resulting category of varieties and rational maps is equivalent to the opposite of the category of finitely generated field extensions of k.

(d) If $V \subseteq Y$ is an open such that a given rational map from Y to X is represented by a pair (V, f), we say that f is **defined on** V. If Y is separated, there exists a largest open $V \subseteq Y$ on which a given rational map is defined.

Definition 5.1.9. A rational map f from Y to X is **birational** if it is dominant and if it admits an inverse (in the category of dominant rational maps), or equivalently if there exist non-empty opens $V \subseteq Y$ and $U \subseteq X$ such that f induces an isomorphism $V \simeq U$, or equivalently if it induces an isomorphism $k(X) \simeq k(Y)$. We say that two varieties X and Y are **birational** if there exists a birational rational map from Y to X. We say that a variety X is **rational** if it is birational to $\mathbb{P}^n$ for some $n \geq 0$.

Example 5.1.10. We shall prove later that the cubic plane curve $V(Y^2 - X^3 - X)$ is not rational.

5.2. Nonsingular varieties

Recall that a Noetherian local ring A is **regular** if

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(A).$$

Here $\mathfrak{m} \subseteq A$ is the unique maximal ideal and $k = A/\mathfrak{m}$ is its residue field. In general, we have $\geq$ instead of equality, and the left-hand side coincides with the minimal number of generators of $\mathfrak{m}$. Every regular ring is a UFD (this is not so easy to show).

Definition 5.2.1. Let X be an algebraic set and let $x \in X$. We say that X is **nonsingular** at x if $\mathcal{O}_{X,x}$ is a regular local ring.

We shall also call x a **smooth** or **regular** point. If every point is nonsingular, we say that X itself is nonsingular/smooth/regular.

Examples 5.2.2. (a) If $\dim(X) \leq 1$, the ring $\mathcal{O}_{X,x}$ is regular if and only if it is a discrete valuation ring (or equal to k in case of isolated points), if and only if it is integrally closed.

(b) The affine space $\mathbb{A}^n$ is nonsingular. Indeed, for every $x = (x_1, \dots, x_n) \in \mathbb{A}^n$ we have $\dim(\mathcal{O}_{X,x}) = n$ and the maximal ideal $\mathfrak{m}$ is generated by n elements $T_i - x_i$.

(c) Let $X = V(f) \subseteq \mathbb{A}^n$ be a hypersurface (where $f \neq 0$) and let $x \in X$. Then X is nonsingular at x if and only if $(\partial f/\partial T_i)(x) \neq 0$ for some i. Proof: Since $\dim(\mathcal{O}_{X,x}) = n-1$, we want $\dim(\mathfrak{m}/\mathfrak{m}^2) = n-1$. If $\mathfrak{n} = (T_1 - x_1, \dots, T_n - x_n) \subseteq k[T_1, \dots, T_n]$ is the ideal corresponding to x, then

$$\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{n}/((\mathfrak{n}^2 + (f))$$

which has dimension either n or $n-1$, and the latter precisely when $f \notin \mathfrak{n}^2$. We can write

$$f(T_1, \dots, T_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i}(x) \cdot (T_i - x) + R$$

where $R \in \mathfrak{n}^2$. Thus $f \in \mathfrak{n}^2$ precisely when $(\partial f/\partial T_i)(x) = 0$ for all i.

Theorem 5.2.3. *Let $X \subseteq \mathbb{A}^n$ be an affine variety and let $I = \mathcal{I}(X) = (f_1, \dots, f_r)$ be its ideal. Then a point $x \in X$ is nonsingular if and only if*

$$\operatorname{rank} \left[\frac{\partial f_j}{\partial T_i}(x) \right] = n - \dim(X).$$

Proof. We did the case $r = 1$. For the general case see Theorem I 5.1 in Hartshorne. □

In general, for two nonsingular points $x \in X$ and $y \in Y$, the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,y}$ are often non-isomorphic even if they have the same dimension. Indeed, we have $k(X) = \operatorname{Frac}(\mathcal{O}_{X,x})$, so $\mathcal{O}_{X,x} \simeq \mathcal{O}_{Y,y}$ only if X and Y are birational. So our intuition from differential geometry that a smooth variety should locally look like $\mathbb{A}^n$, taken too literally, is false. This is because $\mathcal{O}_{X,x}$ is defined in terms of Zariski open neighborhoods of x, which are very large. One can resolve this issue either by considering the étale topology (which we might cover later) or by completing the local ring.

Theorem 5.2.4 (Special case of Cohen's structure theorem)**.** *Let X be an algebraic set and let $x \in X$. Denote by*

$$\widehat{\mathcal{O}}_{X,x} = \varprojlim_{n} \mathcal{O}_{X,x}/\mathfrak{m}_x^n$$

the **completion** *of the local ring $\mathcal{O}_{X,x}$ with respect to its maximal ideal. Then x is a nonsingular point of X if and only if*

$$\widehat{\mathcal{O}}_{X,x} \simeq k[\![T_1, \dots, T_n]\!].$$

More precisely, if $f_1, \dots, f_n \in \mathfrak{m}_x$, we have a unique continuous homomorphism

$$\theta\colon k[\![T_1, \dots, T_n]\!] \longrightarrow \widehat{\mathcal{O}}_{X,x}, \qquad \theta(T_i) = f_i.$$

The map θ is surjective if and only if $\mathfrak{m}_x = (f_1, \dots, f_n)$, and an isomorphism if and only if in addition

$$n = \dim(k[\![T_1, \dots, T_n]\!]) = \dim(\mathcal{O}_{X,x}),$$

which happens precisely if $\mathcal{O}_{X,x}$ is regular.