3. Lecture 3 (Jan 29): Products, separatedness, completeness

Recommended reading: Kempf §3

3.1. Basic facts about algebraic sets

In order to distinguish between regular functions on different spaces, we shall sometimes write $\mathcal{O}_X(U)$ instead of $\mathcal{O}(U)$ for the set of regular functions on an open $U \subseteq X$ of a space with functions X.

Definition 3.1.1. Let X be a space with functions and let $Y \subseteq X$ be a subspace (subset, endowed with the induced topology). The **induced swf structure** on Y is defined as follows: a function defined on an open of Y is regular if locally on U it extends to a regular function on an open of X.

To be completely precise: for an open $V \subseteq Y$ and $f\colon V \to k$, we have $f \in \mathcal{O}_Y(V)$ if and only if there exist opens $U_\alpha \subseteq X$ such that $V \subseteq \bigcup U_\alpha$ and regular functions $f \in \mathcal{O}_X(U_\alpha)$ such that $f(y) = f_\alpha(y)$ for every $y \in V \cap U_\alpha$. One checks easily that Y endowed with $\mathcal{O}_Y$ defined this way is an swf, and that the inclusion $Y \to X$ is a morphism of swf's.

Remark 3.1.2. A reader acquainted with sheaf theory will notice that $\mathcal{O}_Y$ is the image of the morphism of sheaves $i^{-1}(\mathcal{O}_X) \to \prod_{y\in Y} k_y$. Here $i^{-1}(\mathcal{O}_X)$ is the sheaf pull-back of $\mathcal{O}_X$, and $\prod_{y\in Y} k_y$ is the sheaf of (not necessarily continuous) k-valued functions on Y.

In the context of affine algebraic sets, the way we have endowed $X = V(I) \subseteq \mathbb{A}^n$ with an swf structure shows that it is the induced swf structure from $\mathbb{A}^n$.

Definition 3.1.3. A morphism of swf's $Y \to X$ is an **immersion** if it is a homeomorphism onto its image and the swf structure on Y coincides with the induced swf structure on the image. A **closed** (resp. **open**, resp. **locally closed**) immersion is an immersion whose image is closed (resp. open, resp. locally closed) in X.

Example 3.1.4. Consider the map $f\colon \mathbb{A}^1 \to \mathbb{A}^2$ sending t to (t^2,t^3). It is a homeomorphism onto its image, which is the cuspidal curve $C = V(Y^2 - X^3)$. However, it is not an immersion, since the map on coordinate rings is

$$f^*\colon k[X,Y]/(Y^2 - X^3) = \mathcal{O}(C) \longrightarrow \mathcal{O}(\mathbb{A}^1) = k[T], \qquad f^*(X) = T^2,\, f^*(Y) = T^3$$

which is not an isomorphism.

Proposition 3.1.5. *Let X be an algebraic set and let $Y \subseteq X$ be a locally closed subset. Then Y is an algebraic set (when endowed with the induced swf structure).*

Proof. Let $X = U_1 \cup \dots \cup U_n$ be a finite affine open cover of X. Then $V_i = U_i \cap Y$ form an open cover of Y, and each V_i is locally closed in the affine algebraic set U_i. It follows that we may assume that X itself is affine (as an swf which is covered by a finite number of opens which are algebraic sets is an algebraic set).

Suppose first that Y is closed in X. Since X is affine, it is closed in $\mathbb{A}^n$. So Y is closed in $\mathbb{A}^n$ and endowed with the induced swf structure, and hence an affine algebraic set.

Now, suppose that Y is open in X. In this case, since the standard open affines $D(f) \subseteq X$ (for $f \in \mathcal{O}(X)$) form a base of the topology on Y, we can write Y as the union of such opens. Moreover, since X is Noetherian, Y is quasi-compact, and hence a finite number suffices. □

Let us record the following crucial fact which follows from the above proof:

Lemma 3.1.6. *Let X be an algebraic set. Then affine open subsets of X form a base of the topology on X.*

3.2. Products

Recall from the last lecture that if X and Y are *affine* algebraic sets, then $X \times Y$ is an affine algebraic set with coordinate ring
$$\mathcal{O}(X \times Y) \simeq \mathcal{O}(X) \otimes_k \mathcal{O}(Y).$$
A word of warning (see 1.3(d)): $X \times Y$ is the product of X and Y as sets, but not as topological spaces.

Theorem 3.2.1. *Let X and Y be algebraic sets. Then, the product $X \times Y$ of swf's exists and is an algebraic set. Moreover, if X and Y are projective (isomorphic to a closed algebraic subset of $\mathbb{P}^n$ for some n), then so is $X \times Y$.*

Proof. The proof of the first part is straightforward (and boring). Cover $X = U_1 \cup \cdots \cup U_n$ and $Y = V_1 \cup \cdots \cup V_m$ with affine open subsets. Then, as sets
$$X \times Y = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} U_i \times V_j.$$
Each $U_i \times V_j$ has the structure of an affine algebraic set, with coordinate ring $\mathcal{O}(U_i) \otimes \mathcal{O}(V_j)$. We give $X \times Y$ the topology induced by the topologies on $U_i \times V_j$ (so $W \subseteq X \times Y$ is open iff $W \cap (U_i \times V_j)$ is open in $U_i \times V_j$ for all (i, j)), and deem a function $f\colon W \to k$ defined on an open $W \subseteq X \times Y$ regular if its restriction to $W \cap (U_i \times V_j)$ is regular for each pair (i, j). We then need to verify that

(a) Each $U_i \times V_j$ is an open subset of $X \times Y$.

(b) If $W \subseteq X \times Y$ is an open contained in $U_i \times V_j$ for some (i, j), then a function $f\colon W \to k$ is regular if and only if it is regular when treated as a function on an open subset of the affine algebraic set $U_i \times V_j$.

(c) For any swf Z and maps $f\colon Z \to X$, $g\colon Z \to Y$, the resulting map $(f \times g)\colon Z \to X \times Y$ is a morphism of swf's.

(The first two ensure that $X \times Y$ is an an algebraic set, and the last one gives the universal property of the product.) I suggest you prove these statements, and look up the proof in a textbook in case you get stuck.

The assertion about projective algebraic sets is more fun to prove. Note first that it suffices to show that $\mathbb{P}^n \times \mathbb{P}^m$ is projective for every $n, m \geq 0$. Let $X_0, \ldots, X_n$ and $Y_0, \ldots, Y_m$ be their homogeneous coordinates Set $N = (n+1)(m+1) - 1 = nm + n + m$ and consider $P = \mathbb{P}^N$ with homogeneous coordinates W_{ij} ($i = 0, \ldots, n$, $j = 0, \ldots m$). Consider the map (called the **Segre embedding**)
$$\phi\colon \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$$
defined by
$$((X_0 : \cdots : X_n), (Y_0 : \cdots : Y_m)) \mapsto (W_{ij} = X_i Y_j) = \begin{bmatrix} X_0Y_0 & X_0Y_1 & \ldots & X_0Y_m \\ X_1Y_0 & \ldots & & \ldots \\ \ldots & & \ldots & \\ X_nY_0 & \ldots & & X_nY_m \end{bmatrix}$$
Note first of all that this is a well-defined map of sets. Indeed, if we scale either all X_i's or all Y_j's by the same scalar $\lambda \in k^\times$, the result scales by the same factor, and if $X_i \neq 0$ and $Y_j \neq 0$, then $W_{ij} \neq 0$.

Next, we identify the image of this map. A matrix $[W_{ij}] \in \mathbb{P}^N$ is in the image if and only if it is of rank one (it cannot be of rank zero since not all coordinates vanish). On the other hand, this holds if and only if every 2×2 minor of this matrix is zero, and hence the image Q of ϕ is the projective algebraic set defined by the system of homogeneous equations

$$0 = \det \begin{bmatrix} W_{ij} & W_{ij'} \\ W_{i'j} & W_{i'j'} \end{bmatrix} = W_{ij}W_{i'j'} - W_{i'j}W_{ij'}, \qquad i,i' \in \{0,\dots,n\}, \quad j,j' \in \{0,\dots,m\}$$

To check that the map ϕ induces an isomorphism onto Q, we check what happens over the open subset $\mathbb{A}^N \simeq D(W_{00}) \subseteq \mathbb{P}^N$. Its preimage is defined by $X_0 \neq 0 \neq Y_0$ and hence it is equal to the open subset $D(X_0)\times D(Y_0) \simeq \mathbb{A}^n \times \mathbb{A}^m$ of $\mathbb{P}^n \times \mathbb{P}^m$. In the dehomogenized coordinates $w_{ij} = W_{ij}/W_{00}$, $x_i = X_i/X_0$, $y_j = Y_j/Y_0$, the intersection $Q \cap D(W_{00})$ is cut out by the equations $w_{ij} = w_{0j}w_{j0}$ (set $(i',j') = (0,0)$), and the restriction of ϕ is defined by the map of rings

$$k[w_{ij} \,:\, (i,j) \neq (0,0)]/(w_{ij} - w_{0j}w_{i0}) \longrightarrow k[x_1,\dots,x_n,y_1,\dots,y_m], \qquad w_{ij} \mapsto x_i y_j$$

(where we interpret $x_0 = 1$ and $y_0 = 1$) which is an isomorphism, the inverse sending x_i to w_{i0} and y_j to w_{0j}. □

3.3. Separated varieties

Recall (or learn) the following fact from topology. A topological space is Hausdorff if and only if the diagonal

$$\Delta = \{(x,x) \,:\, x \in X\} \quad \subseteq \quad X \times X$$

is a closed subset of $X \times X$.

Definition 3.3.1. An algebraic set X is **separated** if the diagonal $\Delta \subseteq X \times X$ is a closed subset of $X \times X$.

Note that this does **not** imply that X is Hausdorff since the topology on $X \times X$ is not the product topology.

Examples 3.3.2. (a) If we have an injective map $f\colon Y \to X$ and X is separated, then so is Y. Indeed, then $\Delta_Y \subseteq Y \times Y$ is the preimage of the closed subset $\Delta_X \subseteq X \times X$ under the continuous map $f \times f\colon Y \times Y \to X \times X$.

(b) If X is (quasi)affine then X is separated. Indeed, then X admits an injective map to $\mathbb{A}^n$, and $\mathbb{A}^n$ is separated, as its the diagonal in $\mathbb{A}^n \times \mathbb{A}^n$ (with coordinates $X_1,\dots,X_n,Y_1,\dots,Y_n$) is cut out by the equations $X_i = Y_i$ and hence is closed.

(c) The projective space $\mathbb{P}^n$ is separated. (Consequently, every quasi-projective algebraic set is separated.) To see that $\mathbb{P}^n$ is separated, we use the Segre embedding $\mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^{n^2+2n}$. The diagonal is the preimage of the linear subvariety cut out by the equations $W_{ij} = W_{ji}$, and is therefore closed.

(d) (Line with doubled origin) Consider the space with functions X obtained by gluing two copies $U_i = \mathbb{A}^1$ (with coordinate T_i) for $i = 0,1$ along the isomorphism of open subsets $D(T_0) \simeq D(T_1)$ sending T_1 to T_0. There is a natural map $X \to \mathbb{A}^1$ (with coordinate T on the target pulling back to T_i on U_i) which is bijective away from zero, and such that $0 \in \mathbb{A}^1$ has two preimages $0_i \in U_i$. On $U_0 \times U_1$, the diagonal is $V(T_0 - T_1) \setminus \{(0_0, 0_1)\}$ and is not closed. Thus X is not separated.

Lemma 3.3.3. *Let X be a separated algebraic set and let $U,V \subseteq X$ be affine open subsets. Then $U \cap V$ is affine.*

Proof. We have $U \cap V \simeq (U \times V) \cap \Delta$, so $U \cap V$ is a closed subset of the affine algebraic set $U \times V$. □

Example 3.3.4 (Plane with doubled origin)**.** Consider a variant of Example 3.3.2(d) where we replace $U_i = \mathbb{A}^1$ with $\mathbb{A}^2$ with coordinates T_i, U_i and the open subsets $D(T_i)$ with $U_i \setminus \{(0,0)\}$. The resulting space X has two affine opens U_0 and U_1 whose intersection $U_0 \cap U_1$ is isomorphic to $\mathbb{A}^2 \setminus \{(0,0)\}$, which is not affine.

Again, recall from topology that two maps $f, g\colon Y \to X$ to a Hausdorff space X which are equal on a dense subset of Y have to be equal. Here is an algebraic variant:

Lemma 3.3.5. *Let $f, g\colon Y \to X$ be a parallel pair of maps between algebraic sets. If X is separated, then the subset ("equalizer")*

$$\mathrm{Eq}(f,g) = \{y \in Y \,:\, f(y) = g(y)\} \subseteq Y$$

is closed in Y.

Proof. Use the "diagonal trick:" $\mathrm{Eq}(f,g) = (f \times g)^{-1}(\Delta_X)$ is the preimage of the diagonal $\Delta_X \subseteq X \times X$ under the map $f \times g\colon Y \to X \times X$. □

3.4. Complete varieties

Recall (or learn) another fact from topology. A Hausdorff topological space X is compact if and only if for every topological space Y, the projection map

$$\pi_Y : X \times Y \longrightarrow Y$$

is closed (maps closed subsets of $X \times Y$ to closed subsets of Y).

Definition 3.4.1. An algebraic set X is **complete** (a.k.a. **proper**) if X is separated and for every algebraic set Y, the projection map

$$\pi_Y : X \times Y \longrightarrow Y$$

is closed.

Remark 3.4.2. (a) Again, $X \times Y$ does not have the product topology.

(b) The affine space $\mathbb{A}^n$ is not complete for $n \geq 1$. More generally, let X be an algebraic set admitting a function $f \in \mathcal{O}(X)$ which takes infinitely many values. Then X is not complete. Proof: consider $Y = \mathbb{A}^1$ with coordinate T and the closed subset $Z = V(fT - 1) \subseteq X \times Y$. Its projection onto Y is then an infinite subset which does not contain zero, and hence cannot be closed.

(c) We shall prove later that $\mathbb{P}^n$ is complete (and therefore, by Lemma 3.4.3, every projective algebraic set is complete).

(d) The following relative versions of separatedness and completeness are used in algebraic geometry. A morphism $f\colon X \to S$ is **separated** if the diagonal $\Delta \subseteq X \times X$ is a closed subset of the fiber product

$$X \times_S X = \{(x,y) \in X \times X \,:\, f(x) = f(y)\} \subseteq X \times X.$$

A separated morphism $f\colon X \to S$ is **proper** if for every map of algebraic sets $g\colon Y \to S$, the projection map

$$\pi_Y : X \times_S Y \longrightarrow Y$$

is closed. Here $X \times_S Y = \{(x,y) \in X \times Y \,:\, f(x) = g(y)\}$.

(e) A closed subspace of a complete algebraic set is complete.

Another standard fact from topology: if $f\colon Y \to X$ is a map from a compact space Y into a Hausdorff space X, then the image $f(Y) \subseteq X$ is a closed subspace of X (and is therefore both compact and Hausdorff). Algebraic version:

Lemma 3.4.3. *Let $f\colon Y \to X$ be a map from a complete algebraic set Y to a separated algebraic set X. Then $f(Y) \subseteq X$ is closed in X and complete.*

Proof. For the first statement, consider the graph

$$\Gamma_f = \{(y,x) : x = f(y)\} \subseteq Y \times X.$$

It is closed in $Y \times X$, being the equalizer of the projection $\pi_X\colon Y \times X \to X$ and the composition $f \circ \pi_Y\colon Y \times X \to X$ (here we use Lemma 3.3.5). Then $f(Y) = \pi_X(\Gamma_f)$ is closed in X since by assumption (Y complete) the map $X \times Y \to X$ is closed.

It remains to show that $Z = f(X)$ is complete. For this we use the surjective map $Y \to Z$ induced by f. Let W be an algebraic set and let $F \subseteq Z \times W$ be a closed subset. We must show that $\pi_W(F)$ is closed in W. Let $F' \subseteq Y \times W$ be the preimage of F. Since $Y \to Z$ is surjective, so is $F' \to F$, and hence $\pi_W(F) = \pi_W(F')$ is closed in W since $\pi_W\colon Y \times W \to W$ is closed. □

Lemma 3.4.4. *Let X be a complete variety. Then $\mathcal{O}(X) = k$.*

Proof. Let $f \in \mathcal{O}(X)$, we must show that f is constant. Treat f as a morphism $f\colon X \to \mathbb{A}^1$. The image $f(X) \subseteq \mathbb{A}^1$ is then closed and complete by the previous lemma. It is also irreducible (being the image of the irreducible space X), and hence it is a singleton (because $\mathbb{A}^1$ is not complete, see Remark 3.4.2(a)). □

Corollary 3.4.5. *Let X be an algebraic set which is both compact and quasi-affine. Then X is a finite set.*