2. Lecture 2 (Jan 15 and 27): Algebraic sets

Recommended reading: Kempf §1, Hartshorne I.3

2.1. Some general topology

Definition 2.1.1. Let X be a topological space.

(a) We say that X is **irreducible** if for every pair of closed subsets $Y_1, Y_2 \subseteq X$ with $X = Y_1 \cup Y_2$, we have $X = Y_1$ or $X = Y_2$. (Equivalently, every non-empty open subset of X is dense.)

(b) The **dimension** $\dim(X)$ is the supremum of the set of integers $n \geq 0$ for which there exists a chain

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

of distinct closed irreducible subsets of X.

(c) We say that X is **Noetherian** if every decreasing sequence $F_0 \supseteq F_1 \supseteq \ldots$ of closed subsets stabilizes (is eventually constant).

Proposition 2.1.2. *Let $X = V(f_1, \ldots, f_r) \subseteq k^n$ be an affine algebraic set and let $A = k[T_1, \ldots, T_n]/(f_1, \ldots, f_r)$. Then:*

(a) X is irreducible if and only if the coordinate ring

$$\mathcal{O}(X) = k[T_1, \ldots, T_n]/\mathcal{I}(X) = k[T_1, \ldots, T_n]/\sqrt{(f_1, \ldots, f_r)} = A/\sqrt{0}$$

is a domain, or equivalently iff the ideal $\sqrt{(f_1, \ldots, f_r)} = \mathcal{I}(X) \subseteq k[T_1, \ldots, T_n]$ is prime.

(b) X is a Noetherian topological space.

(c) We have $\dim(X) = \dim(\mathcal{O}(X)) = \dim(A)$ *(the latter two denote the* **Krull dimension***).*

Remark 2.1.3. Thus irreducible closed subsets of X correspond to prime ideals of $\operatorname{Spec}(A)$. The smallest irreducible closed subsets are the points of X, which correspond to the largest prime ideals, i.e. the maximal ideals.

Lemma 2.1.4. *Let X be a Noetherian topological space. Then there exist closed irreducible subsets $Z_1, \ldots, Z_r \subseteq X$ such that $Z_i \not\subseteq Z_j$ for $i \neq j$ and*

$$X = Z_1 \cup \cdots \cup Z_r.$$

They are unique up to permutation.

Definition 2.1.5. The closed irreducible subsets $Z_1, \ldots, Z_r \subseteq X$ in the lemma are called the **irreducible components** of X.

Note that we have $\dim(X) = \sup\{\dim Z_1, \ldots, \dim Z_r\}$. In order to say more about dimension, we need to review some results from the **dimension theory** part of commutative algebra (see e.g. Chapter 10 of Atiyah–Macdonald, though we need a bit more).

Let $X \subseteq \mathbb{A}^n$ be an irreducible algebraic set, let $\mathfrak{p} = \mathcal{I}(X) \subseteq k[T_1, \ldots, T_n]$ be the corresponding prime ideal, and let $A = k[T_1, \ldots, T_n]/\mathfrak{p} = \mathcal{O}(X)$.

(1) Let $K = \mathrm{Frac}(A)$ be the field of fractions. Then

$$\dim(X) = \dim(A) = \mathrm{trdeg}(K/k)$$

is the transcendence degree of the extension K/k. In particular, we have $\dim \mathbb{A}^n = n$.

(2) All maximal chains $Z_0 \subsetneq \cdots \subsetneq Z_r$ of irreducible subsets of X have length $r = \dim(X)$.

From this we can deduce that for a locally closed $Y \subseteq \mathbb{A}^n$, we have $\dim(Y) = \dim(\overline{Y})$.

(3) Let $f \in A$ be a nonzero nonunit, and let $Y = V(f) \subseteq X$. Let $Y_1, \ldots, Y_r \subseteq Y$ be the irreducible components of Y. Then

(a) We have $\dim(Y_i) = \dim(X) - 1$ for every $i = 1, \ldots, r$.

(b) Conversely, A is a UFD (unique factorization domain, for example if $X = \mathbb{A}^n$), then every closed irreducible subset $Y \subseteq X$ such that $\dim(Y) = \dim(X) - 1$ is of the form $Y = V(f)$ for some prime element $f \in A$.

2.2. Regular functions

A polynomial $f \in k[T_1, \ldots, T_n]$ defines a function $f\colon k^n \to k$. Similarly, a rational function $f = p/q \in k(T_1, \ldots, T_n)$ with $p, q \in k[T_1, \ldots, T_n]$, $q \neq 0$ defines a function $f = p/q\colon D(q) \to k$ where $D(q) = k^n \setminus V(q)$. Recall that the subsets $D(q)$ for a varying q form a basis of the Zariski topology on $k^n = \mathbb{A}^n$. We define regular functions as those which are *locally* given by a rational function.

Definition 2.2.1. Let $X \subseteq k^n$ be an algebraic set, $U \subseteq X$ an open subset, and let $f\colon U \to k$ be a function. We say that f is a **regular function** on U if every $x \in U$ there exists an open neighborhood $x \in V \subseteq U$ and $p, q \in k[T_1, \ldots, T_n]$ such that for every $y \in V$, we have $q(y) \neq 0$ and

$$f(y) = \frac{p(y)}{q(y)}.$$

We observe first that if f is a regular function on U and $Z \subseteq U$ is a locally closed subset, then $f|_Z$ is a regular function. Moreover, every polynomial $f \in k[T_1, \ldots, T_n]$ defines a regular function on $\mathbb{A}^n$ and hence on every locally closed $U \subseteq \mathbb{A}^n$. For an affine algebraic set $X \subseteq \mathbb{A}^n$ we obtain a map

$$A = \mathcal{O}(X) = k[T_1, \ldots, T_n]/\mathcal{I}(X) \longrightarrow \{\text{regular functions on } X\}.$$

It follows from the Nullstellensatz that this map is injective. Moreover, regular functions on any locally closed subset $U \subseteq \mathbb{A}^n$ form a k-subalgebra of the ring of all functions $U \to k$.

Theorem 2.2.2. *Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set and let $A = k[T_1, \ldots, T_n]/\mathcal{I}(X)$ be its coordinate ring. Then:*

(a) The map

$$A \longrightarrow \{\textit{regular functions on } X\}$$

is an isomorphism of k-algebras.

(b) For $g \in A$, let $U = D(g) \subseteq X$. Then the above map induces an isomorphism

$$A[g^{-1}] \simeq \{\textit{regular functions on } U\}.$$

The proof shows an algebraic variant of "partitions of unity" in differential geometry.

Proof. (a) We need to show that this map is surjective, so let f be a regular function on X and let $X = \bigcup V_i$ be an open cover such that $f|V_i = p_i/q_i$ for $p_i, q_i \in k[T_1, \ldots, T_n]$ with $V_i \subseteq D(q_i)$. We may assume that $V_i = D(g_i)$ for some $g_1, \ldots, g_r \in A$ generating the unit ideal in A. We can simplify this a bit further: replacing g_i with $g_i q_i$ and p_i/q_i with $(p_i g_i)/(q_i g_i)$ we may assume that $g_i = q_i$. Consider the functions

$$f_i = q_i^2 f \colon X \to k.$$

We notice that $f_i = p_i q_i$ for every i (the right-hand side is the function $X \to k$ defined by the element $p_i q_i \in k[T_1, \ldots, T_n]$). Indeed, on V_i we have $f_i = q_i^2 (p_i/q_i) = p_i q_i$, and outside of V_i both sides are zero.

Since $A = (q_1, \ldots, q_r)$, we also have $A = (q_1^2, \ldots, q_r^2)$. Let $a_1, \ldots, a_r \in A$ be such that $1 = \sum a_i q_i^2$. Multiply the last equality by f to get

$$f = \sum_{i=1}^{r} a_i q_i^2 f = \sum_{i=1}^{r} a_i f_i = \sum_{i=1}^{r} a_i p_i q_i \in A.$$

(b) We apply (a) to $V(I, T_{n+1} g - 1) \subseteq \mathbb{A}^{n+1}$. □

2.3. Spaces with functions

Definition 2.3.1. Fix a field k. A **space with functions** (**swf** for short) is a topological space X together with an assignement, for every open $U \subseteq X$, of a k-subalgebra $\mathcal{O}(U)$ of the ring of all functions $U \to k$ (called the ring of **regular functions** on U) such that

(a) Being regular is a local property. That is, if $U = \bigcup U_\alpha$ is an open cover and $f \colon U \to k$ is a function, then $f \in \mathcal{O}(U)$ if and only if $f|_{U_\alpha} \in \mathcal{O}(U_\alpha)$ for each α.

(b) If $U \subseteq X$ is an open subset and $f \in \mathcal{O}(U)$, then the set

$$D(f) = \{x \in U \,:\, f(x) \neq 0\} \subseteq U$$

is an open subset of U and $(f|_{D(f)})^{-1} \in \mathcal{O}(D(f))$.

A **morphism** of spaces with functions is a continuous map $\phi \colon Y \to X$ such that the pullbacks of regular funtions are regular: for every open $U \subseteq X$ and every regular function $f \in \mathcal{O}(U)$, the function $f \circ \phi \in \mathcal{O}(\phi^{-1}(U))$.

Note that a map of swf's $\phi \colon Y \to X$ induces a k-algebra homomorphism

$$\phi^* \colon \mathcal{O}(X) \longrightarrow \mathcal{O}(Y), \qquad \phi^*(f) = f \circ \phi.$$

Remark 2.3.2 (If you know some sheaf theory)**.** Condition (a) means that $\mathcal{O}$ forms a *subsheaf* of the sheaf $\prod_{x \in X} k_x$ of all k-valued functions on X. Condition (b) ensures that the stalks $\mathcal{O}_x = \varinjlim_{x \in U} \mathcal{O}(U)$ for $x \in X$ are local rings, with maximal ideal $\mathfrak{m}_x = \{f \in \mathcal{O}_x \,:\, f(x) = 0\}$.

Examples 2.3.3. (a) Let $k = \mathbb{R}$ or $\mathbb{C}$ and let X be a topological space. Then $\mathcal{O}(U) = C(U; k)$ (continuous functions $U \to k$) gives X the structure of a space with functions.

(b) Similarly with C^∞, analytic, and complex manifolds.

(c) If X is a space with functions and $U \subseteq X$ is an open subset, then U is a space with functions in the obvious way.

(d) Let again k be our chosen algebraically closed field and let $X \subseteq \mathbb{A}^n$ be a locally closed subset. For an open $U \subseteq X$, let $\mathcal{O}(U)$ be the ring of regular functions on U as in Definition 2.3.1. This makes X into a space with functions. Note that by Theorem 2.2.2, the two meanings of $\mathcal{O}(X)$ we have introduced agree.

Theorem 2.3.4. *Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set. Then for every space with functions Y, the pull-back map*

$$\phi \mapsto \phi^* \colon \operatorname{Hom}(Y,X) \longrightarrow \operatorname{Hom}_k(\mathcal{O}(X),\mathcal{O}(Y))$$

is bijective.

Note that by Yoneda's lemma, this determines the swf X if we know the ring $\mathcal{O}(X)$. (There may be other swfs with the same $\mathcal{O}(X)$, but only one of them is an affine algebraic set.)

Proof. **Injectivity** is easy: if $\phi,\psi\colon Y \to X$ are two maps and $y \in Y$ is such that $\phi(y) = x \neq x' = \psi(y)$, we find an $f \in \mathcal{O}(X)$ with $f(x) \neq f(x')$ (for example, one of the coordinates $T_1,\dots,T_n$), and then $\phi^*(f)(y) = f(x) \neq f(x') = \psi^*(f)(y)$, and $\phi^* \neq \psi^*$.

Surjectivity: Let $\phi^*\colon \mathcal{O}(X) \to \mathcal{O}(Y)$ be a k-algebra homomorphism, for which we seek to build the corresponding map of swf's $\phi\colon Y \to X$. For each $y \in Y$, we have the evaluation map $\mathrm{ev}_y\colon \mathcal{O}(Y) \to k$ mapping $f \mapsto f(y)$. Consider the composition

$$\mathcal{O}(X) \xrightarrow{\phi^*} \mathcal{O}(Y) \xrightarrow{\mathrm{ev}_y} k.$$

This defines an element $x \in \operatorname{Hom}_k(\mathcal{O}(X),k)$, which equals X by the Nullstellensatz. We define the map ϕ by $\phi(y) = x$. This defines a map (of sets) $\phi\colon Y \to X$ inducing ϕ^*. Moreover, the pull-back of the basic open set $D(f) \subseteq X$ is $D(\phi^* f) \subseteq Y$, which is open by axiom (b) of the definition of an swf, which shows that ϕ is continuous. We omit the (easy) verification that ϕ is a morphism of swf's. □

Remark 2.3.5. Here is a direct way of reconstructing the swf X from the reduced k-algebra $A = \mathcal{O}(X)$. We set $X = \operatorname{MSpec}(A) = \operatorname{Hom}_k(A,k)$. We give it the induced topology from $\operatorname{Spec}(A)$, in other words generated by the base open sets $D(g)$ for $g \in A$. Finally, we call a function $f\colon U \to k$ defined on an open $U \subseteq X$ regular for every $x \in U$ there exist $g,h \in A$ such that $D(g) \subseteq U$ and $f(y) = h(y)/g(y)$ for every $y \in D(g)$.

Corollary 2.3.6. *The category of* **affine algebraic sets** *(defined as a full subcategory of the category of swf's) is equivalent to the opposite category of the category of* **finitely generated reduced k-algebras***.*

Corollary 2.3.7 (Products of affine algebraic sets)**.** *The category of affine algebraic sets admits products. More precisely, let $X = V(I) \subseteq \mathbb{A}^n$ (with coordinates $T_1,\dots,T_n$) and $Y = V(J) \subseteq \mathbb{A}^m$ (with coordinates $U_1,\dots,U_m$) be two affine algebraic sets. Then*

$$X \times Y = V(I+J) \subseteq \mathbb{A}^{n+m}$$

is the product of X and Y in the category of swf's, and we have

$$\mathcal{O}(X \times Y) = \mathcal{O}(X) \otimes_k \mathcal{O}(Y).$$

Proof. This is straightforward except for the fact that $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is reduced. For this, see Proposition 5.17 in Milne's notes[1] (which also shows that $\mathcal{O}(X \times Y)$ is a domain if $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are domains. □

[1] https://www.jmilne.org/math/CourseNotes/AG.pdf

We are finally able to define algebraic sets (which we will later identify as reduced schemes of finite type over k) and varieties.

Definition 2.3.8. Let k be an algebraically closed field.

(1) An **algebraic set** over k is a space with functions X admitting a finite open cover $X = U_1 \cup \cdots \cup U_n$ where each U_i is isomorphic as an swf to an affine algebraic set.

(2) We say that an algebraic set is a **variety** if it is irreducible.

(3) We say that X is **projective** if it is isomorphic to a projective algebraic set (see below).

(4) We say that X is **quasi-affine** if it is isomorphic to an open subset of an affine algebraic set, and **quasi-projective** if it is isomorphic to an open subset of a projective algebraic set.

We note the key fact that not only can every algebraic set be covered by open affine algebraic sets, but the **affine open subsets form a base** for the topology on X (since if $U \subseteq X$ is affine, then a basis of opens of U is given by the sets $D(g)$ for $g \in \mathcal{O}(U)$).

Example 2.3.9. The projective space $\mathbb{P}^n$ has the standard open cover $U_0 \cup \cdots \cup U_n$ by affine spaces. We call a function on a locally closed subset Z of $\mathbb{P}^n$ regular if its restriction to each $Z \cap U_i$ is regular in the sense of Definition 2.3.1. Thus every locally closed subset of $\mathbb{P}^n$ is an swf and moreover an algebraic set.

Example 2.3.10. The punctured plane $\mathbb{A}^2 \setminus \{0\}$ is quasi-affine but not affine (see below). Similarly, the punctured projective plane $\mathbb{P}^2 \setminus \{P\}$ for a point P is quasi-projective, but neither projective nor quasi-affine.

Lemma 2.3.11. *(a)* $\mathcal{O}(\mathbb{A}^{n+1} \setminus 0) = \mathcal{O}(\mathbb{A}^{n+1})$ *for* $n \geq 1$;

(b) $\mathcal{O}(\mathbb{P}^n) = k$;

(c) Let $X \subseteq \mathbb{P}^n$ *be a projective variety (closed and irreducible subset). Then* $\mathcal{O}(X) = k$.

Proof. (a) The set $U = \mathbb{A}^{n+1} \setminus 0$ is the union of $D(T_i)$, $i = 0, \ldots, n$. We have $\mathcal{O}(D(T_i)) = \mathcal{O}(\mathbb{A}^{n+1})[T_i^{-1}]$. Consider all of these as subrings of $k[T_0^{\pm 1}, T_n^{\pm 1}]$, then $\mathcal{O}(U)$ is their intersection, which equals $k[T_0, \ldots, T_n] = \mathcal{O}(\mathbb{A}^{n+1})$.

(b) For this we use the fact (easy proof omitted) that for an open (or locally closed) $W \subseteq \mathbb{P}^n$, a function $f \colon W \to k$ is regular if and only if $f \circ \pi$ is regular on $\pi^{-1}(W)$ where $\pi \colon U \to \mathbb{P}^n$ is the quotient map. The fact for $W = \mathbb{P}^n$ combined with (a) implies that $\mathcal{O}(\mathbb{P}^n)$ consists of all $f \in k[T_0, \ldots, T_n]$ which are invariant under scaling of the coordinates, i.e. homogeneous of degree zero. But $k[T_0, \ldots, T_n]_0 = k$.

(c) We shall prove this later. □

Example 2.3.12 (Ojanguren)**.** In all examples of algebraic sets we have encountered so far, the ring $\mathcal{O}(X)$ was a finitely generated k-algebra. This is true for affine algebraic sets and projective algebraic sets, but for completely different reasons. In general for an algebraic set X, the ring $\mathcal{O}(X)$ might be non-Noetherian. Here is a simple example, found by Ojanguren. Consider the projective three-space $\mathbb{P}^3$ with homogeneous coordinates $(X : Y : Z : T)$ and the subsets

$$W = V_{\mathbb{P}}(XY) \subseteq \mathbb{P}^3, \qquad L = V_{\mathbb{P}}(X, Z) \subseteq W, \qquad U = X \setminus L.$$

Thus W is the union of two hyperplanes $H_1 = V_{\mathbb{P}}(X), H_2 = V_{\mathbb{P}}(Y) \simeq \mathbb{P}^2$ in $\mathbb{P}^3$ intesecting along the line $V_{\mathbb{P}}(X, Y) \simeq \mathbb{P}^1$. The set $L \simeq \mathbb{P}^1$ is another line, contained in one of the planes H_1 and intersecting the other H_2 in a single point $Q = (0 : 0 : 0 : 1)$. Let us calculate $\mathcal{O}(U)$. A regular function f on U restricts to a regular function f_1 on $H_1 \setminus L \simeq \mathbb{A}^2$ (with coordinates $u = Y/Z$ and $v = T/Z$) and a regular function f_2 on $H_2 \setminus Z \simeq \mathbb{P}^2 \setminus Q$. But $\mathcal{O}(\mathbb{P}^2 \setminus Q) = k$, so f_2 is constant. It follows that

$$\mathcal{O}(U) = \{f \in k[u,v] \,:\, f(0,v) \in k\}$$

which is not Noetherian (as the ideal $v \cdot k[u,v]$ is contained in $\mathcal{O}(U)$ and is an ideal there, generated by vu^n for all $n \geq 0$ but not by any proper subset).

2.4. Problem session (Jan 27)

During the second problem session:

1. We discussed Ojanguren's example (Example 2.3.12).
2. As a prelude to products of (non-affine) algebraic sets and the **Segre embedding** (see Lecture 3), we constructed an isomorphism between $\mathbb{P}^1 \times \mathbb{P}^1$ and the quadric surface

$$Q = V(XY - ZW) \subseteq \mathbb{P}^2.$$

   The map $\mathbb{P}^1 \times \mathbb{P}^1 \to Q$ is given by

$$((u_0 : u_1), (v_0 : v_1)) \quad \mapsto \quad (\underbrace{u_0 v_0}_{X} : \underbrace{u_1 v_1}_{Y} : \underbrace{u_0 v_1}_{Z} : \underbrace{u_1 v_0}_{W}).$$

3. We introduced **algebraic groups**. An algebraic group is an algebraic set G endowed with a structure of a group for which the multiplication and inverse maps

$$\mu \colon G \times G \longrightarrow G, \qquad \iota \colon G \to G$$

   are morphisms of algebraic sets. We gave a list of examples:

   (a) The **additive group** $\mathbb{G}_a$, which is the affine line $\mathbb{A}^1 = k$ and the group structure is given by addition in k.

   (b) The **multiplicative group** $\mathbb{G}_m$, the punctured affine line $\mathbb{A}^1 \setminus 0 = k^\times$, where the group structure is given by multiplication.

   (c) The n-the **roots of unity** $\mu_n \subseteq \mathbb{G}_m$, the subgroup of $\mathbb{G}_m$ cut out by the equation $T^n - 1 = 0$. It is a finite group of order $n/\gcd(n, p^\infty)$ where $p = \mathrm{char}(k)$ if the latter is positive and $p = 1$ otherwise.

   (d) The **general linear group** GL_n of invertible $n \times n$ matrices, which is the open subset $D(\Delta)$ of $\mathbb{A}^{n^2}$ (with coordinates T_{ij}) given by the nonvanishing of the determinant $\Delta = \det[T_{ij}]$. The group structure is given by multiplication of matrices.

   (e) As we shall later learn **elliptic curve** (a smooth cubic in $\mathbb{P}^2$ with a chosen basepoint) has a unique group structure in which the basepoint is the neutral element. Moreover, this group structure is commutative.

   Examples (a)–(d) are affine and (e) is not affine.

   We discussed how **affine algebraic groups** correspond to commutative finite type and reduced **Hopf algebras** over k. Let G be an affine algebraic group and let $A = \mathcal{O}(G)$ be its coordinate ring, a finite type and reduced k-algebra. Then the multiplication μ and inverse ι correspond to k-algebra

$$\mu^* \colon A \to A \otimes_k A, \qquad \iota^* \colon A \to A$$

("comultiplication" and "coinverse"), satisfying the "duals" of the group axioms. For instance, associativity of G is expressed by the commutativity of the square

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mu \times \mathrm{id}} & G \times G \\ {\scriptstyle \mathrm{id} \times \mu} \downarrow & & \downarrow {\scriptstyle \mu} \\ G \times G & \xrightarrow[\mu]{} & G \end{array}$$

which translates into the commutativity of

$$\begin{array}{ccc} A \otimes A \otimes A & \xleftarrow{\mathrm{id} \otimes \mu^*} & A \otimes A \\ {\scriptstyle \mu^* \otimes \mathrm{id}} \uparrow & & \uparrow {\scriptstyle \mu^*} \\ A \otimes A & \xleftarrow[\mu^*]{} & A. \end{array}$$

A k-algebra A endowed with a coidentity $\varepsilon^* \colon A \to k$, a comultiplication $\mu^* \colon A \to A \otimes A$ and a coinverse $\iota^* \colon A \to A$ satisfying these axioms is called a (commutative) **Hopf algebra** over k. The equivalence of categories between affine algebraic sets and the opposite category of reduced finite type k-algebras thus induces an equivalence between affine algebraic groups and the oppositve category of reduced finite type Hopf algebras over k.

We computed that in the examples (a)–(d) above, the comultiplication maps are given by

(a) $T \mapsto T_0 + T_1 \colon k[T] \to k[T_0, T_1]$

(b) $T \mapsto T_0 T_1 \colon k[T, T^{-1}] \to k[T_0, T_0^{-1}, T_1, T_1^{-1}]$

(c) $T \mapsto T_0 T_1 \colon k[T]/(T^n - 1) \to k[T_0, T_1]/(T_0^n - 1, T_1^n - 1)$

(d) $T_{ij} \mapsto \sum_k T_{0,ik} T_{1,kj} \colon k[T_{ij}][\Delta^{-1}] \to k[T_{0,ij}, T_{1,ij}][\Delta_0^{-1}, \Delta_1^{-1}]$.