18. Lecture 18 (Apr 9): Riemann–Roch

Recommended reading: Hartshorne IV 1, Kempf §7–8

18.1. Basic facts about curves

Let C be a smooth and complete one-dimensional variety and let $k(C)$ be its function field.

1. Every rational map from C to a projective (or just complete) variety X is everywhere defined.
2. We have $k(C) = \mathrm{Hom}(C, \mathbb{P}^1)$.
3. Every map $f{:}C \to C'$ to another smooth complete curve is either constant or finite. In the latter case, $f_*\mathcal{O}_C$ is locally free of rank $d = \deg(f) = [k(C) : k(C')]$, and for every divisor D on C' we have $\deg(f^*D) = \deg(f)\cdot\deg(D)$.
4. The curve C is projective (Sketch of proof: a non-constant $f \in k(C)$ gives a finite map $f{:}C \to \mathbb{P}^1$. Let $L = f^*\mathcal{O}(1)$ and consider the graded ring $R = \bigoplus_{n\geq 0} H^0(C, L^n) = \bigoplus_{n\geq 0} H^0(\mathbb{P}^1, (f_*\mathcal{O}_C)(n))$. We can show it is finitely generated, and so $R = k[x_0, \dots, x_n]/I$, $C = V(I) \subseteq \mathbb{P}^n$.)
5. For a non-constant $f \in k(C)$ corresponding to a finite map $f{:}C \to \mathbb{P}^1$, we have $\mathrm{div}(f) = f^*((0) - (\infty))$.
6. The degree of every principal divisor on C is zero. (This follows from (3) and (5).)

18.2. The Riemann–Roch theorem

Let C be a smooth projective curve. We introduce the following notation and terminology:

- $\ell(D) = \dim H^0(C, \mathcal{O}_C(D)) = \dim\{f \in k(C) : \mathrm{div}(f) \geq -D\}$;
- K is any divisor for which $\mathcal{O}_C(K) \simeq \omega_X$ (a **canonical divisor**);
- $g = \ell(K_C)$ is the **genus** of C;
- for a coherent sheaf $\mathcal{F}$ on C we write $\chi(\mathcal{F}) = \dim H^0(C, \mathcal{F}) - \dim H^1(C, \mathcal{F})$ for its **Euler characteristic**.

Theorem 18.2.1. *Let C be a smooth projective curve and D a divisor on C. Then*

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g.$$

Proof. Denote the left-hand side by $\mathrm{LHS}(D)$ and the right-hand side by $\mathrm{RHS}(D)$. By Serre duality:

$$\begin{aligned}\mathrm{LHS}(D) &= \dim H^0(C, \mathcal{O}_C(D)) - \dim H^0(X, \mathcal{O}_C(D)^\vee \otimes \omega_C)\\ &= \dim H^0(C, \mathcal{O}_C(D)) - \dim H^1(C, \mathcal{O}_C(D))^\vee\\ &= \chi(\mathcal{O}_C(D)).\end{aligned}$$

For a point $P \in C$ consider $D' = D + P$. We have a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_C(D) \arrow[r] & \mathcal{O}_C(D') \arrow[r] & k_P \arrow[r] & 0 \end{tikzcd}$$

where k_P is the skyscraper sheaf at P. From this we get

$$\mathrm{LHS}(D') = \chi(\mathcal{O}_C(D')) = \chi(\mathcal{O}_C(D)) + \chi(k_P) = \mathrm{LHS}(D) + 1,$$

and at the same time obviously

$$\mathrm{RHS}(D') = \mathrm{RHS}(D) + 1.$$

Thus the assertion holds for D if and only if it holds for D'. By induction, it suffices to verify the formula for $D = 0$:

$$\mathrm{LHS}(0) = \ell(0) - \ell(K) = 1 - g = \mathrm{RHS}(0). \qquad \square$$

Corollary 18.2.2. *We have* $\deg(\omega_C) = 2g - 2$.

Corollary 18.2.3. *We have* $\ell(D) \geq \deg(D) + 1 - g$*, with equality if* $\deg(D) > 2g - 2$.

Corollary 18.2.4. *Every proper open subset* $U \subseteq X$ *is affine.*

Proof. Consider first the case $U = X \smallsetminus \{P\}$ for a single point P. Then $\ell(nP) > 1$ for $n \gg 0$, so we can find a non-constant rational function $f \in k(C)$ with $\mathrm{div}(f) \geq -nP$. Thus f has a pole at P and no other poles, and hence the corresponding finite map $f: C \to \mathbb{P}^1$ satisfies $f^{-1}(\infty) = \{P\}$, so $U = f^{-1}(\mathbb{A}^1)$ is affine.

For the general case, write $U = X \smallsetminus \{P_1, \ldots, P_r\}$, and for each $i = 1, \ldots, r$ let f_i be a rational function constructed above for $P = P_i$. Take $f = f_1 + \cdots + f_r$, then f has poles precisely at $P_1, \ldots, P_r$, so again for the corresponding map $f: C \to \mathbb{P}^1$ we have $U = f^{-1}(\mathbb{A}^1)$. $\square$

18.3. Curves of genus zero (a.k.a. rational curves)

Lemma 18.3.1. *Let C be a smooth projective curve. The following are equivalent:*

(a) $g(C) = 0$;

(b) there exist $P, Q \in C$ with $P \neq Q$ and $P \sim Q$ (meaning that $P - Q$ is a nonzero principal divisor);

(c) $C \simeq \mathbb{P}^1$.

Proof. (a)⇒(b): By Riemann–Roch, we have $\ell(P - Q) = 1$, so that $P - Q \sim D$ for an effective divisor D. But $\deg(D) = 0$, so $D = 0$.

(b)⇒(c): Suppose that $P - Q = \mathrm{div}(f)$ and let $f: C \to \mathbb{P}^1$ be the corresponding finite map. Then $Q = f^*(\infty)$ and hence $\deg(f) = 1$, so that f is an isomorphism.

(c)⇒(a): We computed that $\omega_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$, so $g = \ell(K) = 0$. $\square$

18.4. Curves of genus one (a.k.a. elliptic curves)

Definition 18.4.1. An **elliptic curve** is a pair $(E, \underline{0})$ where E is a smooth projective curve of genus one and $\underline{0} \in E$ is a point.

Lemma 18.4.2. *Let E be an elliptic curve. We have* $\omega_E \simeq \mathcal{O}_E$.

Proof. We have $\deg(\omega_E) = 2g - 2 = 0$ and $\dim H^0(E, \omega_E) = g = 1$, so that $\omega_E \simeq \mathcal{O}_E$. $\square$

Lemma 18.4.3. *Let* $\mathrm{Pic}^0(E) \subseteq \mathrm{Pic}(E)$ *be the subgroup consisting of divisor classes of degree zero. The map* $E \to \mathrm{Pic}^0(E)$ *mapping* $P \in E$ *to the class of* $P - \underline{0}$ *is bijective.*

Proof. Injectivity follows from Lemma 18.3.1. For surjectivity, let $D = \sum n_i P_i$ be a divisor of degree zero, which we rewrite as

$$D = \sum n_i (P_i - \underline{0}) + \left(\sum n_i\right)\underline{0} = \sum n_i (P_i - \underline{0}).$$

By induction on the number of terms, it suffices to show that for every $P, Q \in E$ there exists a (unique) $R \in P$ such that

$$(P - \underline{0}) + (Q - \underline{0}) \sim (R - \underline{0}).$$

By Riemann–Roch, we have $\ell(D) = \deg(D)$ as long as $\deg(D) > 0$. Apply this to $D = P + Q - \underline{0}$ to get $\ell(P + Q - \underline{0}) = 1$. Thus there exists a unique effective divisor R with $P + Q - \underline{0} \sim R$. Since $\deg(R) = \deg(P + Q - \underline{0}) = 1$, this divisor R is a point, and we are done. ☐

Corollary 18.4.4. *There is a natural structure of a commutative group variety on E with neutral element* $\underline{0}$.

(Technically we did not show that the group structure $E \times E \to E$ is a morphism of varieties.)

More facts about elliptic curves (assuming k is not of characteristic 2 or 3): There exists a finite map $E \to \mathbb{P}^1$ of degree two, ramified at four points, which after a change of coordinates on $\mathbb{P}^1$ can be taken to be $0, 1, \infty, \lambda$. The curve is then isomorphic to the plane curve

$$y^2 z = x(x - z)(x - \lambda z),$$

where the point $\underline{0}$ is sent to $(0 : 1 : 0)$. For three points $P, Q, R \in E$, we have $P + Q + R = 0$ in the group structure on E if and only if there exists a line $\ell \subseteq \mathbb{P}^2$ such that $\ell \cap \mathbb{P}^2 = P + Q + R$ as divisors.

18.5. Bonus: proof of Serre duality

We shall outline Kempf's ingenious proof of Serre duality for curves. As we have seen in both the proof of Riemann–Roch and in its applications, important information about an invertible sheaf $\mathcal{L}$ on a smooth projective curve C is encoded by the boundary map

$$\delta_P : k \longrightarrow H^1(C, \mathcal{L})$$

induced by the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L} \arrow[r] & \mathcal{L}(P) \arrow[r] & k_P \arrow[r] & 0 \end{tikzcd}$$

for a chosen point $P \in C$: the map δ_P is zero if and only if there is a section of $\mathcal{L}(P)$ which is not a section of $\mathcal{L}$. The idea behind the proof is to consider all of these maps at the same time as the point P varies. As we shall see, taken together they assemble into a map of vector bundles on C.

In order to state this precisely, we first need an important correction: the quotient $\mathcal{L}(P)/\mathcal{L}$ is only non-canonically isomorphic to the skyscraper k_P at P; rather, it is the skyscraper at P with value the fiber of $\mathcal{L}(P)$ at P[1]. For example, for $\mathcal{L} = \mathcal{O}_C$, this quotient is $\mathfrak{m}_P^{-1}/\mathcal{O}_{C,P}$; this space is dual to $\mathfrak{m}_P/\mathfrak{m}_P^2$, which is the fiber of ω_C at P. In general, it will be this tensored with $\mathcal{L}$. So it makes sense to expect that there exists a map of locally free sheaves on C of the form

$$\delta : \mathcal{L} \otimes \omega_C^{-1} \longrightarrow H^1(C, \mathcal{L}) \otimes_k \mathcal{O}_C;$$

whose fiber at P is the map δ_P. Here the source can be identified with $\mathcal{H}om(\omega_C, \mathcal{L})$ and the target is the free sheaf with fiber $H^1(C, \mathcal{L})$. In particular, if $\mathcal{L} = \omega_C$, the map δ produces an element τ of

$$H^0(C, H^1(C, \omega_C) \otimes_k \mathcal{O}_C) = H^0(C, \mathcal{O}_C) \otimes_k H^1(C, \omega_C) = H^1(C, \omega_C).$$

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[1] Warning: by accident, this fiber should be denoted by $\mathcal{L}(P)(P)$. Since we are on a curve, a point can be considered a divisor, and so two notational conventions are in conflict: $\mathcal{F}(x) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ for the fiber of a coherent sheaf $\mathcal{F}$ at a point $x \in X$, and $\mathcal{L}(D) = \mathcal{L} \otimes \mathcal{O}_X(D)$ for an invertible sheaf $\mathcal{L}$ and a divisor D on X.

This turns out to be a basis element (i.e. the map δ for $\mathcal{L} = \omega_C$ is an isomorphism), yielding the isomorphism $H^1(C,\omega_C) \xrightarrow{\sim} k$ which is part of the formulation of Serre duality.

We relegate the definition of the map δ to the next subsection (see Proposition 18.6.1). Here we show how to complete the proof of Serre duality assuming its existence.

We begin with some preliminaries about cohomology of invertible sheaves on curves.

Lemma 18.5.1. *Let $\mathcal{L}$ be an invertible sheaf on a smooth projective curve C, and let D be a divisor on C for which $\mathcal{L} \simeq \mathcal{O}_X(D)$. Having made this choice, we regard $\mathcal{L}$ as a subsheaf of the constant sheaf $\mathcal{K}_X$.*

(a) The exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{K}_C \longrightarrow \mathcal{K}_C/\mathcal{L} \longrightarrow 0 \tag{18.5.1}$$

is a flabby resolution of $\mathcal{L}$.

(b) Consequently, we have

$$H^1(C,\mathcal{L}) \simeq \operatorname{coker}(k(C) \to \Gamma(C,\mathcal{K}_C/\mathcal{L}))\,.$$

(c) The group $H^1(C,\mathcal{L})$ is zero if and only if, for every effective divisor E and every $P \in C$ we have

$$\dim\Gamma(C,\mathcal{L}(E+P)) = \dim\Gamma(C,\mathcal{L}(E)) + 1. \tag{18.5.2}$$

Proof. (a) The sheaf $\mathcal{K}_C$ is the constant sheaf with value $k(C)$, and hence is flabby as C is irreducible. On the other hand, we can write

$$\mathcal{K}_C = \varinjlim_{E\geq 0} \mathcal{L}(E),$$

and hence $\mathcal{K}_C/\mathcal{L} = \varinjlim_{E\geq 0} \mathcal{L}(E)/\mathcal{L}$ (by exactness of direct limits). Now each quotient sheaf $\mathcal{L}(E)/\mathcal{L}$ is a direct sum of skyscraper sheaves, and hence is flabby. Since C is noetherian, taking sections commutes with direct limits of sheaves, and we conclude that $\mathcal{K}_C/\mathcal{L}$ is flabby. In fact, it is isomorphic to the direct sum of the skyscraper sheaves $k(C)/\mathcal{L}_P$ at all $P \in C$.

(b) Follows from (a) by Lecture 15, Lemma 15.2.2(b).

(c) The condition is equivalent to the claim that

$$\dim\Gamma(C,\mathcal{L}(E)) = \dim\Gamma(C,\mathcal{L}) + \deg E \qquad \text{for all } E \geq 0. \tag{18.5.3}$$

The exact sequence (18.5.1) is the inductive limit of the sequences

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(E) \longrightarrow \mathcal{L}(E)/\mathcal{L} \longrightarrow 0. \tag{18.5.4}$$

Our is thus equivalent to the statement that these sequences remain exact after taking global sections. Taking inductive limits is exact, so condition (18.5.3) implies that the exact sequence (18.5.1) is exact after applying global sections, which means precisely that $H^1(C,\mathcal{L}) = 0$. Conversely, if $H^1(C,\mathcal{L}) = 0$, then the cohomology exact sequence associated to (18.5.4) shows that $\Gamma(C,\mathcal{L}(E)) \to \Gamma(C,\mathcal{L}(E)/\mathcal{L})$ is surjective, as desired. □

Lemma 18.5.2. *Let $P_1,\dots,P_r$ be distinct points on C and let $\mathcal{L}$ be an invertible sheaf. Then the boundary map $k^r \to H^1(C,\mathcal{L})$ induced by the sequence*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(P_1+\cdots+P_r) \longrightarrow \oplus_{i=1}^r k_{P_i} \longrightarrow 0 \tag{18.5.5}$$

equals the direct sum of the maps $\delta_{P_1},\dots,\delta_{P_r}$.

Proof. For every effective divisor E, we can regard $\Gamma(C,\mathcal{L}(E)/\mathcal{L})$ as a subspace of $\Gamma(C,\mathcal{K}_C/\mathcal{L})$, and then $\delta_E:\Gamma(C,\mathcal{L}(E)/\mathcal{L}) \to H^1(C,\mathcal{L})$ is the restriction of the surjection $\delta:\Gamma(C,\mathcal{K}_C/\mathcal{L}) \to H^1(C,\mathcal{L})$ to that subspace. In the situation at hand, if $E = P_1 + \cdots + P_r$, the subspace $\Gamma(C,\mathcal{L}(E)/\mathcal{L}) \subseteq \Gamma(C,\mathcal{K}_C/\mathcal{L})$ is the direct sum of the one-dimensional subspaces $\Gamma(C,\mathcal{L}(P_i)/\mathcal{L})$ for $i = 1,\ldots,r$. □

The very existence of the map δ has an important consequence.

Corollary 18.5.3. *If* $\deg(\mathcal{L}) > \deg(\omega_C)$ *then* $H^1(C,\mathcal{L}) = 0$.

Proof. The assumption implies that $\mathcal{L} \otimes \omega_C^{-1}$ has positive degree, and hence there are no nonzero maps $\mathcal{L} \otimes \omega_C^{-1} \to \mathcal{O}_C$. Consequently, the map δ for $\mathcal{L}$ is zero (as the target is non-canonically the direct sum of copies of $\mathcal{O}_C$). Therefore $\delta(P) = \delta_P$ is zero for all $P \in C$. The same argument applies to $\mathcal{L}(D)$ for any effective divisor D. By Lemma 18.5.1(c), this implies that $H^1(C,\mathcal{L}) = 0$. □

Corollary 18.5.4. *The map* δ *for* $\mathcal{L} = \omega_C$ *gives an isomorphism* $\mathcal{O}_C = \omega_C \otimes \omega_C^{-1} \xrightarrow{\sim} H^1(C,\omega_C) \otimes_k \mathcal{O}_C$. *Consequently,* $H^1(C,\omega_C)$ *is canonically isomorphic to* k.

Proof. Consider the set of integers d for which there exists an invertible sheaf $\mathcal{L}$ of degree d with $H^1(C,\mathcal{L}) \neq 0$. By Corollary 18.5.3, the set is also bounded from above. We can therefore find an invertible sheaf $\mathcal{L}$ with $H^1(C,\mathcal{L}) \neq 0$ such that $\deg(\mathcal{L})$ is maximal (if $H^1(C,\mathcal{L}) = 0$ for all $\mathcal{L}$ (a case which never happens and is easily ruled out), take any $\mathcal{L}$ in what follows). Then $H^1(C,\mathcal{L}(P)) = 0$ for every $P \in C$, and consequently $\delta_P \neq 0$ for every P. It follows that the map $\delta:\mathcal{L} \otimes \omega_C^{-1} \to H^1(C,\mathcal{L}) \otimes_k \mathcal{O}_C$ is an isomorphism: the invertible sheaf $\mathcal{L} \otimes \omega_C^{-1}$ is isomorphic to the free sheaf $H^1(C,\mathcal{L}) \otimes_k \mathcal{O}_C$. Then $\mathcal{L} \simeq \omega_C$, $H^1(C,\mathcal{L})$ is one-dimensional, and (taking $\mathcal{L} = \omega_C$) that we have a canonical isomorphism $H^1(C,\omega_C) \simeq k$. □

Since $H^0(C,\mathcal{O}_C) = k$, for any two vector spaces V and W we have $\mathrm{Hom}(V,W) \xrightarrow{\sim} \mathrm{Hom}(V \otimes_k \mathcal{O}_C, W \otimes_k \mathcal{O}_C)$. Thus to prove Serre duality, we want to show that the map

$$\alpha:\mathrm{Hom}(\mathcal{L},\omega_C) \longrightarrow \mathrm{Hom}(H^1(C,\mathcal{L}) \otimes_k \mathcal{O}_C, H^1(C,\omega_C) \otimes_k \mathcal{O}_C), \qquad \phi \mapsto H^1(\phi) \otimes 1.$$

By naturality of the map δ, for a map $\phi:\mathcal{L} \to \omega_C$, we have a commutative square

$$\begin{tikzcd}
\mathcal{L} \otimes \omega_C^{-1} \arrow[r, "\delta"] \arrow[d, "\phi\otimes 1"'] & H^1(C,\mathcal{L}) \otimes_k \mathcal{O}_C \arrow[d, "H^1(\phi)\otimes 1"] \\
\omega_C \otimes \omega_C^{-1} \arrow[r, "\sim"] & H^1(C,\omega_C) \otimes_k \mathcal{O}_C.
\end{tikzcd}$$

Since the bottom map is an isomorphism, we can treat the left map $\phi \otimes 1$ as the (pre)composition of the right map $H^1(\phi) \otimes 1$ with δ. We thus have a candidate for an inverse to α, namely

$$\beta:\mathrm{Hom}(H^1(C,\mathcal{L}),H^1(C,\omega_C)) \longrightarrow \mathrm{Hom}(\mathcal{L} \otimes \omega_C^{-1}, \omega_C \otimes \omega_C^{-1}) = \mathrm{Hom}(\mathcal{L},\omega_C), \quad \psi \mapsto ((\psi \otimes 1) \circ \delta) \otimes \omega_C.$$

We have just observed that the composition $\beta \circ \alpha$ is the identity, in particular α is injective.

To finish, it suffices to show β is injective. Suppose $\psi:H^1(C,\mathcal{L}) \to H^1(C,\omega_C)$ is such that $(\psi \otimes 1) \circ \delta = 0$. This means that $\delta_P \circ \psi = 0$ for every $P \in C$. Therefore it remains to show the following:

Lemma 18.5.5. *The space* $H^1(C,\mathcal{L})$ *is spanned by the images of* δ_P *for all* $P \in C$.

Proof. For distinct points $P_1,\ldots,P_r$, Lemma 18.5.2 implies that the sum of the images of $\delta_{P_1},\ldots,\delta_{P_r}$ equals the image of the boundary map for (18.5.5), which is the kernel of the next map

$$H^1(C,\mathcal{L}) \longrightarrow H^1(C,\mathcal{L}(P_1 + \cdots + P_r)).$$

It therefore suffices to find $P_1,\ldots,P_r$ for which the latter group is zero. By Corollary 18.5.3, this will be the case as long as $r > \deg(\omega_C) - \deg(\mathcal{L})$. □

18.6. Construction of the map δ

In order to make sense of $\mathcal{L}(P)$ for a varying $P \in C$, we make use of the diagonal $\Delta \subseteq C \times C$, which is a divisor on the smooth projective surface $C \times C$. Let $p, q: C \times C \to C$ be the two projections

$$\begin{tikzcd} & C \arrow[d, "\Delta"] \arrow[ddl, bend right, "\mathrm{id}"'] \arrow[ddr, bend left, "\mathrm{id}"] & \\ & C \times C \arrow[dl, "p"] \arrow[dr, "q"'] & \\ C & & C \end{tikzcd}$$

and consider the invertible sheaf $q^*\mathcal{L}$ and its twist $q^*\mathcal{L}(\Delta) = q^*\mathcal{L} \otimes \mathcal{O}_{C\times C}(\Delta)$. The restriction of the latter to $p^{-1}(P) = \{P\} \times C \simeq C$ is $\mathcal{L}(P)$. On $C \times C$, we have the short exact sequence

$$0 \longrightarrow q^*\mathcal{L} \longrightarrow q^*\mathcal{L}(\Delta) \longrightarrow q^*\mathcal{L}(\Delta)|_\Delta \longrightarrow 0.$$

Recall that (identifying the diagonal Δ with C via p) we have

$$\omega_C = \mathcal{J}_\Delta/\mathcal{J}_\Delta^2 = \mathcal{J}_\Delta|_\Delta = \mathcal{O}_{C\times C}(-\Delta)|_\Delta$$

and hence the rightmost sheaf $q^*\mathcal{L}(\Delta)|_\Delta$ is identified with $\Delta_*(\mathcal{L} \otimes \omega_C^{-1})$, which is the source of our postulated map δ.

In order to construct δ, we need to consider "cohomology in a family," i.e. the derived functors $R^i p_*$ of the first projection $p: C \times C \to C$. They are defined in the same way as cohomology (say, using flabby resolutions), but replacing global sections with $p_*(-)$, and turn short exact sequences on the source $C \times C$ into long exact sequences of sheaves on the target C. In particular, we get the boundary map

$$\delta: \mathcal{L} \otimes \omega_C^{-1} = p_*(p^*\mathcal{L}(\Delta)|_\Delta) \longrightarrow R^1 p_* p^* \mathcal{L}.$$

As we shall see, the target is $H^1(C, \mathcal{L}) \otimes_k \mathcal{O}_C$, and the fiber of δ at $P \in C$ is δ_P.

Proposition 18.6.1. *There is a functorial way of associating, to every invertible sheaf $\mathcal{L}$ on C, a map of coherent sheaves*

$$\delta: \mathcal{L} \otimes \omega_C^{-1} \longrightarrow H^1(C, \mathcal{L}) \otimes_k \mathcal{O}_C$$

such that for every $P \in C$ we have a commutative square

$$\begin{tikzcd} (\mathcal{L} \otimes \omega_C^{-1})(P) \arrow[r, "\delta(P)"] \arrow[d, "\simeq"'] & (H^1(C, \mathcal{L}) \otimes_k \mathcal{O}_C)(P) \arrow[d, equal] \\ k \arrow[r, "\delta_P"'] & H^1(C, \mathcal{L}). \end{tikzcd}$$

(In words: the fiber of δ at P is δ_P, up to a choice of basis of $(\mathcal{L} \otimes \omega_C^{-1})(P)$.)

Proof. As we have mentioned, this follows from the general machinery of higher direct images. Let us construct the map by hand. Let $C = U_0 \cup U_1$ be an affine open cover of C and let $U_{01} = U_0 \cap U_1$ (which is also affine). (As we have seen, any non-constant rational function on C gives a finite map $f: C \to \mathbb{P}^1$, and we can take $U_0 = C \smallsetminus f^{-1}(\infty)$ and $U_1 = C \smallsetminus f^{-1}(0)$.) As in the construction of the Čech complex, the inclusions $j_i\colon C\times U_i \hookrightarrow C\times C$ are affine, and for a coherent sheaf $\mathcal{F}$ on $C\times C$, we have an exact sequence of quasi-coherent sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \underbrace{j_{0,*}\mathcal{F}|_{C\times U_0} \oplus j_{1,*}\mathcal{F}|_{C\times U_1}}_{C^0(\mathcal{F})} \longrightarrow \underbrace{j_{01,*}\mathcal{F}|_{C\times U_{01}}}_{C^1(\mathcal{F})} \longrightarrow 0.$$

Applying p_* we obtain a complex of sheaves on C, exact on the left

$$0 \longrightarrow p_*\mathcal{F} \longrightarrow p_*C^0(\mathcal{F}) \longrightarrow p_*C^1(\mathcal{F}) \longrightarrow 0$$

Moreover, for any affine open $U\subseteq C$, the restriction of the above sequence to U is the (complex of sheaves associated to) the Čech complex of $\mathcal{F}|_{U\times C}$ and the open cover of $U\times C$ by $U\times U_i$.

Applying this to the terms of the short exact sequence

$$0 \longrightarrow q^*\mathcal{L} \longrightarrow q^*\mathcal{L}(\Delta) \longrightarrow \Delta_*(\mathcal{L}\otimes\omega_C^{-1}) \longrightarrow 0$$

we obtain an exact diagram of the form

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & p_*q^*\mathcal{L} \arrow[r] \arrow[d] & p_*C^0(q^*\mathcal{L}) \arrow[r, "\beta"] \arrow[d] & p_*C^1(q^*\mathcal{L}) \arrow[d] \\
0 \arrow[r] & p_*q^*\mathcal{L}(\Delta) \arrow[r] \arrow[d] & p_*C^0(q^*\mathcal{L}(\Delta)) \arrow[r] \arrow[d] & p_*C^1(q^*\mathcal{L}(\Delta)) \arrow[d] \\
0 \arrow[r] & p_*\Delta_*(\mathcal{L}\otimes\omega_C^{-1}) \arrow[r] & p_*C^0(\Delta_*(\mathcal{L}\otimes\omega_C^{-1})) \arrow[r, "\gamma"'] \arrow[d] & p_*C^1(\Delta_*(\mathcal{L}\otimes\omega_C^{-1})) \arrow[d] \\
 & & 0 & 0
\end{tikzcd}$$

(the two rightmost columns are exact since the compositions $p\circ j_i$ are affine). Snake lemma yields a map

$$\delta\colon \mathcal{L}\otimes\omega_C^{-1} = p_*\Delta_*(\mathcal{L}\otimes\omega_C^{-1}) \longrightarrow \operatorname{coker}(\beta).$$

On any affine $U\subseteq C$, the top row is the Čech complex of $\mathcal{L}$ tensored with $\mathcal{O}(U)$. Thus $\operatorname{coker}(\beta)\simeq H^1(C,\mathcal{L})\otimes_k\mathcal{O}_C$, and we obtain the desired map

$$\delta\colon \mathcal{L}\otimes\omega_C^{-1} = p_*\Delta_*(\mathcal{L}\otimes\omega_C^{-1}) \longrightarrow H^1(C,\mathcal{L})\otimes_k\mathcal{O}_C.$$

Clearly, the construction is functorial in $\mathcal{L}$. (Technically speaking, we have not checked that it is also independent of the choice of the open cover.)

For a point $P\in C$, we make a further pull-back of this diagram to P. The columns remain exact since the stalks of modules on the bottom at P are torsion-free and hence flat. Moreover, we have $\operatorname{coker}(\beta)(P) = \operatorname{coker}(\beta(P))$. Finally, we need to argue that $\ker(\gamma)(P) = \ker(\gamma(P))$. Since the bottom row is simply the sheafy Čech complex of $\mathcal{L}\otimes\omega_C^{-1}$, the map γ is surjective. Since its target again is a flat $\mathcal{O}_C$-module, we can argue as for the left-exactness of the columns. The outcome of this analysis is that we have a commutative square

$$\begin{tikzcd}
(\mathcal{L}\otimes\omega_C^{-1})(P) \arrow[r, "\delta(P)"] \arrow[d, "\simeq"'] & (H^1(C,\mathcal{L})\otimes_k\mathcal{O}_C)(P) \arrow[d, "\simeq"] \\
k \arrow[r, "\delta_P"'] & H^1(C,\mathcal{L})
\end{tikzcd}$$

as desired. □